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BOUNDED SOLUTIONS OF LINEAR PERTURBED DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract. Conditions for the existence of solutions bounded on the entire real axis for a linear weakly perturbed differential equation in a Banach space,

$$\dot{x} = A(t)x + \varepsilon A_1(t)x + f(t), \quad f(t) \in BC(\mathbb{R}, \mathbf{B}),$$
$$x\colon \mathbb{R} \to \mathbf{B}, \ x(\cdot) \in BC^1(\mathbb{R}, \mathbf{B}),$$

are obtained under the assumption that the corresponding linear unperturbed homogeneous equation is exponentially dichotomous on both semi-axes. Examples of the existence of bounded solutions for countable systems of differential equations are considered.

1. Statement of the problem

In a Banach space $\mathbf{B}$, we consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + \varepsilon A_1(t)x(t) + f(t), \tag{1}$$

where the vector function $f(t)$ acts from $\mathbb{R}$ into the Banach space $\mathbf{B}$,

$$f(t) \in BC(\mathbb{R}, \mathbf{B}) := \big\{ f(\cdot) : \mathbb{R} \to \mathbf{B}, f(\cdot) \in C(\mathbb{R}), |||f||| = \sup_{t \in R} \|f(t)\| < \infty \big\},$$

$BC(\mathbb{R}, \mathbf{B})$ is the Banach space of functions continuous and bounded on $\mathbb{R}$; the operator-valued functions $A(t)$ and $A_1(t)$ are strongly continuous [1, p. 141] with the norm $|||A||| = \sup_{t\in\mathbb{R}} \|A(t)\| < \infty$, $|||A_1||| = \sup_{t\in\mathbb{R}} \|A_1(t)\| < \infty$, and the

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solution $x = x(t, \varepsilon)$ of the equation

$$x(t,\varepsilon) = x_0 + \int_{t_0}^{t} \big(A(s)x(s) + \varepsilon A_1(s)x(s) + f(s)\big)\,\mathrm{d}s$$

is continuously differentiable at every point $t \in \mathbb{R}$ and satisfies Eq. (1) everywhere in $\mathbb{R}$. We seek the solution $x(t,\varepsilon)$ of Eq. (1) in the Banach space $BC^1(\mathbb{R}, \mathbf{B})$ of functions continuously differentiable on $\mathbb{R}$ and bounded together with their derivative, ε is a small parameter.

We determine conditions for the existence of solutions $x(\cdot,\varepsilon) \in BC^1(\mathbb{R}, \mathbf{B})$ of Eq. (1) under the assumption that the corresponding unperturbed homogeneous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) \tag{2}$$

admits an exponential dichotomy [2],[3] on the semi-axes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors P and Q, respectively, i.e., there exist projectors $P(P^2 = P)$ and $Q(Q^2 = Q)$ and constants $k_{1,2} \geq 1$ and $\alpha_{1,2} > 0$ such that the estimates

$$\begin{aligned} \big\|U(t)PU^{-1}(s)\big\| &\leq k_1 e^{-\alpha_1(t-s)}, \qquad t \geq s, \\ \big\|U(t)(E-P)U^{-1}(s)\big\| &\leq k_1 e^{\alpha_1(t-s)}, \quad s \geq t, \qquad \text{for all} \quad t,s \in \mathbb{R}_+, \end{aligned}$$

and

$$\begin{aligned} \big\|U(t)QU^{-1}(s)\big\| &\leq k_2 e^{-\alpha_2(t-s)}, \qquad t \geq s, \\ \big\|U(t)(E-Q)U^{-1}(s)\big\| &\leq k_2 e^{\alpha_2(t-s)}, \quad s \geq t, \qquad \text{for all} \quad t,s \in \mathbb{R}_- \end{aligned}$$

hold, where $U(t) = U(t,0)$ is the evolution operator of Eq. (2) such that

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = A(t)U(t), \quad U(0) = E \text{ is the identity operator [1, p. 145].}$$

2. Preliminaries

Now we formulate the following result, which is proved in [4] for the unperturbed nonhomogeneous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + f(t)\,. \tag{3}$$

