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APPROXIMATIONS BY DARBOUX FUNCTIONS IN THE BAIRE ONE CLASS

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ABSTRACT. In the paper [*Sur la premiére dérivée*, Trans. Amer. Math. Soc. (N.S.) **2** (1940), 17–23], Z. Zahorski designed a way of constructing the real semi-continuous functions. Throughout the present paper, it is shown that a modification of Zahorski's approach is usefull for approximation of a Baire one function by a Darboux Baire one function.

A. B. Gurevič [3] and L. Mišík [6] investigated methods how to approximate the function Baire α class, $\alpha > 1$, by a Darboux function Baire α class. Gurevič claims a simple and a short modification of Mišík's method for the case $\alpha = 1$. Since his simplification suffers from several errors, A. M. Bruckner, J. G. Ceder and R. Keston [2] revised Gurevič and Mišík's theorem for the case $\alpha = 1$.

In paper [7], Z. Zahorski explains how to construct a semi-continuous real function by utilizing a certain system of closed sets $\{P_\lambda, \lambda \geq 1\}$ where each set P_λ is associated with a constant function $f_\lambda = \frac{1}{\lambda}$. Paper [5] modifies Zahorski's idea and associates a certain system of closed sets $\{P_\lambda, \lambda \geq 1\}$ with a certain system of continuous real functions $\{f_\lambda, \lambda \geq 1\}$ resulting in a theorem on approximation of a semi-continuous function by a Darboux semi-continuous function. The present paper explains how to apply this methodology in order to prove a theorem considering approximation of the function of Baire one class by the function of Darboux Baire one class while obtaining richer information than in [2].

We deal with the classes of real functions defined on interval $[0,1]$. The symbols C, D and B_1 stand for the class of continuous, Darboux and Baire one functions, respectively. DB_1 denotes $D \cap B_1$, C_f denotes the set of points of continuity of the function f and $f \restriction F$ denotes the restriction of the function f on the set F. We will say that a point x is a bilateral c-point of a set A if and only

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if the sets $(x; x+\delta) \cap A$ and $(x-\delta; x) \cap A$ have the cardinality of continuum for all $\delta > 0$, that is

$$\operatorname{card}\bigl((x; x+\delta) \cap A\bigr) = \operatorname{card}\bigl((x-\delta; x) \cap A\bigr) = c\,.$$

We will say that the set A is bilaterally c-dense in the set B ($B \subset_c A$) if and only if each point $x \in B$ is a bilateral c-point of the set A.

Let the function $f\colon [0,1] \longrightarrow \mathbb{R}$ be from Baire one class, and let $\{P_\lambda, \lambda \geq 1\}$ be the system of closed nowhere dense subsets of $[0,1]$ such that for all $\lambda_1 < \lambda_2$, the set $P_{\lambda_1} \subset_c P_{\lambda_2}$. Obviously, the set

$$E = \bigcup_{\lambda \geq 1} P_\lambda$$

is of first category set in $[0,1]$. Since the function $f \in B_1$, then there exists a sequence of continuous functions f_n, $n = 1, 2, \dots$, which converges on $[0,1]$ to the function f. For each $\lambda \geq 1$, let the closed set P_λ be associated with the continuous function

$$f_\lambda = f_n + (\lambda - n)(f_{n+1} - f_n), \qquad \text{for} \quad \lambda \in [n, n+1)\,.$$

Define a function g such that

$$\begin{aligned} g(x) &= f(x), && \text{for} \quad x \notin E, \\ g(x) &= f_{\lambda(x)}(x), && \text{for} \quad x \in E, \quad \text{where } \lambda(x) = \inf\{\lambda; x \in P_\lambda\}\,. \end{aligned}$$

Then, the function g satisfies two following lemmas:

Lemma 1. *The set C_g is residual in $[0,1]$.*

P r o o f. Let $B_{n,k}$ be the set of all $x \in [0,1]$, for which there exists a neighbourhood $O(x)$ such that

$$|f_m(y) - f_p(y)| < \frac{1}{k}, \qquad \text{for all} \quad m, p > n,\ y \in O(x).$$

Fix $k \in \mathbb{N}$. We will show: if $J \subset [0,1]$ is arbitrary interval, then there is an index n_J and an open interval $J_{n_J,k} \subset J$ such that $J_{n_J,k} \subset B_{n_J,k}$.

