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SIMILAR FUNCTIONS AND THEIR PROPERTIES

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ABSTRACT. We are working with two topological notions of similarity of functions. We show, that these notions can be used to investigate some important properties of functions. Some types of generalized continuity are investigated. New optimization results are presented, too.

1. Introduction

In [6], two topological notions of similarity of functions were defined. It was shown that these notions were a topological generalization of some natural and well-known relations between functions. In this article we provide new results concerning generalized continuity, in particular, the quasicontinuity. Moreover, new optimization results are presented, too. All these results should also illustrate that the above mentioned notions of similarity of functions—the continuous similarity and the strong similarity—can serve as a useful tool when investigating the properties of functions from the topological point of view.

2. Two basic notions

In what follows, we will use these notions concerning topological spaces and functions: a net of points, a limit of a net, a net of functions, uniform convergence, pointwise convergence (see, e.g., [4] or [5]).

First, we introduce the notion of the continuous similarity. The definition was introduced in [6], in this article we replace the original cumbersome expression "the degree of continuity of g at x is equal or greater than the degree of continuity of f at x" with a simpler expression "g is f-continuous at the point x".

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Definition 2.1. Let X, Y, Z be topological spaces, let $f\colon X \to Y$, $g\colon X \to Z$ be functions.

Let x be from X. We say that g is f-continuous at x if for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements from X converging to x the following holds:

If the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y, then the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z. We denote this by $c_t^x(g) \geq c_t^x(f)$.

Let A be a subset of X. We say that g is f-continuous on A if, for every x from A, $c_t^x(g) \geq c_t^x(f)$ holds true. We denote this by $c_t^A(g) \geq c_t^A(f)$. Of course, for a particular x, the expressions $c_t^x(g) \geq c_t^x(f)$ and $c_t^{\{x\}}(g) \geq c_t^{\{x\}}(f)$ describe the same situation. When $c_t^X(g) \geq c_t^X(f)$ is true, we write simply $c_t(g) \geq c_t(f)$ and we say that g is f-continuous.

Let A be a subset of X. We say that f and g are continuously similar on A if $c_t^A(g) \geq c_t^A(f)$, and $c_t^A(f) \geq c_t^A(g)$ hold true at the same time. We denote this situation by writing $c_t^A(g) = c_t^A(f)$. If $A = X$ we also write $c_t(g) = c_t(f)$ or, for the sake of simplicity, $f \sim g$; we say that f and g are continuously similar.

(Obviously, the f-continuity of a function g per se does not guarantee, that g will automatically have all nice properties of f. Considering any type of generalized continuity of f, it must always be examined and proven whether this type of continuity will be inherited by g or not.)

Remark 2.2. To sum up, $f \sim g$ means that for every convergent net $\{x_\gamma\}_{\gamma\in\Gamma}$ from X, the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y if and only if the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z.

In general, we can see immediately that if f is continuous on a subset A of X and $c_t^A(g) \geq c_t^A(f)$ is true, then g is continuous on A, too. Moreover, if g is not continuous at a point x from X and $c_t^x(g) \geq c_t^x(f)$ holds true, then f is not continuous at x.

Now, the second notion of similarity of functions will be defined. In [6], the notion was defined in a way, that the domain X of f and g was supposed to be a topological space, however, the topological structure on X was not used in the definition. Therefore, we will suppose that X is simply a set.

Definition 2.3. Let X be a nonempty set and let Y, Z be topological spaces. Let $f\colon X \to Y$, $g\colon X \to Z$ be functions.

Let A be a subset of X. We say that g is f-constant on A if for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements from A the following holds:

If the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y, then the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z.

We denote this by $c_s^A(g) \geq c_s^A(f)$. If $A = X$, we also write $c_s(g) \geq c_s(f)$.

Let A be a subset of X. We say that f and g are strongly similar on A if $c_s^A(g) \geq c_s^A(f)$ and $c_s^A(f) \geq c_s^A(g)$ is true at the same time. We denote this situation by writing $c_s^A(g) = c_s^A(f)$. If $A = X$, we also write $c_s(g) = c_s(f)$ or, for the sake of simplicity, $f \approx g$. We say that f and g are strongly similar.

Remark 2.4. To sum up, $f \approx g$ means that for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ from X, $\big\{f(x_\gamma)\big\}_{\gamma\in\Gamma}$ converges in Y if and only if the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z. In our first definition we only considered some nets $\{x_\gamma\}_{\gamma\in\Gamma}$ from X, namely the convergent ones. Now, the same relationship between f and g must be verified–
–for all nets from X.

