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POINTWISE DENSITY TOPOLOGY WITH RESPECT TO ADMISSIBLE σ-ALGEBRAS

Magdalena Górajska — Jacek Hejduk

ABSTRACT. The paper presents a pointwise density topology with respect to admissible σ-algebras on the real line. The properties of such topologies, including the separation axioms, are studied.

1. Preliminaries

Let $\mathbb{R}$ be the set of reals, $\mathbb{N}$ the set of positive integer numbers and $\mathbb{Q}$ the set of rational numbers. By λ we shall denote the Lebesgue measure over $\mathbb{R}$. The capitals $\mathcal{L}$ and $\mathbb{L}$ denote the σ-algebra of Lebesgue measurable sets on $\mathbb{R}$ and the σ-ideal of Lebesgue null sets. Let $\mathcal{T}_{nat}$ be the natural topology on $\mathbb{R}$. If $\mathcal{T}$ is a topology on $\mathbb{R}$, then we fix the notation:

$\mathcal{B}(\mathcal{T})$ – σ-algebra of all Borel sets with respect to $\mathcal{T}$,

$\mathcal{B}_a(\mathcal{T})$ – σ-algebra of all sets having the Baire property with respect to $\mathcal{T}$,

$\mathbb{K}(\mathcal{T})$ – σ-ideal of all meager sets with respect to $\mathcal{T}$.

If $\mathcal{T} = \mathcal{T}_{nat}$, then we use the following short form symbols: $\mathcal{B}$, $\mathcal{B}_a$, $\mathbb{K}$. Let A' stand for the complement of A in $\mathbb{R}$. By χ_A we shall denote the characteristic function of a set $A \subset \mathbb{R}$.

2. Introduction

Recall that $x_0 \in \mathbb{R}$ is a density point of the set $A \in \mathcal{L}$ if and only if

$$\lim_{h\to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

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Observe that the above condition is equivalent to the following one

$$\lim_{n\to\infty} \frac{\lambda(A \cap [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}])}{\frac{2}{n}} = 1.$$

Putting $nA = \{na : a \in A\}$ and $A - x_0 = \{a - x_0 : a \in A\}$ for every $n \in \mathbb{N}$, the above condition can by written in the form

$$\lim_{n\to\infty} \lambda\big(n(A - x_0) \cap [-1, 1]\big) = 2$$

which means that the sequence of characteristic functions $\{\chi_{n(A-x_0)\cap[-1,1]}\}_{n\in\mathbb{N}}$ converges in measure λ to $\chi_{[-1,1]}$ (cf. [5]). Using various kinds of convergences of the above sequence, we get different type of density point and finally the different type of density topology in the family at Lebesgue measurable sets. For example, the classical density topology $\mathcal{T}_d$ (cf. [5]), the simple density topology $\mathcal{T}_s$ (cf. [7]) and density topology $\mathcal{T}_{fin}$ generated by convergence everywhere except for a finite set (cf. [3]).

3. Pointwise density point

Following the paper [2], we consider the concept of a pointwise density point for every set on $\mathbb{R}$ (denoted briefly by p-density point).

Definition 1. Let $A \subset \mathbb{R}$. We shall say that

a) 0 is a p-density point of a set A if and only if the sequence $\big\{\chi_{nA\cap[-1,1]}\big\}_{n\in\mathbb{N}}$ is convergent everywhere to the function $\chi_{[-1,1]}$,

b) $x \in \mathbb{R}$ is a p-density point of a set A if and only if 0 is a p-density point of $A - x$,

c) $x \in \mathbb{R}$ is a p-dispersion point of a set A if and only if x is a p-density point of A'.

The above definition has the following characterization of a p-density point.

Proposition 2. *Let $A \subset \mathbb{R}$. Then*

a) 0 *is a p-density point of a set A if and only if* $[-1, 1] \subset \liminf_{n\to\infty} nA$,

b) $x \in \mathbb{R}$ *is a p-density point of a set A if and only if* $[-1, 1] \subset \liminf_{n\to\infty} n(A - x)$,

c) $x \in \mathbb{R}$ *is a p-dispersion point of a set A if and only if* $[-1, 1] \subset \liminf_{n\to\infty} n(A' - x)$.