Let

$$A_n = \left\{x \in J;\ |f_m(x) - f_p(x)| < \frac{1}{2k},\ \text{for all } m, p > n\right\}.$$

Since

$$J = \bigcup_{n=1}^{\infty} A_n,$$

there exist an index n_J and an open interval $J_{n_J,k} \subset J$ such that A_{n_J} is dense in $J_{n_J,k}$. We consider arbitrary $y \in J_{n_J}$ and arbitrary positive integers $m, p > n_J$. The functions f_m and f_p are continuous, thus, for $\varepsilon = \frac{1}{4k}$, there exists a point $x \in A_{n_J} \cap J_{n_J,k}$, such that

$$|f_m(x) - f_m(y)| < \varepsilon \wedge |f_p(x) - f_p(y)| < \varepsilon.$$

Hence,

$$\begin{aligned}|f_m(y) - f_p(y)| &\le |f_m(y) - f_m(x)| + |f_m(x) - f_p(x)| + |f_p(x) - f_p(y)| \\ &< \varepsilon + \frac{1}{2k} + \varepsilon = \frac{1}{k},\end{aligned}$$

that is $J_{n_J,k} \subset B_{n_J,k}$. Let the set G_k be the union of intervals of type $J_{n_J,k}$. It is obvious that each of the sets G_k, $k = 1, 2, \dots$ is open and dense in $[0,1]$. Moreover, for each $x \in G_k$, there exist an index $n(x)$ and an open interval $J_{n(x),k}$ such that $x \in J_{n(x),k} \subset G_k$, and

$$|f_m(y) - f_p(y)| < \frac{1}{k}, \qquad \text{for all} \quad m, p > n(x), \quad \text{for all} \quad y \in J_{n(x),k}.$$

We define the set $G = \bigcap_{k=1}^{\infty} G_k$. The set G is residual in $[0,1]$. Since the set E is a set of first category, the set $G \setminus E$ is residual in $[0,1]$ as well. We prove that

(i) $G \subset C_f$.

Let $x_0 \in G$ and let ε be an arbitrary positive real number. If a natural number $k > \frac{3}{\varepsilon}$ is chosen, then there exists an open interval $J_{n(x_0),k}$ such that

$$|f_m(x) - f_p(x)| < \frac{1}{k}, \qquad \text{for all} \quad m, p > n(x_0), \quad \text{for all} \quad x \in J_{n(x_0),k}.$$

The sequence $f_m(x)$ converges to $f(x)$. Thus,

$$|f(x) - f_p(x)| \le \frac{1}{k}, \qquad \text{for all} \quad p > n(x_0), \quad \text{for all} \quad x \in J_{n(x_0),k}.$$

Fix $p > n(x_0)$. The continuity of the function f_p implies the existence of the neighbourhood $O(x_0)$ of the point x_0 such that

$$|f_p(x) - f_p(x_0)| < \frac{1}{k}, \qquad \text{for all} \quad x \in O(x_0).$$

Then for each $x \in J_{n(x_0),k} \cap O(x_0)$, the inequality

$$\begin{aligned}|f(x_0) - f(x)| &\le |f(x_0) - f_p(x_0)| + |f_p(x_0) - f_p(x)| + |f_p(x) - f(x)| \\ &< \frac{1}{k} + \frac{1}{k} + \frac{1}{k} < \varepsilon\end{aligned}$$

holds. In other words, the function f is continuous at the point $x_0 \in G$ and $G \subset C_f$.

(ii) $G \setminus E \subset C_g$.

Let $x_0 \in G \setminus E$ and let symbols ε, k, $J_{n(x_0),k}$, $O(x_0)$ have the same meaning as in (i). Moreover, let $O(x_0) \cap P_{n(x_0)} = \emptyset$. Assume that arbitrary

$$x \in J_{n(x_0),k} \cap O(x_0)$$

is chosen.