Theorem 1. *Suppose that the homogeneous equation* (2) *admits an exponential dichotomy on the semi-axes* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P *and* Q, *respectively. If the operator*

$$D = P - (E - Q) : \mathbf{B} \to \mathbf{B} \tag{4}$$

acting from the Banach space $\mathbf{B}$ *onto itself is invertible in the generalized sense* [5, p. 26], *then*

(i) *in order that solutions of* Eq. (3) *bounded on the entire real axis exist, it is necessary and sufficient that the function* $f(t) \in BC(\mathbb{R}, \mathbf{B})$ *satisfies the condition*

$$\int_{-\infty}^{+\infty} H(t)\, f(t)\, \mathrm{d}t = 0\,, \tag{5}$$

where

$$H(t) = \mathcal{P}_{N(D^*)} Q U^{-1}(t) = \mathcal{P}_{N(D^*)} (E - P) U^{-1}(t)\,,$$

(ii) *under condition* (5), *solutions bounded on the entire axis of* Eq. (3) *have the form*

$$x(t, c) = U(t) P \mathcal{P}_{N(D)} c + (Gf)(t), \qquad \text{for all} \quad c \in \mathbf{B}\,, \tag{6}$$

where

$$\big(G[f]\big)(t) = U(t) \begin{cases} \int\limits_0^t PU^{-1}(s) f(s)\, \mathrm{d}s - \int\limits_t^\infty (E-P) U^{-1}(s) f(s)\, \mathrm{d}s \\ + PD^- \left[\int\limits_0^\infty (E-P) U^{-1}(s) f(s)\, \mathrm{d}s + \int\limits_{-\infty}^0 QU^{-1}(s) f(s)\, \mathrm{d}s \right], & t \geq 0, \\ \int\limits_{-\infty}^t QU^{-1}(s) f(s)\, \mathrm{d}s - \int\limits_t^0 (E-Q) U^{-1}(s) f(s)\, \mathrm{d}s \\ +(E-Q) D^- \left[\int\limits_0^\infty (E-P) U^{-1}(s) f(s)\, \mathrm{d}s + \int\limits_{-\infty}^0 QU^{-1}(s) f(s)\, \mathrm{d}s \right], & t \leq 0 \end{cases} \tag{7}$$

is the generalized Green operator of the problem for solutions bounded on the entire axis, D^- *is the generalized inverse of* D, $\mathcal{P}_{N(D)} = E - D^- D$ *and* $\mathcal{P}_{N(D^*)} = E - DD^-$, c *is an arbitrary constant element of the Banach space* $\mathbf{B}$.

Remarks. I. The condition that the operator D is invertible in the generalized sense means [5],[6] the following. An operator $D : \mathbf{B} \to \mathbf{B}$ is invertible in the generalized sense if and only if

(i) the operator D is a normally resolvable operator $\Longleftrightarrow \overline{\operatorname{Im} D} = \operatorname{Im} D$;

(ii) the subspace $\ker D$ has a direct complement in $\mathbf{B}$;

(iii) the subspace $\operatorname{Im} D$ has a direct complement in $\mathbf{B}$.

II. In a finite-dimensional case, where $\mathbf{B} = \mathbb{R}^n$ and the operator D is an $n \times n$-dimensional matrix, properties (i), (ii), (iii) are always satisfied (because the subspaces $\ker D$ and $\ker D^* = coker D$ are finite-dimensional) and from Theorem 1 we obtain the well-known Palmer's lemma [3] and results [5], [8],[9].

III. If we consider Eq. (3) in $\mathbb{R}^n$ under the assumption that the corresponding linear homogeneous equation is exponentially dichotomous on both semi-axes, then operator

$$(Lx)(t) = \dot{x}(t) - A(t)x(t)$$

must be only Fredholm operator [3]. In the case when we consider this equation in the Banach space B under the same conditions we have considerably greater variants. The operator $(Lx)(t)$ may be

1) or **a normally resolvable operator** $\iff$ $\overline{\operatorname{Im} L} = \operatorname{Im} L$; $\big(\dim\ker(L) = \dim\ker(L^*) = \infty\big)$;
2) or **a d-normally resolvable operator** (according to the classification by S. G. Krein [7]: $\big(\dim\ker(L^*) = d < \infty, \dim\ker(L) = \infty\big)$;
3) or **an n-normally resolvable operator** $\big(\dim\ker(L) = n < \infty, \dim\ker(L^*) = \infty\big)$;
4) or **a Fredholm operator** $\big(\dim\ker(L) < \infty \ \dim\ker(L^*) < \infty\big)$.

