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ON THE DIFFERENTIAL EQUATION WITH POWER COEFFICIENTS AND PROPORTIONAL DELAYS

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ABSTRACT. The main result presents the description of some asymptotic properties of the delay differential equation

$$\dot{y}(t) = \sum_{j=0}^{m} a_j t^{\alpha_j} y(\lambda_j t), \quad t > 0\,,$$

where a_j, α_j, λ_j are real scalars such that $a_j \neq 0$ $(j = 0, \ldots, m)$, $\lambda_0 = 1$ and $0 < \lambda_j < 1$ $(j = 1, \ldots, m)$. We show that the asymptotic behaviour of the solutions essentially depends on the sign of a_0 and, as an auxiliary result, we find a solution of a special functional equation.

1. Introduction

The aim of this paper is to discuss the asymptotic behaviour of the solutions of the differential equation with rescaling

$$\dot{y}(t) = \sum_{j=0}^{m} a_j t^{\alpha_j} y(\lambda_j t), \qquad t > 0\,, \tag{1.1}$$

where a_j, α_j, λ_j are real scalars such that $a_j \neq 0$ $(j = 0, \ldots, m)$, $\lambda_0 = 1$ and $0 < \lambda_j < 1$ $(j = 1, \ldots, m)$.

The qualitative investigation of (1.1) is of great importance because of interesting applications as well as for theoretical reasons. The list of diverse applications of linear differential equations with proportional delays can be found in

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[14]. In particular, equations with power (or more generally polynomial) coefficients and proportional delays appear in control theory in the context of the remote control problem of a moving away object, in the case when the distance between the control panel and the object cannot be neglected.

The theoretical study of (1.1) started at the beginning of the seventies, when T. Kato and J. B. McLeod [12] described the asymptotic and oscillatory properties of (1.1) with $m = 1$ and $\alpha_0 = \alpha_1 = 0$. Later, many papers generalizing these results have appeared. Among numerous papers we can mention at least those by E. B. Lim [15], L. Pandolfi [18], A. Iserles [10] and [11], Y. Liu [16], G. Makay and J. Terjeki [17], [2], [3] and many others. For further related results we refer, e.g., to papers by O. Arino and M. Pituk [1], J. Diblík [6] or I. Győri and M. Pituk [8]. In particular, the asymptotic investigation of differential equations with polynomial coefficients and linearly compressed arguments was the subject of papers published by G. Derfel [4] and G. Derfel and F. Vogl [5].

The recapitulation of some relevant results will be carried out in Section 2. Here we also outline, how these results can motivate us in the asymptotic investigation of equation (1.1). As it might be expected, we have to distinguish the cases a_0 positive (Section 3) and a_0 negative (Section 4).

As it is customary, by a solution of (1.1) we understand a real valued function y, which is defined and continuous on an interval $[\sigma^*, \infty)$, where $\sigma^* > 0$, continuously differentiable on $[\sigma, \infty)$, where $\sigma = \sigma^* \max(\lambda_1^{-1}, \dots, \lambda_m^{-1})$ and satisfying (1.1) on $[\sigma, \infty)$. We note that all the results of this paper remain valid also for solutions of (1.1) defined on $[0, \infty)$.

2. Some auxiliary results

In this section, we first recall some related results concerning the delay equation (1.1). If we consider (1.1) with constant coefficients and a single delay, i.e.,

$$\dot{y}(t) = a_0 y(t) + a_1 y(\lambda t), \qquad 0 < \lambda < 1, \qquad t > 0\,, \tag{2.1}$$

then we have (see ([12]).

Theorem 2.1. (i) *Let $a_0 > 0$, $a_1 \neq 0$. Then for any solution y of* (2.1) *there exists a constant L such that*

$$\lim_{t\to\infty} \exp\{-a_0 t\} y(t) = L\,.$$

(ii) *Let $a_0 < 0$, $a_1 \neq 0$. Then*

$$y(t) = O(t^\gamma), \quad \text{as } t \to \infty, \qquad \gamma = \frac{\log \frac{|a_1|}{-a_0}}{\log \lambda^{-1}}$$

for any solution y of (2.1).