If $x \notin E$, then (i) implies

$$|g(x_0) - g(x)| = |f(x_0) - f(x)| < \varepsilon,$$

and if $x \in E$, then for concrete positive integer $p_x > n(x_0)$ and $\alpha \in [0,1)$,

$$g(x) = f_{p_x}(x) + \alpha\big(f_{p_x+1}(x) - f_{p_x}(x)\big).$$

Therefore,

$$\begin{aligned}|g(x_0) - g(x)| &= \Big|f(x_0) - \Big(f_{p_x}(x) + \alpha\big(f_{p_x+1}(x) - f_{p_x}(x)\big)\Big)\Big| \\ &\le |f(x_0) - f(x)| + |f(x) - f_{p_x}(x)| + \alpha\,|f_{p_x+1}(x) - f_{p_x}(x)| \\ &< \varepsilon + \frac{1}{k} + \alpha\frac{1}{k} < 2\varepsilon.\end{aligned}$$

It was shown that for every $x \in O(x_0) \cap J_{n(x_0),k}$, the inequality $|g(x_0) - g(x)| < 2\varepsilon$ holds which implies the continuity of the function g at arbitrary point $x_0 \in G \setminus E$. Therefore, $G \setminus E \subset C_g$. Since the set $G \setminus E$ is residual in $[0,1]$, the set C_g is residual in $[0,1]$, too. □

Lemma 2. *The function g is Baire one.*

Proof. According to [1], $g \in B_1$ if and only if each nonempty perfect set $P \subset [0,1]$ contains a point $x_0 \in P$ such that the function $g \restriction P$ is continuous at x_0. Let P be a nonempty perfect subset of interval $[0,1]$. Two cases can be assumed: $P \cap E$ is the set of first category in the set P or $P \cap E$ is the set of the second category in the set P.

If $P \cap E$ is the set of first category in the set P, then it is sufficient to replace interval $[0,1]$ with the set P in the proof of Lemma 1. As a result, the set of the points of continuity of the function $g \restriction P$ forms a residual subset of the set P.

If $P \cap E$ is the set of second category in the set P, then, for a certain $n \in \mathbb{N}$, the set P_n is not nowhere dense in P. Therefore, there exists an open interval $J \subset [0,1]$ such that the set P_n is dense in $P \cap J$. Let $\lambda_0 = \inf\{\lambda; P_\lambda \text{ is dense in } P \cap J\}$. If $\lambda_0 = 1$, then $g \restriction P \cap J = f_1 \restriction P_1 \cap J$ is a continuous function. Therefore, the function $g \restriction P$ is continuous at each point $x_0 \in P \cap J$. The presence of $\lambda_0 > 1$ implies the existence of the point $x_0 \in P \cap J$ such that $x_0 \notin P \cap P_\lambda \cap J$ for $\lambda < \lambda_0$ and $x_0 \in P \cap P_\lambda \cap J$ for $\lambda_0 < \lambda$. Let i be a positive integer, $i < \lambda_0 \le i+1$ and $\varepsilon > 0$ be an arbitrary real number, $\varepsilon < \lambda_0 - i$. Obviously,

$$g(x_0) = f_i(x_0) + (\lambda_0 - i)\big(f_{i+1}(x_0) - f_i(x_0)\big).$$

The functions f_i, f_{i+1} are continuous, and therefore, there exist a neighbourhood $O(x_0) \subset J$ of the point x_0 and a constant M such that

$$\begin{aligned}|f_i(x) - f_i(x_0)| &< \varepsilon, \\ |f_{i+1}(x) - f_{i+1}(x_0)| &< \varepsilon, \qquad \text{for all} \quad x \in O(x_0),\end{aligned}$$

and

$$|f_{i+1}(x) - f_i(x)| < M, \qquad \text{for all} \quad x \in [0,1].$$

Since $x_0 \notin P_{\lambda_0-\varepsilon}$, it can be required that $O(x_0) \cap P_{\lambda_0-\varepsilon} = \emptyset$. If $x \in O(x_0) \cap P$, then

$$g(x) = f_i(x) + (\lambda - i)\big(f_{i+1}(x) - f_i(x)\big), \qquad \text{for certain} \quad \lambda \in (\lambda_0 - \varepsilon; \lambda_0]\,.$$