3. Main result

Let us find conditions on the perturbation term $A_1(t)$ under which Eq. (1) has at least one solution bounded on the entire axis in the space $BC^1(\mathbb{R}, \mathbf{B})$, and show that this problem can be solved using the operator

$$B_0 = \int_{-\infty}^{+\infty} H(t)A_1(t)U(t)P\mathcal{P}_{N(D)}\,\mathrm{d}t : \mathbf{B} \to \mathbf{B} \tag{8}$$

constructed from the perturbed coefficient of Eq. (1).

Theorem 2. *Let the homogeneous equation* (2) *admit an exponential dichotomy on the semi-axes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors P and Q, respectively, and the operator $D = P - (E - Q) : \mathbf{B} \to \mathbf{B}$ acting from the Banach space $\mathbf{B}$ onto itself be invertible in the generalized sense. Assume that following conditions are satisfied:*

1) *the operator* B_0 *is invertible in the generalized sense;*

2) $\mathcal{P}_{N(B_0^*)}\mathcal{P}_{N(D^*)}Q = 0$.

a) If for $f \in BC(\mathbb{R}, \mathbf{B})$ *the unperturbed equation* (3) *does not have solutions bounded on the entire real axis, then equation* (1) *possesses at least one solution bounded on* $\mathbb{R}$ *in the form of a part of the Laurent series in powers of the small parameter,*

$$x(t,\varepsilon) = \sum_{i=-1}^{+\infty} \varepsilon^i x_i(t)\,, \tag{9}$$

convergent for sufficiently small $\varepsilon \in (0, \varepsilon_*]$.

b) If for $f \in BC(\mathbb{R}, \mathbf{B})$ *the unperturbed equation* (3) *has bounded solutions, then equation* (1) *possesses at least one solution bounded on* $\mathbb{R}$ *in the form of the series*

$$x(t,\varepsilon) = \sum_{i=0}^{+\infty} \varepsilon^i x_i(t)\,, \tag{10}$$

convergent for sufficiently small $\varepsilon \in [0, \varepsilon_*]$.

Proof. We substitute series (9) in equation (1) and equate the coefficients of the same powers of ε. The problem of determining the coefficient $x_{-1}(t)$ of the term with ε^{-1} in series (9) is reduced to the problem of finding bounded solutions of the homogeneous equation

$$\frac{\mathrm{d}x_{-1}(t)}{\mathrm{d}t} = A(t)x_{-1}(t)\,.$$

By virtue of Theorem 1, this equation has a family of bounded solutions

$$x_{-1}(t, c_{-1}) = U(t)P\mathcal{P}_{N(D)}c_{-1}\,,$$

for any given element $c_{-1} \in \mathbf{B}$, which will be defined below.

The problem of determining the coefficient $x_0(t)$ of the term with ε^0 in series (9) reduces to the problem of finding bounded solutions of the equation

$$\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} = A(t)x_0(t) + A_1(t)x_{-1}(t, c_{-1}) + f(t)\,. \tag{11}$$

Using Theorem 1, a criterion of solvability of problem (11) has the form

$$\int_{-\infty}^{+\infty} H(t)\bigl\{A_1(t)x_{-1}(t, c_{-1}) + f(t)\bigr\}\,\mathrm{d}t = 0\,, \tag{12}$$

from which we, finally, have the operator equation

$$B_0 c_{-1} = -\int\limits_{-\infty}^{+\infty} H(t)f(t)\,\mathrm{d}t\,.$$

Under conditions 1) and 2) this equation is solvable and one of its solutions has the form

$$c_{-1} = -B_0^- \int\limits_{-\infty}^{+\infty} H(t)f(t)\,\mathrm{d}t\,,$$

where B_0^- is the generalized inverse of the operator B_0.