Considering the nonautonomous equation (2.1), i.e.,

$$\dot{y}(t) = a_0(t)y(t) + a_1(t)y(\lambda t), \qquad 0 < \lambda < 1, \qquad t > 0\,, \tag{2.2}$$

we can recall the following result derived in [17], which is relevant with the respect to the asymptotic investigation of (1.1).

Theorem 2.2. *Consider equation* (2.2)*, where the functions $a_0, a_1 \in C\big((0,\infty)\big)$ are such that $|a_1(t)| \le -Qa_0(t)$ for a suitable $Q > 0$ and all t large enough and $a_0(t) \le -mt^{-\alpha}$ for all t large enough, where $m > 0$ and $\alpha < 1$ are suitable constants. If y is a solution of* (2.2)*, then*

$$y(t) = O(t^\gamma) \qquad \text{as } t \to \infty, \qquad \gamma = \frac{\log Q}{\log \lambda^{-1}}\,.$$

As we remarked above, the asymptotic investigation of differential equations with power coefficients and proportional delays was the subject of papers [4] and [5]. We recall here the following asymptotic result from [4], which we reformulate for equation (1.1). Let us emphasize that the assumption $\lambda_0 = 1$ is now replaced by $0 < \lambda_0 < 1$.

Theorem 2.3. *Consider equation* (1.1)*, where a_j, α_j, λ_j are real scalars such that $a_j \neq 0$ and $0 < \lambda_j < 1$ $(j = 0, \dots, m)$. Then there exists $\tilde{\delta} > \frac{1}{2\log \lambda_j^{-1}}$ $(j = 0, \dots, m)$ such that any solution y of* (1.1) *satisfies the estimate*

$$y(t) = O\big(\exp\{\delta \log^2 t\}\big) \qquad \text{as } t \to \infty\,,$$

where $\delta > \tilde{\delta}$.

Our aim is to generalize or extend some of the previous results to equation (1.1). For some preliminary ideas it can be useful to note that the functions $\exp\{a_0 t\}$ and t^γ, $\gamma = \frac{\log \frac{|a_1|}{-a_0}}{\log \lambda^{-1}}$ appearing in the asymptotic relations of Theorem 2.1 are solutions of equations

$$\dot{y}(t) = a_0 y(t), \qquad\qquad t > 0$$

and

$$a_0\omega(t) + |a_1|\omega(\lambda t) = 0, \qquad t > 0\,,$$

respectively. A similar interpretation can be done also for the function t^γ appearing in the O-term presented in Theorem 2.2. These connections lead us to the question, whether the asymptotic behaviour of solutions of (1.1) can be related (under specified sign condition imposed on a_0) to the behaviour of a solution of the differential equation

$$\dot{y}(t) = a_0 t^{\alpha_0} y(t), \qquad t > 0$$

or to the behaviour of a solution of the mth order functional equation

$$\sum_{j=1}^{m} |a_j| t^{\alpha_j} \omega(\lambda_j t) = -a_0 t^{\alpha_0} \omega(t), \qquad t > 0\,. \tag{2.3}$$

As we shall see in the next sections, this conjecture is true and, moreover, instead of the functional equation (2.3) we can consider the corresponding inequality

$$\sum_{j=1}^{m} |a_j| t^{\alpha_j} \omega(\lambda_j t) \le -a_0 t^{\alpha_0} \omega(t), \qquad t > 0\,. \tag{2.4}$$

Hence, our next step is to find a particular solution of the functional inequality (2.4) so that the corresponding asymptotic result could be effective. To simplify the process of solving (2.4) it is enough to find a solution of the first order functional equation

$$t^{\beta} \varphi(\lambda t) = b \varphi(t), \qquad t > 0 \tag{2.5}$$

with suitable reals $b > 0$, $0 < \lambda < 1$ and β. There is no general method how to explicitly solve linear functional equations in a single variable; solutions of these equations are usually expressed by means of iterative formulas (see, e.g., M. Kuczma et al. [13]). However, in some particular cases, it is possible to find a solution in the required explicit form.