For arbitrary $x \in O(x_0) \cap P$, the inequality

$$\begin{aligned}
&|g(x_0) - g(x)| \\
&= \big|f_i(x_0) + (\lambda_0 - i)\big(f_{i+1}(x_0) - f_i(x_0)\big) - f_i(x) - (\lambda - i)\big(f_{i+1}(x) - f_i(x)\big)\big| \\
&\leq |f_i(x_0) - f_i(x)| + (\lambda_0 - \lambda)\,|f_{i+1}(x_0) - f_i(x_0)| \\
&\quad + (\lambda - i)\,\big|\big(f_{i+1}(x_0) - f_{i+1}(x)\big) - \big(f_i(x_0) - f_i(x)\big)\big| \\
&< \varepsilon + \varepsilon M + (\lambda - i)\,2\varepsilon < \varepsilon\,(3 + M)
\end{aligned}$$

holds; it implies the continuity of the function $g \restriction P$ at the point x_0. □

Theorem 3 ([4]). *Each uncountable Borel set contains a nonempty perfect set.*

It is easy to show that every perfect set P has the cardinality of continuum, and moreover, each point of a perfect set P, except for a countable set of boundary points of contiguous intervals of the set P, is bilateral c-point of the set P.

Lemma 4. *Let E be a nonempty Borel set and let E^* be a set of all points $x \in E$ such that x is a bilateral c-point of the set E. Then, the set $E \setminus E^*$ is countable.*

P r o o f. Let S be a system of all closed intervals $I \subset [0,1]$ such that

$$\operatorname{card}(I \cap E) < c.$$

It is easy to see that

$$E^* = E \setminus \bigcup_{I \in S} I \quad \text{and} \quad E \setminus E^* = \bigcup_{I \in S} (I \cap E)\,.$$

Apparently,

$$\operatorname{card}\left(\bigcup_{I \in S} (I \cap E)\right) < c,$$

because in the opposite case, according to Theorem 3, there exists a nonempty perfect subset P of the set $\cup\,(I \cap E)$, $I \in S$. Then, there exists a bilateral c-point x_0 of the set P and an interval $I_0 \in S$ such that $x_0 \in I_0 \cap P$. The set $I_0 \cap P$ is nonempty perfect, $I_0 \cap P \subset I_0 \cap E$, therefore $\operatorname{card}(I_0 \cap P) = \operatorname{card}(I_0 \cap E) = c$, which contradicts the definition of S, especially $I_0 \in S$. □

Remark 5. The sets E and E^* from Lemma 4 satisfy the following assertions:

x is a bilateral c-point of E if and only if x is a bilateral c-point of E^*,

E^* is bilateral c-dense in itself, that is $E^* \subset_c E^*$.

Lemma 6. *Let F be a nowhere dense closed set and E^* a Borel set such that $F \subset_c E^*$. Then, there is an F_σ set $P \subset E^*$ of first category such that $F \cup P$ is closed nowhere dense, and $F \cup P \subset_c E^*$.*

P r o o f. Let F be a nowhere dense closed set, E^* a Borel set, and $F \subset_c E^*$. If $I_n = (a_n, b_n)$, $n = 1, 2, \dots$ is the sequence of contiguous intervals of the set F, then for each $i = 1, 2, \dots$ the following:

$$\operatorname{card}\left(E^* \cap \left(a_n, a_n + \frac{|I_n|}{2^i}\right)\right) = c,$$
$$\operatorname{card}\left(E^* \cap \left(b_n - \frac{|I_n|}{2^i}, b_n\right)\right) = c$$

holds.