If conditions 1) and 2) are satisfied, then equation (11) possesses a family of solutions bounded on $\mathbb{R}$,

$$x_0(t,c_0) = U(t)P\mathcal{P}_{N(D)}c_0 + \Big(G\big[A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot)\big]\Big)(t), \quad \text{for all} \quad c_0 \in \mathbf{B}\,,$$

where c_0 is an arbitrary constant element of the Banach space $\mathbf{B}$ determined, in the next step of the process, from the condition of solvability of the problem of determining of the coefficient $x_1(t)$ of series (9); $\big(G[*]\big)(t)$ is the generalized Green operator (7) of the problem for bounded solutions of equation (3).

The problem of determining the coefficient $x_1(t)$ of the term with ε^1 in series (9) reduces to the problem of determining bounded solutions of the equation

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = A(t)x_1(t) + A_1(t)x_0(t,c_0)\,. \tag{13}$$

Using the criterion of solvability of this equation,

$$\int\limits_{-\infty}^{+\infty} H(t)A_1(t)U(t)P\mathcal{P}_{N(D)}\,\mathrm{d}t\;\; c_0 + \int\limits_{-\infty}^{+\infty} H(t)A_1(t)\Big(G\big[A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot)\big]\Big)(t)\,\mathrm{d}t\; = 0$$

and conditions 1) and 2), we determine the constant $c_0 \in \mathbf{B}$ as

$$c_0 = -B_0^- \int\limits_{-\infty}^{+\infty} H(t)A_1(t)\Big(G\big[A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot)\big]\Big)(t)\,\mathrm{d}t\,.$$

Thus, equation (13) possesses a family of solutions bounded on $\mathbb{R}$,

$$x_1(t,c_1) = U(t)P\mathcal{P}_{N(D)}c_1 + \Big(G\big[A_1(\cdot)x_0(\cdot,c_0)\big]\Big)(t), \qquad \text{for all} \quad c_1 \in \mathbf{B}\,.$$

As above, one can easily show (by induction) that, under conditions 1) and 2), the problem of determining the coefficients $x_i(t)$ of the terms with ε^i in series (9) is reduced to the problem of finding solutions of the equation

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = A(t)x_i(t) + A_1(t)x_{i-1}(t, c_{i-1})\,. \tag{14}$$

If conditions 1) and 2) are satisfied, then the family of solutions, bounded on the entire real axis, of equation (14) has the form

$$x_i(t, c_i) = U(t)P\mathcal{P}_{N(D)}c_i + \Big(G\big[A_1(\cdot)x_{i-1}(\cdot, c_{i-1})\big]\Big)(t)\,,$$

where c_i is given by the formula

$$c_i = -B_0^- \int\limits_{-\infty}^{+\infty} H(t)A_1(t)\Big(G\big[A_1(\cdot)x_{i-1}(\cdot, c_{i-1}) + f(\cdot)\big]\Big)(t)\,\mathrm{d}t\,.$$

We now show that, for sufficiently small $\varepsilon \in (0, \varepsilon_*]$, series (9) converges for $t \in \mathbb{R}$. First, we recall some inequalities,

$$\begin{array}{rcll} \|H(t)\| & = & \|\mathcal{P}_{N(D^*)}(E-P)U^{-1}(t)\| \le k_1 p e^{-\alpha_1 t}, & t \ge 0\,, \\ \|H(t)\| & = & \|\mathcal{P}_{N(D^*)}QU^{-1}(t)\| \le k_2 p e^{\alpha_2 t}, & t \le 0\,, \end{array}$$

where $p = \max\{\|P_{N(D)}\|, \|P_{N(D^*)}\|\}$. These estimates follows from the inequalities,

$$\begin{array}{rcll} \|U(s)PU^{-1}(t)\| & \le & k_1 e^{-\alpha_1(s-t)}, & s \ge t\,, \\ \|U(s)(E-P)U^{-1}(t)\| & \le & k_1 e^{-\alpha_1(t-s)}, & s \le t\,, \end{array}$$

for all $t, s \in \mathbb{R}_+$, and

$$\begin{array}{rcll} \|U(s)QU^{-1}(t)\| & \le & k_2 e^{-\alpha_2(s-t)}, & s \ge t\,, \\ \|U(s)(E-Q)U^{-1}(t)\| & \le & k_2 e^{-\alpha_2(t-s)}, & s \le t\,, \end{array}$$

for all $t, s \in \mathbb{R}_-$.