Equation (2.5) was solved in the particular case $\beta = 1$ in the paper by M. L. Heard [9], where this equation arised in the context of the qualitative investigation of an advanced differential equation. The method of solving is based on the Mellin integral transformation $\Phi(s) = \int_0^\infty \varphi(t) t^{s-1}\,\mathrm{d}t$ and its properties. Although we can modify this method to obtain a solution of (2.5), we shall proceed in a different way and propose an alternative method of solving (2.5) which seems to be much more simple that the method described in [9].

One of the most important equations of linearization is that of Abel. Particularly, in the case of (2.5) its form becomes

$$\rho(\lambda t) = \rho(t) - 1$$

and admits the solution $\rho(t) = \frac{\log t}{\log \lambda^{-1}}$. If we set $s = \frac{\log t}{\log \lambda^{-1}}$, $\psi(s) = \varphi(t)$ in (2.5), then $\psi(s)$ defines a solution of the difference equation

$$\psi(s) = \frac{\psi(s-1)}{b\lambda^{\beta s}}\,. \tag{2.6}$$

Now considering this relation with s replaced successively by $s-1$, $s-2, \dots$, we can conclude that equation (2.6) admits the solution

$$\psi(s) = \frac{1}{b^s \lambda^{\beta\binom{s+1}{2}}}\,.$$

We note that another way of solving (2.6) is to take the logarithm in (2.6) to obtain a linear difference equation with constant coefficients and a polynomial forcing term. Then applying the method of undetermined coefficients (see, e.g., S. Elaydi [7]) we can arrive at the same form of ψ.

Substituting this back we get the required solution of (2.5) in the form

$$\varphi(t) = \exp\left\{\frac{\beta \log^2 t}{2\log\lambda^{-1}} + \left(\frac{\beta}{2} - \frac{\log b}{\log \lambda^{-1}}\right)\log t\right\}. \tag{2.7}$$

This solution is eventually nondecreasing provided $\beta > 0$ or $\beta = 0$ and $b \le 1$, and it is eventually decreasing otherwise (i.e., if $\beta < 0$ or $\beta = 0$ and $b > 1$).

Now we consider relations (2.4), (2.5) and put

$$\beta := \max(\alpha_1, \dots, \alpha_m) - \alpha_0, \quad b := \frac{-a_0}{\sum\limits_{j=1}^{m} |a_j|} \tag{2.8}$$

and

$$\lambda := \begin{cases} \max(\lambda_1, \dots, \lambda_m) & \text{if} \quad \beta > 0 \\ & \quad\;\; \text{or } \beta = 0 \quad \text{and} \quad b \le 1\,, \\ \min(\lambda_1, \dots, \lambda_m) & \text{if} \quad \beta < 0 \\ & \quad\;\; \text{or } \beta = 0 \quad \text{and} \quad b > 1\,. \end{cases} \tag{2.9}$$

Summarizing our previous calculations we can deduce

Proposition 2.4. *The function φ given by* (2.7) *with the above specified parameters β, b and λ defines an infinitely many differentiable and eventually monotonic solution of* (2.4) *on* $(0, \infty)$.

3. Asymptotic behaviour in the case $a_0 > 0$

In this section, we mention some asymptotic properties of equation (1.1) provided the coefficient a_0 is positive. The following result shows that the presence of delayed terms has no essential influence on the asymptotic behaviour of (1.1).