According to Theorem 3, there exist nonempty perfect sets

$$A_i^n \subset E^* \cap \left(a_n, a_n + \frac{|I_n|}{2^i}\right),$$
$$B_i^n \subset E^* \cap \left(b_n - \frac{|I_n|}{2^i}, b_n\right), \qquad i = 1, 2, \dots$$

We denote $P_i^n = A_i^n \cup B_i^n$. It can be assumed that P_i^n are nowhere dense perfect sets. It is easy to see that

$$F \subset_c P = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} P_i^n \subset E^*,$$

and, moreover, $F \cup P$ is closed nowhere dense set, $F \cup P \subset_c E^*$. □

Lemma 7. *Let a set F be of type F_σ of first category and let a Borel set E be bilaterally c-dense in the set F. Then, there exists an F_σ-set $E^* \subset E$ of first category bilaterally c-dense in itself such that $F \subset_c E^*$.*

P r o o f. By Remark 5, we can assume that the set E is bilaterally c-dense in itself. Let

$$F = \bigcup_{n=1}^{\infty} F_n,$$

where $F_n, n = 1, 2, \dots$ are nowhere dense closed sets, $F_1 \subset F_2 \subset F_3 \subset \dots$ According to Lemma 6, there exists an F_σ-set $P_1 \subset E$ of first category such that $F_1 \subset_c P_1$. Moreover, the set $F_1 \cup P_1$ is closed nowhere dense. Applying the same reason to the set $F_2 \cup P_1$, we show that there exists an F_σ-set $P_2 \subset E$ of first category such that $F_2 \cup P_1 \subset_c P_2$. Moreover, the set $F_2 \cup P_1 \cup P_2$ is closed nowhere dense. We continue this procedure and show that there exists a sequence of first category sets $P_n \subset E$, $n = 1, 2, \ldots$ of type F_σ such that $F_n \subset_c P_n$ and $P_n \subset_c P_{n+1}$. Apparently, the set

$$E^* = \bigcup_{n=1}^{\infty} P_n$$

satisfies assertions from Lemma 7. □

Theorem 8. *Let a function $f \in B_1$ and let a Borel set E be bilaterally c-dense in the set of points of discontinuity of the function f. Then, there exists a function $g \in DB_1$ such that $\{x; f(x) \neq g(x)\} \subset E$.*

Proof. Let a sequence of continuous functions f_n, $n = 1, 2, \ldots$ converges on interval $[0, 1]$ to the function f and let $\bigcup_{n=1}^{\infty} F_n$, where $F_1 \subset F_2 \subset F_3 \subset \ldots$ are closed nowhere dense sets, be a set of points of discontinuity of the function f. Additionally, according to Lemma 7, it can be assumed that the set E is of type F_σ of first category bilaterally c-dense in itself, $\bigcup_{n=1}^{\infty} F_n \subset_c E$. Otherwise, the set E can be replaced with its subset having these properties. Then,

$$E = \bigcup_{n=1}^{\infty} E_n, \qquad E_1 \subset E_2 \subset E_3 \subset \ldots,$$

where E_n, $n = 1, 2, \ldots$ are closed sets. We choose a sequence of positive real numbers ε_n, $n = 1, 2, \ldots$, $\varepsilon_n \to 0$. A sequence of positive numbers $\delta_n \to 0$ can be assigned to the sequence ε_n such that for every $x_1, x_2 \in [0, 1]$,

$$|x_1 - x_2| < \delta_n \Rightarrow |f_n(x_1) - f_n(x_2)| < \varepsilon_n. \tag{$*$}$$

According to Lemma 2 in [5], if a Borel set E is bilaterally c-dense in itself and X is a closed subset of E, then there exists a perfect set P such that $X \subset_c P \subset E$. Following the paper [5], we define a system of perfect sets

$$P_1 \subset_c P_2 \subset_c P_3 \subset_c \cdots \subset_c E,$$

where $(E_n \cup P_{n-1}) \subset_c P_n$, for all $n = 2, 3, \ldots$ Clearly, $E = \bigcup_{n=1}^{\infty} P_n$, and since $\bigcup_{n=1}^{\infty} F_n \subset_c E$, we can require that

$$\forall\, x \in F_n \text{ there is } a, b \in P_n \text{ such that } a < x < b \wedge b - a < \delta_n. \tag{$**$}$$