In particular, we can see that if X is a topological space and the functions f and g have values in a complete metric space (Y, d) and (Z, ρ), respectively, and if there exist two positive constants K and L such that, for all points t, s from an open neighbourhood of x (for all points t and s from a set A),

$$d\big(f(t), f(s)\big) \leq K\cdot \rho\big(g(t), g(s)\big) \leq L\cdot d\big(f(t), f(s)\big)$$

holds true, then f and g are continuously similar at x (f and g are strongly similar on A).

In [6], it was shown that if f and g are strongly similar then, for each x from X, the sets $f^{-1}\big(f(x)\big)$ and $g^{-1}\big(g(x)\big)$ are equal.

3. Generalized continuity, function spaces

The below defined notions were examined for example in [1], [2], [10]–[12].

Definition 3.1. Let (X, T) be a topological space. We say that a set $V \subset X$ is α-open if and only if there exist an open set $O \in T$ and a nowhere dense set S such that $V = O \setminus S$. The system of all α-open sets in (X, T) is denoted by T_α. T_α defines a new topology on X.

Let (Y, τ) be a topological space. Let x be from X. We say that a function $f\colon (X, T) \to (Y, \tau)$ is α-continuous at x if, for each $W \in \tau$ such that $f(x) \in W$, there exists an $V \in T_\alpha$ such that $x \in V$ and $f(V) \subset W$ is true.

Let X, Y be topological spaces. A function $f\colon X \to Y$ is said to be quasicontinuous at x from X if and only if for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$, there exists a nonempty open set $O \subset U$ such that $f(O) \subset V$.

Let $X = \mathbb{R}$, let Y be a topological space. A function $f\colon X \to Y$ is said to be left (right) hand sided quasicontinuous at a point x from $\mathbb{R}$ if for every $\delta > 0$ and for every open neighbourhood V of $f(x)$ there exists an open nonempty set $W \subset (x-\delta, x)$ ($W \subset (x, x+\delta)$) such that $f(W) \subset V$. A function f is bilaterally quasicontinuous at x if it is both left and right hand sided quasicontinuous at this point.

It is known [6] that a function g continuously similar to a quasicontinuous function f is also quasicontinuous. Now, we will show that the relation "being continuously similar" preserves the α-continuity of functions. The following theorem shows even more.

Theorem 3.2. *Let X, Y, Z be topological spaces and $f\colon X \to Y$, $g\colon X \to Z$ the functions. Let f be α-continuous at x. If g is f-continuous at x, then g is α-continuous at x.*

Proof. Denote by T the topology on X and by T_α the α-topology induced by T. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary net that converges in X to x with respect to the topology T_α. To prove the α-continuity of g in x, we need to prove that the net $\big\{g(x_\gamma)\big\}_{\gamma\in\Gamma}$ converges in Z.

Since the function f is α-continuous at x, the net $\big\{f(x_\gamma)\big\}_{\gamma\in\Gamma}$ converges in Y. Moreover, since the topology T_α is finer than T, the net $\{x_\gamma\}_{\gamma\in\Gamma}$ converges to x also with respect to the topology T. These facts and the fact that g is f-continuous at x imply that the net $\big\{g(x_\gamma)\big\}_{\gamma\in\Gamma}$ converges in Z. □

Definition 3.3 ([8]). Let X, Y be topological spaces. Let $\mathcal{A}$ be a system of nonempty subsets of X. A function $f\colon X \to Y$ is said to be $\mathcal{A}$-continuous at x from X if and only if for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$, there exists a set S from $\mathcal{A}$ such that $S \subset U$ and $f(S) \subset V$.

In the proof of the following theorem, we are going to work with special nets constructed from other nets. First, we will modify an indexed set of a net in the following way:

Let Γ be a directed set. $\Gamma^{‘}$ will stand for an indexed set defined as follows

$$(*) \quad \Gamma^{‘} = \big\{(\gamma, 1); \gamma \in \Gamma\big\} \cup \big\{(\gamma, 2); \gamma \in \Gamma\big\}$$

and $\Gamma^{‘}$ is equipped with a preorder defined by

$\forall \alpha, \beta \in \Gamma$ if $\alpha < \beta$ then $(\alpha, 1) < (\alpha, 2) < (\beta, 1) < (\beta, 2)$. It is easy to check that $\Gamma^{‘}$ is a directed set.

To prove the following theorem, we need a special kind of a net which we are going to define now.