Set $\Phi_p(A) = \{x \in \mathbb{R} : x$ is a p-density point of $A\}$ for all $A \subset \mathbb{R}$. The operation Φ_p fulfills the following conditions.

Theorem 3. *For any sets A, $B \subset \mathbb{R}$ and for all $y \in \mathbb{R}$ the following properties hold:*

(1) $\Phi_p(\emptyset) = \emptyset$, $\Phi_p(\mathbb{R}) = \mathbb{R}$,

(2) $\Phi_p(A \cap B) = \Phi_p(A) \cap \Phi_p(B)$,

(3) $\Phi_p(A) \subset A$,

(4) $\Phi_p(A) + y = \Phi_p(A + y)$.

Obviously, we have that if $A \in \mathbb{L}$ or $A \in \mathbb{K}$, then $\Phi_p(A) = \emptyset$. The paper [2] contains a result that there exists a set $A \in \mathcal{L}$ such that $\Phi_p(A) \notin \mathcal{L}$ and there exists a set $B \in \mathcal{B}_a$ such that $\Phi_p(B) \notin \mathcal{B}_a$. The main idea of this paper is to apply an operator Φ_p for the purpose of getting topology on the real line.

Theorem 4. *Let S be a σ-algebra on $\mathbb{R}$ containing $\mathcal{B}$ and $\mathcal{T}_{pS} = \big\{A \in S : A \subset \Phi_p(A)\big\}$. Then*

a) $\emptyset$, $\mathbb{R} \in \mathcal{T}_{pS}$,

b) $\forall_{A,B \in \mathcal{T}_{pS}}$; $A \cap B \in \mathcal{T}_{pS}$,

c) $\mathcal{T}_{nat} \subset \mathcal{T}_{pS}$.

P r o o f. Condition a) and b) are consequence of Theorem 3. Let $V \in \mathcal{T}_{nat}$. Obviously, $V \in S$. Suppose that $V \neq \emptyset$ and let $x \in V$. There exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset V - x$. Then

$$\liminf_{n \to \infty} n\,(-\varepsilon, \varepsilon) \subset \liminf_{n \to \infty} n\,(V - x)\,.$$

Therefore, $[-1, 1] \subset \liminf_{n\to\infty} n(V - x)$, which means that $x \in \Phi_p(V)$. Finally, $V \in \mathcal{T}_{pS}$. □

Definition 5. We shall say that σ-algebra S of subsets of $\mathbb{R}$ is admissible if $\mathcal{B} \subset S$ and the family

$$\mathcal{T}_{pS} = \big\{A \in S\colon A \subset \Phi_p(A)\big\}$$

is a topology on $\mathbb{R}$.

Evidently, σ-algebra $2^{\mathbb{R}}$ is admissible and we shall denote

$$\mathcal{T}_p = \big\{A \subset \mathbb{R} : A \subset \Phi_p(A)\big\}.$$

Theorem 6. *If a σ-algebra S contains $\mathcal{B}$ and there exists an operator $\Phi\colon S \to \mathbb{R}$ such that $\Phi_p(A) \subset \Phi(A)$ for every $A \in S$ and the family $\big\{A \in S\colon A \subset \Phi(A)\big\}$ is closed under arbitrary unions, then S is admissible.*

P r o o f. Following Theorem 4, it is sufficient to show that if $\{A_t\}_{t\in T} \subset \mathcal{T}_{pS}$, then $\bigcup_{t\in T} A_t \in \mathcal{T}_{pS}$. According to the assumption, $\bigcup_{t\in T} A_t \in S$. Then, by monotonicity of operator Φ_p, we get that $\bigcup_{t\in T} A_t \in \mathcal{T}_{pS}$. □

Corollary 7. *If Φ_d and $\Phi_{\mathcal{I}}$ denote the density operator and $\mathcal{I}$-density operator* (cf. [6]), *respectively, then $\Phi_p(A) \subset \Phi_d(A)$ for $A \in \mathcal{L}$ and $\Phi_p(A) \subset \Phi_{\mathcal{I}}(A)$ for every $A \in \mathcal{B}_a$. Since*

$$\{A \in \mathcal{L} \colon A \subset \Phi_d(A)\} \quad \text{and} \quad \{A \in \mathcal{B}_a \colon A \subset \Phi_{\mathcal{I}}(A)\}$$

stand for topologies, we conclude that the algebras $\mathcal{L}$ and $\mathcal{B}_a$ are admissible. Also, it is easy to observe that the σ-algebra $\mathcal{L} \cap \mathcal{B}_a$ is admissible.