Consequently, for all i, n, m (where $i = 1, 2, \ldots$; $n = 1, 2, \ldots$; $0 < m < 2^n$), a perfect set $P_{i+\frac{m}{2^n}}$ such that

$$P_i \subset_c P_{i+\frac{1}{2^n}} \subset_c P_{i+\frac{2}{2^n}} \subset_c \cdots \subset_c P_{i+\frac{m}{2^n}} \subset_c P_{i+\frac{m+1}{2^n}} \subset_c \cdots \subset_c P_{i+1}.$$

can be found.

Finally, for each real $\lambda \geq 1$, $i \leq \lambda < i + 1$, the closed set P_λ is defined as

$$P_\lambda = \bigcap_{\lambda \leq i + \frac{m}{2^n}} P_{i+\frac{m}{2^n}}.$$

For such a defined system of closed sets P_λ, $\lambda \geq 1$, the following holds:

$$\text{if} \quad \lambda_1 < \lambda_2 \Rightarrow P_{\lambda_1} \subset_c P_{\lambda_2}.$$

Let the functions f_λ, $\lambda \geq 1$ and g be defined as in the preface of the paper. By Lemma 2, the function $g \in B_1$ and the set $\{x; f(x) \neq g(x)\} \subset E$. We will show that the function g has Darboux property. It is sufficient to show [1, p. 9] that for each $x_0 \in [0,1]$ there exist sequences $x_n \uparrow x_0$, $y_n \downarrow x_0$, $n = 1, 2, \ldots$ such that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(y_n) = g(x_0).$$

We utilize the following fact: if $P_\lambda \subset_c P_{\lambda_0}$, and an interval (a,b) is a contiguous interval of P_{λ_0} associated with the function f_{λ_0}, then $\lambda < \lambda_0$ implies $a \notin P_\lambda$, $b \notin P_\lambda$, and by definition of g, we get $g(a) = f_{\lambda_0}(a)$, $g(b) = f_{\lambda_0}(b)$.

If $x_0 \in E$, then there exists $\lambda_0 \geq 1$ such that the point $x_0 \in P_\lambda$ for each $\lambda > \lambda_0$ and $x_0 \notin P_\lambda$ for each $\lambda < \lambda_0$. Since

$$P_{\lambda_0} \subset_c P_{\lambda_0 + \frac{1}{n}}, \qquad n = 1, 2, \ldots,$$

we choose sequences $x_n \uparrow x_0$, $y_n \downarrow x_0$ such that

$$x_n, y_n \in P_{\lambda_0 + \frac{1}{n}} \wedge x_n, y_n \notin P_\lambda, \qquad \text{for} \quad \lambda < \lambda_0 + \frac{1}{n},$$

and therefore,

$$g(x_n) = f_{\lambda_0 + \frac{1}{n}}(x_n) \wedge g(y_n) = f_{\lambda_0 + \frac{1}{n}}(y_n).$$

Since the sequence of functions $f_{\lambda_0 + \frac{1}{n}}$ uniformly converges on interval $[0,1]$ to the function f_{λ_0}, it follows that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f_{\lambda_0 + \frac{1}{n}}(x_n) = f_{\lambda_0}(x_0) = g(x_0);$$

by the same arguments,

$$\lim_{n\to\infty} g(y_n) = g(x_0).$$

If $x_0 \notin E$, two cases can be considered: either $x_0 \in C_f$ or $x_0 \notin C_f$.

In the first case,

$$\lim_{x\to x_0} f(x) = f(x_0) = g(x_0),$$

and since $f(x) = g(x)$ on a residual set, the existence of the sequences $x_n \uparrow x_0$, $y_n \downarrow x_0$, $n = 1, 2, \ldots$ is clear.