Suppose $\big\{(x_\gamma)\big\}_{\gamma\in\Gamma}$ is a net of points of a set X. Let a be a point from X. The symbol $\{x_\gamma, a\}$ will denote the special net $\{x_\gamma, a\} = \{y_{\gamma^{‘}}\}_{\gamma^{‘}\in\Gamma^{‘}}$ where $\Gamma^{‘}$ is defined as in (*) and, for all γ from Γ, we have $y_{(\gamma,1)} = x_\gamma$, $y_{(\gamma,2)} = a$. We can immediately see that the net $\{x_\gamma\}_{\gamma\in\Gamma}$ is a subnet of $\{x_\gamma, a\}$ and that the constant net $\{y_{(\gamma,2)}\}_{\gamma\in\Gamma}$ is a subnet of $\{x_\gamma, a\} = \{y_{\gamma^{‘}}\}_{\gamma^{‘}\in\Gamma^{‘}}$, too. Moreover, we can see that if $\{x_\gamma\}_{\gamma\in\Gamma}$ converges to a, then $\{x_\gamma, a\}$ converges to a, too. This will be used at the very end of the proof of the following theorem.

Theorem 3.4. *Let X, Y, Z be Hausdorff topological spaces. Let $f\colon X \to Y$, $g\colon X \to Z$ be functions. Let x be a point from X. Let $\mathcal{A}$ be a system of nonempty subsets of X and let f be $\mathcal{A}$-continuous at x. If g is f-continuous at x, then g is $\mathcal{A}$-continuous at x, too.*

Proof. We proceed by contradiction. Suppose that g is not $\mathcal{A}$-continuous at x. Because of this, there exists an open neighbourhood W of $g(x)$ and an open neighbourhood U of x such that for any subset S of U such that $S \in \mathcal{A}$ there exists a point s from S such that $g(s) \in Y \setminus W$. In other words, no set $A \in \mathcal{A}$ is a subset of the set $g^{-1}(W) \bigcap U$. Denote by Γ the family of all open neighbourhoods of x contained in U, and by A the family of all open neighbourhoods of $f(x)$.

Define $B = \Gamma \times A$. Define a partial order "$\leq$" on B by

$$\forall (\gamma_1, \alpha_1), (\gamma_2, \alpha_2) \in B (\gamma_1, \alpha_1) \leq (\gamma_2, \alpha_2) \quad \text{if and only if} \quad \gamma_2 \subseteq \gamma_1 \text{ and } \alpha_2 \subseteq \alpha_1.$$

It is easy to see that B so equipped is a directed set.

For each $\beta \in B$, $\beta = (\gamma, \alpha)$, the following holds (since γ is an open neighbourhood of x and α is an open neighbourhood of $f(x)$ and f is $\mathcal{A}$-continuous at x): There exists a set $S_\beta \in \mathcal{A}$ such that $S_\beta \subset \gamma \bigcap U$ and $f(S_\beta) \subset \alpha$. Since S_β cannot be a subset of $g^{-1}(W) \bigcap U$, there exists a point x_β from S_β such that $g(x_\beta) \in Y \setminus W$. At the same time, $f(x_\beta) \in \alpha$.

We have just constructed a net of points $\{x_\beta\}_{\beta \in B}$. It is easy to see that this net has the following properties:

(1) $\lim_{\beta \in B} x_\beta = x$,

(2) $\lim_{\beta \in B} f(x_\beta) = f(x)$,

(3) $\forall \beta \in B\ g(x_\beta) \in Y \setminus W$.

Now, consider the "alternate" net $\{x_\beta, x\}$ and the corresponding nets $\big\{f(x_\beta), f(x)\big\}$ and $\big\{g(x_\beta), g(x)\big\}$. Because of (1), the net $\{x_\beta, x\}$ converges to x. Because of (2), the net $\big\{f(x_\beta), f(x)\big\}$ converges to $f(x)$. However, the convergence of this net and the fact that g is f-continuous at x imply that the net $\big\{g(x_\beta), g(x)\big\}$ is convergent, too. Of course, the net $\big\{g(x_\beta), g(x)\big\}$ has the same limit as any of its subnets. We can see that it has a constant subnet with constant values equal to $g(x)$, so the net $\big\{g(x_\beta), g(x)\big\}$ converges to $g(x)$. By the same reasoning, we obtain that the net $\big\{g(x_\beta)\big\}_{\beta \in B}$ converges to $g(x)$.