Theorem 3.1. *Consider equation* (1.1), *where a_j, α_j, λ_j are real scalars such that $a_0 > 0$, $\lambda_0 = 1$, $a_j \neq 0$ and $0 < \lambda_j < 1$ $(j = 1, \dots, m)$. Further assume that either $\alpha_0 > -1$ and α_j is arbitrary or $\alpha_0 \le -1$ and $\alpha_j < -1$ $(j = 1, \dots, m)$.*

If $\alpha_0 \neq -1$, then for any solution y of (1.1) *there exists a constant L (depending on y) such that*

$$\lim_{t\to\infty} \exp\left\{-\frac{a_0 t^{\alpha_0+1}}{\alpha_0+1}\right\} y(t) = L\,. \tag{3.1}$$

Conversely, for any real L there exists a solution y of (1.1) *such that* (3.1) *holds.*

If $\alpha_0 = -1$, then for any solution y of (1.1) *there exists a constant L such that*

$$\lim_{t\to\infty} \frac{y(t)}{t^{\alpha_0}} = L\,. \tag{3.2}$$

Conversely, for any real L there exists a solution y of (1.1) *such that* (3.2) *holds.*

The proof of Theorem 3.1 essentially utilizes known results concerning the existence of asymptotic equilibrium for functional differential equations. For our next purposes it is enough to recall the following result which is due to M. Pituk [19].

Theorem 3.2. *Consider the equation*

$$\dot{z}(t) = \sum_{j=1}^{m} p_j(t) z\big(\tau_j(t)\big), \qquad t \in [t_0, \infty), \tag{3.3}$$

where $p_j, \tau_j \in C\big([t_0,\infty)\big)$, $\tau_j(t) \le t$ for all $t \in [t_0,\infty)$ and $\inf\big\{\tau_j(t),\, t \in [t_0,\infty)\big\} > -\infty$ $(j = 1, \dots, m)$. If $\sum\limits_{j=1}^{m} \int\limits_{t_0}^{\infty} |p_j(t)|\,\mathrm{d}t$ converges, then every solution z of (3.3) *tends to a finite limit as $t \to \infty$.*

Moreover, if $\sum\limits_{j=1}^{m} \int\limits_{t_0}^{\infty} |p_j(t)|\,\mathrm{d}t < 1$ and L is a nonzero constant, then there exists a function z satisfying (3.3) *on $[t_0,\infty)$ and tending to L as $t \to \infty$.*

Proof of Theorem 3.1. Let y be a solution of (1.1). If $\alpha_0 \neq -1$, we set $z(t) = \exp\Big\{-\frac{a_0 t^{\alpha_0+1}}{\alpha_0+1}\Big\} y(t)$. Then the function z is a solution of

$$\dot{z}(t) = \sum_{j=1}^{m} a_j t^{\alpha_j} \exp\left\{\frac{a_0 t^{\alpha_0+1}}{\alpha_0+1}(\lambda_j - 1)\right\} z(\lambda_j t)\,, \tag{3.4}$$

i.e., (3.4) is equation (3.3) with $p_j(t) = a_j t^{\alpha_j} \exp\Big\{\frac{a_0 t^{\alpha_0+1}}{\alpha_0+1}(\lambda_j - 1)\Big\}$ $(j = 1, \dots, \dots, m)$. If $\alpha_0 = -1$, we set $z(t) = y(t)/t^{a_0}$. Then z is a solution of

$$\dot{z}(t) = \sum_{j=1}^{m} a_j \lambda_j^{a_0} t^{\alpha_j} z(\lambda_j t)\,,$$

which is equation (3.3) with $p_j(t) = a_j \lambda_j^{a_0} t^{\alpha_j}$ $(j = 1, \dots, m)$.

By a simple calculation we can check that in both cases $\sum\limits_{j=1}^{m} \int\limits_{\sigma}^{\infty} |p_j(t)|\,\mathrm{d}t$ converges for any $\sigma > 0$. Of course, considering σ large enough we have $\sum\limits_{j=1}^{m} \int\limits_{\sigma}^{\infty} |p_j(t)|\,\mathrm{d}t < 1$. Now the both assertions follow from Theorem 3.2. $\square$

4. Asymptotic behaviour in the case $a_0 < 0$

This section discusses the asymptotic behaviour of solutions of (1.1) provided a_0 is negative. As a main result, we confirm our conjecture posed in Section 2, namely that the asymptotic bound of the solutions of (1.1) can be expressed by means of a solution of the inequality (2.4).