In the second case, there exists n_0 such that $x_0 \in F_n$ for every $n \geq n_0$. According to $(**)$, there exist $x_n, y_n \in P_n$ such that $x_n < x_0 < y_n$, $y_n - x_n < \delta_n$, and because P_n are perfect nowhere dense sets, it might be assumed that the points x_n, y_n are boundary points of some contiguous intervals of P_n: $f_n(x_n) = g(x_n)$ and $f_n(y_n) = g(y_n)$. Sequences $x_n \uparrow x_0$, $y_n \downarrow x_0$. Moreover, $(*)$ and $(**)$ implies that $|x_n - x_0| < \delta_n$ and hence

$$|g(x_n) - f_n(x_0)| = |f_n(x_n) - f_n(x_0)| < \varepsilon_n.$$

Since $f_n(x_0) \to f(x_0) = g(x_0)$ and $\varepsilon_n \to 0$ for $n \to \infty$, the following

$$\lim_{n\to\infty} g(x_n) = g\left(x_0\right).$$

holds. Similarly,

$$\lim_{n\to\infty} g(y_n) = g\left(x_0\right),$$

which means that the function $g \in DB_1$. $\square$

In [2], A. M. Bruckner, J. G. Ceder and R. Keston proved a theorem on approximation of a function $f \in B_1$ by a function in the class DB_1:

Theorem 9. *Let f be a Baire one function on an interval I and let E be of first category subset of I. There exists a function $g \in DB_1$ such that $f = g$ except on a first category set of measure zero which is disjoint from E and such that the function $f - g$ is in DB_1.*

Authors of Theorem 9 guarantee the equality $f = g$ on a predetermine set of first category. The next theorem is a modification of Theorem 9. The difference is that the equality $f = g$ can be guaranteed on a predetermined residual set as well. The proof is based on Theorem 8.

Theorem 10. *Let f be a Baire one function on an interval I and let a Borel set $E \subset C_f$ be bilaterally c-dense in the set of points of discontinuity of the function f. Then there exists a function $g \in DB_1$ such that $\{x \in I;\ f(x) \neq g(x)\} \subset E$, and the function $f - g$ is in DB_1.*

Proof. Again, by Lemma 7, there exists an F_σ-set $E^* \subset E$ of first category bilaterally c-dense in itself, bilaterally c-dense in the set of points of discontinuity of the function f. Theorem 8 implies the existence of a function $g \in DB_1$ such that the set $\{x \in I;\ f(x) \neq g(x)\} \subset E^*$. The function $f - g \in B_1$. Then it suffices to prove that the function $f - g$ has Darboux property. Let us consider an arbitrary point $x_0 \in I$.

Since the set $\{x \in I;\ f(x) = g(x)\}$ is residual in I and that $f(x_0) = g(x_0)$, there exist sequences $x_n \uparrow x_0$, $y_n \downarrow x_0$, $n = 1, 2, \dots$, $f(x_n) = g(x_n)$, $f(y_n) = g(y_n)$. Thus,

$$\lim_{n\to\infty} (f-g)\left(x_n\right) = \lim_{n\to\infty} (f-g)\left(y_n\right) = (f-g)\left(x_0\right) = 0.$$

If $f(x_0) \neq g(x_0)$, then $x_0 \in C_f$. The function $g \in DB_1$, hence there exists sequences $x_n \uparrow x_0$, $y_n \downarrow x_0$, $n = 1, 2, \dots$ such that

$$\lim_{n\to\infty} g\left(x_n\right) = \lim_{n\to\infty} g\left(y_n\right) = g\left(x_0\right).$$

The function f is continuous at the point x_0. Then,

$$\lim_{n\to\infty} f\left(x_n\right) = \lim_{n\to\infty} f\left(y_n\right) = f\left(x_0\right).$$

From the foregoing, it follows that

$$\lim_{n\to\infty}(f-g)(x_n)=\lim_{n\to\infty}(f-g)(y_n)=(f-g)(x_0)\,.$$

According to [1, Th. 1.1], $f-g\in DB_1$. $\square$

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