There exists a set $A \subset \mathbb{R}$ such that $A \notin \mathcal{B}_a$, $A \notin \mathcal{L}$ and $\Phi_p(A) = A$ (see [1, Example 1.5.1]). It implies that $\mathcal{T}_p \neq \mathcal{T}_p \cap \mathcal{B}_a$ and $\mathcal{T}_p \neq \mathcal{T}_p \cap \mathcal{L}$.

Problem 8. *Is it true that $\mathcal{T}_p \cap \mathcal{B}_a \neq \mathcal{T}_p \cap \mathcal{L}$?*

Theorem 9. *The algebra $\mathcal{B}$ is not admissible.*

Proof. By [2, Proposition 21] and [4, Lemma 2.7] there exists a nonempty perfect set $F \subset \mathbb{R}$ such that $\Phi_p\big((\mathbb{R}\backslash F) \cup \{x\}\big) = (\mathbb{R}\backslash F) \cup \{x\}$ for every $x \in F$. Let $C \subset F$ and $C \notin \mathcal{B}$. Define the family $\big\{(\mathbb{R}\backslash F) \cup \{x\} : x \in C\big\}$. For each $x \in C$ we have $\big((\mathbb{R}\backslash F) \cup \{x\}\big) \in \mathcal{B}$ and $(\mathbb{R}\backslash F) \cup \{x\} \in \mathcal{T}_{p\mathcal{B}}$. Since $(\mathbb{R}\backslash F) \cap C = \emptyset$, then

$$\bigcup_{x \in C} \big((\mathbb{R}\backslash F) \cup \{x\}\big) = (\mathbb{R}\backslash F) \cup C.$$

The set $\bigcup_{x \in C} \big((\mathbb{R}\backslash F) \cup \{x\}\big)$ is not a Borel set. It means that $\mathcal{T}_{p\mathcal{B}}$ is not a topology on $\mathbb{R}$. □

Observation 10. *There exists the smallest (within the meaning of inclusion) admissible σ-algebra S_0 of subsets of $\mathbb{R}$.*

Problem 11. *Is it true that $\mathcal{T}_p \cap S_0 \neq \mathcal{T}_p \cap \mathcal{B}_a \cap \mathcal{L}$?*

Let S be an admissible σ-algebra on $\mathbb{R}$.

Property 12. *If $\mathcal{T}_H$ is Hashimoto topology of the form*

$$\mathcal{T}_H = \{A \subset \mathbb{R} \colon A = U \setminus P,\ U \in \mathcal{T}_{nat},\ P \text{ is countable}\},$$

then $\mathcal{T}_H \setminus \mathcal{T}_{pS} \neq \emptyset$.

Proof. Let $P = \bigcup_{n=1}^{\infty} \{\frac{1}{n}\}$. Then $\mathbb{R} \setminus P \in \mathcal{T}_H$. Clearly, $\mathbb{R} \setminus P \in S$, $0 \in \mathbb{R} \setminus P$ but 0 is not p-density point of $\mathbb{R} \setminus P$. Consequently, $\mathbb{R} \setminus P \notin \mathcal{T}_{pS}$. □

Theorem 13. *The topology $\mathcal{T}_{pS}$ is essentially stronger than the natural topology.*

Proof. By Theorem 4, we have $\mathcal{T}_{nat} \subset \mathcal{T}_{pS}$. It was proved in [1] that $\Phi_p(\mathbb{R}\backslash\mathbb{Q}) = \mathbb{R} \setminus \mathbb{Q}$. So, $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_{pS} \setminus \mathcal{T}_{nat}$. □

Corollary 14. *The space $(\mathbb{R}, \mathcal{T}_{pS})$ is Hausdorff.*

The following properties of the topological space $(\mathbb{R}, \mathcal{T}_{pS})$ can be proved similarly as in the case of admissible σ-algebra $\mathcal{L}$.