Theorem 4.1. *Consider equation* (1.1)*, where a_j, α_j, λ_j are real scalars such that $a_0 < 0$, $\lambda_0 = 1$, $a_j \neq 0$ and $0 < \lambda_j < 1$ $(j = 1, \dots, m)$. Let β, b and λ be defined as in* (2.8), (2.9) *and let $\Lambda := \max(\lambda_1, \dots, \lambda_m)$.*

If either $\Lambda = \lambda$ (i.e., $\max(\alpha_1, \dots, \alpha_m) > \alpha_0$ or $\max(\alpha_1, \dots, \alpha_m) = \alpha_0$ and $\sum\limits_{j=1}^{m} |a_j| + a_0 \geq 0$) or $\alpha_0 > -1$, then

$$y(t) = O\left(\exp\left\{\frac{\beta \log^2 t}{2\log \lambda^{-1}} + \left(\frac{\beta}{2} - \frac{\log b}{\log \lambda^{-1}}\right)\log t\right\}\right) \qquad \text{as } t \to \infty$$

for any solution y of (1.1).

If $\max(\alpha_1, \dots, \alpha_m) = \alpha_0 = -1$, $\sum\limits_{j=1}^{m} |a_j| + a_0 < 0$ *and* $a_0 \le -2 \log b / (\log \lambda^{-1})$, *then*

$$y(t) = O\left(\exp\left\{\frac{\beta \log^2 t}{2 \log \lambda^{-1}} + \left(\frac{\beta}{2} + \frac{1}{\log \Lambda^{-1}} - \frac{\log b}{\log \lambda^{-1}}\right) \log t\right\}\right) \qquad \text{as } t \to \infty$$

for any solution y *of* (1.1).

P r o o f. Let y satisfy (1.1) on $[\sigma, \infty)$, $\sigma > 0$. We introduce the substitution

$$s = \frac{\log t}{\log \Lambda^{-1}}, \qquad z(s) = \frac{y(t)}{\varphi(t)},$$

where φ is given by (2.7). Then z is a solution of

$$\begin{aligned} z'(s) = &\left[a_0 \Lambda^{-\alpha_0 s} \Lambda^{-s} \log \Lambda^{-1} - \frac{\big(\varphi(\Lambda^{-s})\big)'}{\varphi(\Lambda^{-s})}\right] z(s) \\ &+ \sum_{j=1}^{m} a_j \Lambda^{-\alpha_j s} \Lambda^{-s} \log \Lambda^{-1} \frac{\varphi(\lambda_j \Lambda^{-s})}{\varphi(\Lambda^{-s})} z(s - c_j)\,, \end{aligned} \tag{4.1}$$

where "$'$" means the derivative with the respect to s and $c_j := \log \lambda_j^{-1} / \log \Lambda^{-1} \ge 1$ $(j = 1, \dots, m)$. Relation (4.1) is equivalent to

$$\begin{aligned} &\left[z(s) \varphi(\Lambda^{-s}) \exp\left\{-a_0 \frac{\Lambda^{-(\alpha_0+1)s}}{\alpha_0 + 1}\right\}\right]' \\ &= \sum_{j=1}^{m} a_j \Lambda^{-\alpha_j s} \Lambda^{-s} \log \Lambda^{-1} \varphi(\lambda_j \Lambda^{-s}) \exp\left\{-a_0 \frac{\Lambda^{-(\alpha_0+1)s}}{\alpha_0 + 1}\right\} z(s - c_j)\,. \end{aligned} \tag{4.2}$$

Now let $\eta > 0$ be such that φ is monotonic for all $t \ge \eta$ and put $s_0 := \max\left(\frac{\log \sigma}{\log \Lambda^{-1}}, \frac{\log \eta}{\log \Lambda^{-1}}\right)$, $s_{-1} := s_0 - \max(c_1, \dots, c_m)$, $J_0 := [s_{-1}, s_0]$, $J_k := [s_0 + k - 1, s_0 + k]$ and $S_k := \sup\left\{|z(s)|,\ s \in \bigcup\limits_{i=0}^{k} J_i\right\}$ $(k = 1, 2, \dots)$. Then $[s_0, \infty) = \bigcup\limits_{k=1}^{\infty} J_k$ and $s - c_j \in \bigcup\limits_{i=0}^{k} J_i$ for every $s \in J_{k+1}$ $(j = 1, \dots, m)$.