Using these inequalities, we obtain the following estimate for the generalized Green operator:

$$\|(G[f])(t)\| \leq \begin{cases} \left\{ \int\limits_0^t k_1 e^{-\alpha_1(t-s)}\,\mathrm{d}s + \int\limits_t^{+\infty} k_1 e^{\alpha_1(t-s)}\,\mathrm{d}s \right. \\ \left. + k_1 N \left[\int\limits_{-\infty}^0 k_2 e^{\alpha_2 s}\,\mathrm{d}s + \int\limits_0^{+\infty} k_1 e^{-\alpha_1 s}\,\mathrm{d}s \right] \right\} \|f(t)\|, & t \geq 0\,, \\ \left\{ \int\limits_{-\infty}^t k_2 e^{-\alpha_2(t-s)}\,\mathrm{d}s + \int\limits_t^0 k_2 e^{\alpha_2(t-s)}\,\mathrm{d}s\,, \right. \\ \left. + k_2 N \left[\int\limits_{-\infty}^0 k_2 e^{\alpha_2 s}\,\mathrm{d}s + \int\limits_0^{+\infty} k_1 e^{-\alpha_1 s}\,\mathrm{d}s \right] \right\} \|f(t)\|, & t \leq 0\,. \end{cases}$$

Finally,

$$\|(G[f])(t)\| \leq K\|f(t)\|\,,$$

where

$$N = \|D^-\|, \quad K = \max_{i=1,2} k_i \left(\frac{1}{\alpha_i} + Nk \right), \quad k = \frac{k_2}{\alpha_2} + \frac{k_1}{\alpha_1}\,.$$

Further, we have

$$\begin{aligned} \|c_1\| &\leq baK \left\{ \int\limits_{-\infty}^0 k_2 p e^{\alpha_2}\,\mathrm{d}s + \int\limits_0^{+\infty} k_1 p e^{-\alpha_1 s}\,\mathrm{d}s \right\} \|A_1(t) x_0(t, c_0)\| \\ &= bpa^2 kK \|x_0(t, c_0)\|\,, \end{aligned}$$

$$\|x_1(t, c_1)\| \leq k\|c_1\| + Ka\|x_0(t, c_0)\| \leq aK(abplk + 1)\|x_0(t, c_0)\|\,,$$

where

$$b = \|B_0^-\|, a = \|A_1(t)\|, l = \max\{k_1 p, k_2 p\}.$$

Finally,

$$\|c_i\| \leq (aK)^i abpk(abplk + 1)^{i-1}\|x_0(t, c_0)\|\,,$$
$$\|x_i(t, c_i)\| \leq \big[aK(abplk + 1)\big]^i \|x_0(t, c_0)\| (i = 1, 2, \ldots)\,.$$

Thus, for all $t \in \mathbb{R}$ series (9) is majorized by the series

$$\varepsilon^{-1}\|x_{-1}(t, c_{-1})\| + \sum_{i=0}^{+\infty} \big[\varepsilon aK(abplk + 1)\big]^i \|x_0(t, c_0)\|\,.$$

In this case, $\|x_{-1}(t, c_{-1})\|$ and $\|x_0(t, c_0)\|$ are bounded. Therefore, for $t \in \mathbb{R}$ and all $\varepsilon \in (0, \varepsilon_*]$ series (9) is convergent for $\varepsilon_* < \big[aK(abplk + 1)\big]^{-1}$. This completes the proof of item a) and similarly [5], [8] we can prove item b). Similarly results may be proved for difference equations [13], [14]. □