Theorem 15 (cf. [2]). *The space $(\mathbb{R}, \mathcal{T}_{pS})$ does not possess the Lindelöf property.*

Theorem 16 (cf. [2]). *The space $(\mathbb{R}, \mathcal{T}_{pS})$ is not separable.*

Theorem 17 (cf. [2]). *The space $(\mathbb{R}, \mathcal{T}_{pS})$ is not the first countable.*

Theorem 18. *If $S \subset \mathcal{L}$ is an admissible σ-algebra, then the space $(\mathbb{R}, \mathcal{T}_{pS})$ is not normal.*

Proof. On the contrary, suppose that $(\mathbb{R}, \mathcal{T}_{pS})$ is normal. Let $A = \mathbb{Q} + \sqrt{2}$. The sets $\mathbb{Q}$, A are $\mathcal{T}_{pS}$-closed. Obviously, $\mathbb{Q} \cap A = \emptyset$. By the Urysohn Lemma, there exists a continuous function $f\colon (\mathbb{R}, \mathcal{T}_{pS}) \to (\mathbb{R}, \mathcal{T}_{nat})$ such that

$$f(x) = 0 \quad \text{for } x \in \mathbb{Q} \qquad \text{and} \qquad f(x) = 1 \quad \text{for } x \in A.$$

The function f is discontinuous everywhere. Thus the set of discontinuity points is residual. Since $\mathcal{T}_{pS} \subset \mathcal{T}_d$, it means that f is approximately continuous and the set of discontinuity points can be at most the set of first category (cf. [5]). This contradiction completes the proof. □

Theorem 19. *If $S \subset \mathcal{B}_a$ is an admissible σ-algebra, then the space $(\mathbb{R}, \mathcal{T}_{pS})$ is not regular.*

Proof. The set $\mathbb{Q}$ is a $\mathcal{T}_{pS}$-closed. Observe that $\mathcal{T}_{pS}$ is contained in the $\mathcal{I}$-density topology $\mathcal{T}_{\mathcal{I}}$. It is known that the set $\mathbb{Q}$ is $\mathcal{T}_{\mathcal{I}}$-closed and cannot be separated in $\mathcal{T}_{\mathcal{I}}$ from any point $x \in \mathbb{R} \setminus \mathbb{Q}$ (cf. [1]). It means that $\mathcal{T}_{pS}$ is not regular. □

Lemma 20. *If S is an admissible σ-algebra and $\mathcal{K}(\mathcal{T}_{pS}) \subset \mathbb{K}$, then $\mathcal{K}(\mathcal{T}_{pS}) = \mathbb{K}$ and $\mathcal{B}_a \subset \mathcal{B}_a(\mathcal{T}_{pS})$.*

Proof. We show that $\mathbb{K} \subset \mathcal{K}(\mathcal{T}_{pS})$. Let $A \in \mathbb{K}$. It suffices to assume that A is closed and nowhere dense with respect to the natural topology. Evidently, $A \in \mathcal{B}_a(\mathcal{T}_{pS})$. Therefore, $A = V \vartriangle Z$, where $V \in \mathcal{T}_{pS}$ and $Z \in \mathcal{K}(\mathcal{T}_{pS})$. Since $\mathcal{K}(\mathcal{T}_{pS}) \subset \mathbb{K}$, then $Z \in \mathbb{K}$. Hence, $V \in \mathbb{K}$. It implies that $V \subset \Phi_p(V) = \emptyset$ and $A = Z$. Hence, $A \in \mathcal{K}(\mathcal{T}_{pS})$. Finally, $\mathbb{K} = \mathcal{K}(\mathcal{T}_{pS})$. The last equality and inclusion $\mathcal{T}_{nat} \subset \mathcal{T}_{pS}$ imply that $\mathcal{B}_a \subset \mathcal{B}_a(\mathcal{T}_{pS})$. □

Theorem 21. *If $S \subset \mathcal{B}_a$ is an admissible σ-algebra, then $\mathcal{K}(\mathcal{T}_{pS}) = \mathbb{K}$ and $\mathcal{B}_a = \mathcal{B}_a(\mathcal{T}_{pS})$.*