Now we consider arbitrary $\tilde{s} \in J_{k+1}$ and integrate (4.2) over $[s_0 + k, \tilde{s}]$. Then

$$\begin{aligned} z(\tilde{s}) =& \frac{\varphi\big(\Lambda^{-(s_0+k)}\big)}{\varphi(\Lambda^{-\tilde{s}})} \exp\Big\{\frac{a_0}{\alpha_0+1}\Big(\Lambda^{-(\alpha_0+1)\tilde{s}} - \Lambda^{-(\alpha_0+1)(s_0+k)}\Big)\Big\} z(s_0+k) \\ &+ \int\limits_{s_0+k}^{\tilde{s}} \sum_{j=1}^{m} a_j \Lambda^{-\alpha_j s}\Lambda^{-s}\log\Lambda^{-1}\frac{\varphi(\lambda_j\Lambda^{-s})}{\varphi(\Lambda^{-\tilde{s}})} \\ &\times \exp\Big\{\frac{a_0}{\alpha_0+1}\Big(\Lambda^{-(\alpha_0+1)\tilde{s}} - \Lambda^{-(\alpha_0+1)s}\Big)\Big\} z(s - c_j)\,\mathrm{d}s\,. \end{aligned}$$

Taking the absolute value and using (2.4) with t replaced by Λ^{-s} we arrive to

$$\begin{aligned} |z(\tilde{s})| \le& S_k\Bigg(\frac{\varphi\big(\Lambda^{-(s_0+k)}\big)}{\varphi(\Lambda^{-\tilde{s}})} \exp\Big\{\frac{a_0}{\alpha_0+1}\Big(\Lambda^{-(\alpha_0+1)\tilde{s}} - \Lambda^{-(\alpha_0+1)(s_0+k)}\Big)\Big\} \\ &+ \int\limits_{s_0+k}^{\tilde{s}} -a_0\Lambda^{-\alpha_0 s}\Lambda^{-s}\log\Lambda^{-1}\frac{\varphi(\Lambda^{-s})}{\varphi(\Lambda^{-\tilde{s}})} \\ &\times \exp\Big\{\frac{a_0}{\alpha_0+1}\Big(\Lambda^{-(\alpha_0+1)\tilde{s}} - \Lambda^{-(\alpha_0+1)s}\Big)\Big\}\,\mathrm{d}s\Bigg) \\ \le& S_k\Bigg(\frac{\varphi(\Lambda^{-(s_0+k)})}{\varphi\big(\Lambda^{-\tilde{s}}\big)} \exp\Big\{\frac{a_0}{\alpha_0+1}\Big(\Lambda^{-(\alpha_0+1)\tilde{s}} - \Lambda^{-(\alpha_0+1)(s_0+k)}\Big)\Big\} \\ &+ \frac{\exp\big\{\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)\tilde{s}}\big\}}{\varphi(\Lambda^{-\tilde{s}})} \int\limits_{s_0+k}^{\tilde{s}} \varphi(\Lambda^{-s})\frac{\mathrm{d}}{\mathrm{d}s}\Big[\exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\}\Big]\,\mathrm{d}s\Bigg). \end{aligned} \tag{4.3}$$

Denote $I_1 := \int\limits_{s_0+k}^{\tilde{s}} \varphi(\Lambda^{-s})\frac{\mathrm{d}}{\mathrm{d}s}\Big[\exp\big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\big\}\Big]\,\mathrm{d}s$ and integrate this by parts to get

$$\begin{aligned} I_1 =& \Big[\varphi(\Lambda^{-s})\exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\}\Big]_{s_0+k}^{\tilde{s}} \\ &- \int\limits_{s_0+k}^{\tilde{s}} \frac{\mathrm{d}}{\mathrm{d}s}\big[\varphi(\Lambda^{-s})\big]\exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\}\,\mathrm{d}s\,. \end{aligned}$$