4. Example

We now illustrate the assertions proved above. Consider the countable system of differential equations,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + \varepsilon A_1(t)x(t) + f(t)\,, \tag{15}$$

where $A(t)$ and $A_1(t)$ are operators in the form of a countably-dimensional matrix that, for every real value t, acts on the Banach space $\mathbf{B} = m$ [11], [12] with the norm

$$|||A|||_{\mathbf{B}} = \sup_{t\in\mathbb{R}, i,j\in\mathbb{N}} |a_{ij}(t)| < \infty\,,$$

and

$$\begin{array}{rcl} x(t) = col\big\{x_1(t), x_2(t), \dots\, x_k(t), \dots\big\} & \in & BC^1(\mathbb{R}, m)\,, \\ f(t) = col\big\{f_1(t), f_2(t), \dots, f_k(t), \dots\big\} & \in & BC(\mathbb{R}, m) \end{array}$$

are countable vector-columns. We consider system (15) with the operator

$$A(t) = \begin{pmatrix} -th\ t & 0 & 0 & 0 & 0 & \dots \\ 0 & th\ t & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 0 & \dots \\ . & . & . & . & . & \dots \end{pmatrix}. \tag{16}$$

The evolution operator of system (2), (16) has the form

$$U(t) = \begin{pmatrix} 2/(e^t + e^{-t}) & 0 & 0 & 0 & \dots \\ 0 & (e^t + e^{-t})/2 & 0 & 0 & \dots \\ 0 & 0 & e^{-t} & 0 & \dots \\ . & . & . & . & \dots \end{pmatrix}.$$

The homogeneous system (2), (16) is exponentially dichotomous on the semi-axes $\mathbb{R}_+$ and $\mathbb{R}_-$ with the projectors

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ . & . & . & . & . & \dots \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ . & . & . & . & . & \dots \end{pmatrix},$$

respectively. Thus, we have

$$D = P - (E - Q) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ . & . & . & . & . & \dots \end{pmatrix} = D^* = D^t \,.$$

It is easy to verify that $\dim N(D) = \dim N(D^*) = 2$ and D is a Fredholm operator of index zero.

The projectors of the operator D and its adjoint operator onto the null spaces are defined as follows:

$$\mathcal{P}_{N(D)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ . & . & . & . & \dots \end{pmatrix} = \mathcal{P}_{N(D^*)} \,.$$

In this case

$$\mathcal{P}_{N(D^*)} Q = \begin{pmatrix} 0 & 0 & \dots \\ 0 & 1 & \dots \\ . & . & \dots \end{pmatrix} ,$$

$$P \mathcal{P}_{N(D)} = \begin{pmatrix} 1 & 0 & \dots \\ 0 & 0 & \dots \\ . & . & \dots \end{pmatrix}$$

and the operator B_0^* has the form

$$B_0^* = B_0^t = \int\limits_{-\infty}^{+\infty} H(t) A_1(t) U(t) P P_{N(D)} \, \mathrm{d}t \ = \begin{pmatrix} 0 & \int\limits_{-\infty}^{+\infty} \frac{4a_{21}(t)}{(e^t + e^{-t})^2} \, \mathrm{d}t & 0 & \dots \\ 0 & 0 & 0 & \dots \\ . & . & . & \dots \end{pmatrix} .$$

If

$$\int\limits_{-\infty}^{+\infty} \frac{a_{21}(t)}{(e^t + e^{-t})^2} \, \mathrm{d}t \ \neq 0 \,,$$

then

$$\mathcal{P}_{N(B_0^*)} = \operatorname{diag}\{1, 0, 1, \cdots, 1, \cdots\} \quad \text{and} \quad \mathcal{P}_{N(B_0^*)}\mathcal{P}_{N(D^*)}Q = 0 .$$

Thus, according to Theorem 2, system (15), (16) possesses at least one solution, moreover in our case system (15), (16) has a unique solution bounded on $\mathbb{R}$. For example, if $a_{21}(t) = const \neq 0$, then for an arbitrary (except $a_{21}(t)$) elements $a_{ij}(t) \in BC(\mathbb{R}, \mathbb{R})$ of the perturbed countable matrix $A_1(t)$, this condition is always satisfied.

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