Proof. Let $A \in \mathcal{K}(\mathcal{T}_{pS})$. It suffices to assume that A is a $\mathcal{T}_{pS}$-nowhere dense closed set. We now prove that $A \in \mathbb{K}$. From the assumption we obtain $\mathcal{T}_{pS} \subset \mathcal{B}_a$ and $A \in \mathcal{B}_a$. The set A having the Baire property has the form $A = V \vartriangle Z$, where $V \in \mathcal{T}_{nat}$ and $Z \in \mathbb{K}$. We show that $V = \emptyset$. Let us suppose that $V \neq \emptyset$. By Theorem 13 we obtain $V \in \mathcal{T}_{pS}$. Since A is $\mathcal{T}_{pS}$-nowhere dense, there exists a nonempty $V_1 \in \mathcal{T}_{pS}$ such that $V_1 \subset V$ and $A \cap V_1 = \emptyset$. Since $\mathcal{T}_{pS} \subset \mathcal{T}_{p\mathcal{B}_a}$, we have $V_1 \in \mathcal{T}_{p\mathcal{B}_a}$. As $V_1 \neq \emptyset$, we infer that $V_1 \notin \mathbb{K}$. Since

$$Z = A \triangle V = A \triangle \big[(V \backslash V_1) \cup V_1\big] \supset V_1,$$

we get a contradiction with the fact that $Z \in \mathbb{K}$ and $V_1 \notin \mathbb{K}$. Finally, $V = \emptyset$ and that $A = Z$. Therefore, $A \in \mathbb{K}$, so $\mathcal{K}(\mathcal{T}_{pS}) \subset \mathbb{K}$. From Lemma 20 we obtain

$$\mathbb{K} = \mathcal{K}(\mathcal{T}_{pS}) \quad \text{and} \quad \mathcal{B}_a \subset \mathcal{B}_a(\mathcal{T}_{pS}).$$

Since $S \subset \mathcal{B}_a$, we have $\mathcal{T}_{pS} \subset \mathcal{B}_a$. From the equality $\mathbb{K} = \mathcal{K}(\mathcal{T}_{pS})$ we get $\mathcal{B}_a(\mathcal{T}_{pS}) \subset \mathcal{B}_a$. Finally, $\mathcal{B}_a(\mathcal{T}_{pS}) = \mathcal{B}_a$. □

Corollary 22. *If S_0 is the smallest admissible σ-algebra, then*

$$\mathbb{K} = \mathcal{K}(\mathcal{T}_{pS}) \quad \textit{and} \quad \mathcal{B}_a(\mathcal{T}_{pS}) = \mathcal{B}_a.$$

The paper will be finished by presenting some algebraic properties of $\mathcal{T}_{pS}$-open set.

Definition 23. We shall say that σ-algebra S of subset S of $\mathbb{R}$ is invariant if for every $A \in S$ and any $\alpha, y \in \mathbb{R}$, $\alpha A + y \in S$.

Theorem 24. *Let S be an admissible and invariant σ-algebra on $\mathbb{R}$. Then*

a) $\forall A \in \mathcal{T}_{pS}\ \forall y \in \mathbb{R}\ A + y \in \mathcal{T}_{pS}$,

b) $\forall A \in \mathcal{T}_{pS}\ \forall |m| \geq 1\ mA \in \mathcal{T}_{pS}$,

c) $\forall |m| < 1\ \exists A \in \mathcal{T}_{pS}\ mA \notin \mathcal{T}_{pS}$.

P r o o f. Property a) follows immediately from condition 5 in Theorem 3. Property b) is a consequence of the definition of a pointwise density point. Property 25 in [2] contains a result that for every $|m| < 1$ there exists a Borel set $A \subset \Phi_p(A)$, such that the set mA is not a subset of $\Phi_p(mA)$. It means that $A \in \mathcal{T}_{pS}$ and $mA \notin \mathcal{T}_{pS}$, which proves the last statement c). □

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Magdalena Górajska
Center of Mathematics and Physics
Technical University of Łódź
Politechniki 22
PL–90-238 Łódź
POLAND
E-mail: magdalena.gorajska@p.lodz.pl

Jacek Hejduk
Faculty of Mathematics and
Computer Science
University of Łódź
Stefan Banach 22
PL–90-238 Łódź
POLAND
E-mail: hejduk@math.uni.lodz.pl