If $\max(\alpha_1,\dots,\alpha_m) > \alpha_0$ or $\max(\alpha_1,\dots,\alpha_m) = \alpha_0$ and $\sum\limits_{j=1}^{m} |a_j| + a_0 \ge 0$, then $\varphi(\Lambda^{-s})$ is nondecreasing for all $s \ge s_0$, hence

$$I_1 \le \Big[\varphi(\Lambda^{-s}) \exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\}\Big]_{s_0+k}^{\tilde{s}} .$$

Substituting this estimate into (4.3) we have $|z(\tilde{s})| \le S_k$, hence $S_{k+1} \le S_k$ and (S_k) is bounded as $k \to \infty$.

If $\varphi(\Lambda^{-s})$ is decreasing for all $s \ge s_0$, then

$$I_1 = \Big[\varphi(\Lambda^{-s}) \exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\}\Big]_{s_0+k}^{\tilde{s}} + I_2 ,$$

where

$$\begin{aligned} I_2 &:= \int\limits_{s_0+k}^{\tilde{s}} \frac{\mathrm{d}}{\mathrm{d}s}\big[-\varphi(\Lambda^{-s})\big] \exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\}\,\mathrm{d}s \\ &= \int\limits_{s_0+k}^{\tilde{s}} \frac{-\varphi'(\Lambda^{-s})}{\varphi'(\Lambda^{-s}) - a_0\Lambda^{-\alpha_0 s}\varphi(\Lambda^{-s})} \frac{\mathrm{d}}{\mathrm{d}s}\Big[\varphi(\Lambda^{-s}) \exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\}\Big]\,\mathrm{d}s . \end{aligned}$$

Our next step is to estimate I_2. A repeated integration by parts yields

$$\begin{aligned} I_2 =& \Big[\varphi(\Lambda^{-s}) \exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\} \frac{-\varphi'(\Lambda^{-s})}{\varphi'(\Lambda^{-s}) - a_0\Lambda^{-\alpha_0 s}\varphi(\Lambda^{-s})}\Big]_{s_0+k}^{\tilde{s}} \\ &+ \int\limits_{s_0+k}^{\tilde{s}} \varphi(\Lambda^{-s}) \exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\} \frac{\mathrm{d}}{\mathrm{d}s}\Big[\frac{\varphi'(\Lambda^{-s})}{\varphi'(\Lambda^{-s}) - a_0\Lambda^{-\alpha_0 s}\varphi(\Lambda^{-s})}\Big]\,\mathrm{d}s \\ \le& \Big[\varphi(\Lambda^{-s}) \exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\} \frac{-\varphi'(\Lambda^{-s})}{\varphi'(\Lambda^{-s}) - a_0\Lambda^{-\alpha_0 s}\varphi(\Lambda^{-s})}\Big]_{s_0+k}^{\tilde{s}} \\ &+ \varphi(\Lambda^{-\tilde{s}}) \exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)\tilde{s}}\Big\} \Big[\frac{\varphi'(\Lambda^{-s})}{\varphi'(\Lambda^{-s}) - a_0\Lambda^{-\alpha_0 s}\varphi(\Lambda^{-s})}\Big]_{s_0+k}^{\tilde{s}} \\ =& \Big[\varphi(\Lambda^{-s}) \exp\Big\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Big\}\Big]_{s_0+k}^{\tilde{s}} \\ &\times \frac{-\varphi'(\Lambda^{-(s_0+k)})}{\varphi'(\Lambda^{-(s_0+k)}) - a_0\Lambda^{-\alpha_0(s_0+k)}\varphi(\Lambda^{-(s_0+k)})} , \end{aligned}$$

where we utilized the monotonicity of the term $\varphi(\Lambda^{-s})\exp\bigl\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\bigr\}$ following either from the assumption $\alpha_0 > -1$ or the from the assumptions $\alpha_0 = -1$, $\beta = 0$, $b > 1$ and $a_0 \le -2\log b/(\log\lambda^{-1})$. The estimate of I_2 implies that

$$I_1 \le \left[\varphi(\Lambda^{-s})\exp\Bigl\{-\frac{a_0}{\alpha_0+1}\Lambda^{-(\alpha_0+1)s}\Bigr\}\right]_{s_0+k}^{\tilde{s}} \times\left(1+\frac{-\varphi'\bigl(\Lambda^{-(s_0+k)}\bigr)}{\varphi'\bigl(\Lambda^{-(s_0+k)}\bigr)-a_0\Lambda^{-\alpha_0(s_0+k)}\varphi\bigl(\Lambda^{-(s_0+k)}\bigr)}\right).$$

Substituting this into (4.3) we obtain

$$|z(\tilde{s})| \le S_k\left(1-\frac{\varphi'\bigl(\Lambda^{-(s_0+k)}\bigr)}{\varphi'\bigl(\Lambda^{-(s_0+k)}\bigr)-a_0\Lambda^{-\alpha_0(s_0+k)}\varphi\bigl(\Lambda^{-(s_0+k)}\bigr)}\right),$$

hence

$$\begin{aligned} S_{k+1} &\le S_k\left(1-\frac{\varphi'\bigl(\Lambda^{-(s_0+k)}\bigr)}{\varphi'(\Lambda^{-(s_0+k)})-a_0\Lambda^{-\alpha_0(s_0+k)}\varphi\bigl(\Lambda^{-(s_0+k)}\bigr)}\right) \\ &\le S_1\prod_{i=1}^{k}(1-d_i)\,, \end{aligned}$$

where

$$d_i := \frac{\varphi'\bigl(\Lambda^{-(s_0+i)}\bigr)}{\varphi'\bigl(\Lambda^{-(s_0+i)}\bigr)-a_0\Lambda^{-\alpha_0(s_0+i)}\varphi\bigl(\Lambda^{-(s_0+i)}\bigr)}\,.$$

To estimate the product we utilize the inequality $\prod\limits_{i=1}^{k}(1-d_i) < \exp\Bigl\{-\sum\limits_{i=1}^{k}d_i\Bigr\}$ and calculate

$$d_i = \frac{pi+q}{pi+q+r\Lambda^{-(1+\alpha_0)i}}\,,$$

where $p = \beta\log\Lambda^{-1}$, $q = ps_0 + \frac{\beta}{2}\log\lambda^{-1} - \log b$, $r = -a_0\log\lambda^{-1}\Lambda^{-(1+\alpha_0)s_0}$.

If $\alpha_0 > -1$, then $\sum\limits_{i=1}^{k} d_i$ converges as $k\to\infty$, hence (S_k) is bounded as $k\to\infty$. If $\alpha_0 = -1$, $\beta = 0$, $b > 1$ and $a_0 \le -2\log b/(\log\lambda^{-1})$, then $d_i = d \in [-1, 0)$ for all i, hence $-\sum\limits_{i=1}^{k} d_i \le k$ and $S_k = O\bigl(\exp\{k\}\bigr)$ as $k\to\infty$.

Summarizing our previous calculations we note that the boundedness of (S_k) implies the boundedness of z, hence

$$y(t) = O\bigl(\varphi(t)\bigr) \qquad \text{as } t\to\infty$$

for any solution y of (1.1). Similarly, if $S_k = O\big(\exp\{k\}\big)$ as $k \to \infty$, then $z(s) = O\big(\exp\{s\}\big)$ as $s \to \infty$ and

$$y(t) = O\left(\varphi(t) \exp\Big\{\frac{\log t}{\log \Lambda^{-1}}\Big\}\right) \qquad \text{as } t \to \infty\,.$$

□

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