However, this is a contradiction, because, for each point x_β, we have $g(x_\beta) \in Y \setminus W$; on the other hand, W is an open neighbourhood of the "would be limit" $g(x)$. □

We are almost ready to give a characterization of some spaces of $\mathcal{A}$-continuous functions. All we need is to combine the result of the preceding theorem with the following result proved in [6]:

Theorem 3.5. *Let X be a topological space and let $(Y, d), (Z, \varrho)$ be complete metric spaces. Let $h\colon X \to Y$, $f\colon X \to Z$ be functions. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to Z. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ converge uniformly to f. Let x be a point of X, let A be a subset of X. Then:*

(i) *If for all γ from Γ, f_γ is h-continuous at x and if h is continuous at x, then f is continuous at x.*
(ii) *If for all γ from Γ, f_γ is h-continuous on A, then f is h-continuous on A.*

The following assertion is a corollary of the two preceding theorems:

Corollary 3.6. *Let X be a topological space and let $(Y, d), (Z, \varrho)$ be complete metric spaces. Let $h\colon X \to Y$, $f\colon X \to Z$ be functions. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to Z. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ converge uniformly to f. Let x be a point of X, let A be a subset of X. Let h be $\mathcal{A}$-continuous at x ($\mathcal{A}$-continuous at all points from A). Then, if for all γ from Γ, f_γ is h-continuous at x (f_γ is h-continuous on A), then f is $\mathcal{A}$-continuous at x (f is $\mathcal{A}$-continuous at all points from A).*

Remark 3.7. Let X be a topological space and let (Y, d) be a Fréchet space. Using the last two theorems, we can see that if f from X to Y is a function, then for any f_1, f_2 from X to Y such that f_1 and f_2 are f-continuous at a point x, any linear combination of these two function is f-continuous at x, too. (This is just because of the "continuous behavior" of linear combination in Fréchet spaces. More concretely, if two nets $\{f_1(x_\gamma)\}_{\gamma\in\Gamma}$ and $\{f_2(x_\gamma)\}_{\gamma\in\Gamma}$ converge in Y, then every net of the form $\{c_1 f_1(x_\gamma) + c_2 f_2(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y, too.)

This means that if f is $\mathcal{A}$-continuous at a point x, any linear combination of two f-continuous functions f_1 and f_2 is $\mathcal{A}$-continuous at x, too. Moreover, as we have proved, the property "being f-continuous at a point x" is preserved under the uniform convergence. This means that all functions from X to Y, that are f-continuous at x, form a linear subspace of the space of all functions from X to Y. Moreover, this subspace is closed with respect to the operation of the uniform convergence. We will formulate this result in the following theorem.

In what follows, if f is a function from a topological space X into a topological space Y, by $\mathcal{C}_f$ we will denote the set of all f-continuous functions from X to Y.

Theorem 3.8. *Let X be a topological space and let (Y, d) be a Fréchet space.*

(i) *Let $f\colon X \to Y$ be a function. Then the set $\mathcal{C}_f$ is a nonempty linear subspace of the linear space of all functions from X to (Y, d). Moreover, $\mathcal{C}_f$ is closed with respect to the uniform convergence.*

(ii) *Let $\mathcal{A}$ be a system of nonempty subsets of X. Then the set of all functions from X to Y that are $\mathcal{A}$-continuous at x from X (at all points s from a subset $S \subset X$) is a union of a system of nonempty linear spaces of functions such that each of these spaces contains all continuous functions from X to Y and is closed with respect to the uniform convergence.*

Proof. (i) $\mathcal{C}_f$ is nonempty, because $f \in \mathcal{C}_f$. For the rest, let us see the preceding remark.

(ii) Put $\mathcal{C}_{\mathcal{A},S} = \{f\colon X \to Y;\ f$ is $\mathcal{A}$-continuous at all points s from $S\}$.
Then, $\mathcal{C}_{\mathcal{A},S} = \bigcup_{f \in C_{\mathcal{A},S}} \mathcal{C}_f$. □

Because of the generality of $\mathcal{A}$-continuity, we have just characterized a wider range of systems of functions. Concretely, if X, Y are topological spaces, $\mathcal{A}$ is a system of nonempty subsets of X, and if a function $f\colon X \to Y$ is $\mathcal{A}$-continuous at a point x (at all points of a set $S \subset X$), then f is

(1) continuous at x if $\mathcal{A} = \{U;\ U$ is open in X and $x \in U\}$,
(2) α-continuous at x if $\mathcal{A} = \{O;\ O$ is α-open in X and $x \in O\}$,
(3) quasicontinuous at x (or on S) if $\mathcal{A} = \{U;\ U$ is open in $X\}$.
Moreover, suppose that $X = \mathbb{R}$. Then f is
(4) left (right) hand sided quasicontinuous at x if

$$\mathcal{A} = \big\{V; V = (a, b) \text{ and } a < b < x\big\}$$

($\mathcal{A} = \{V;\ V = (a, b)$ and $x < a < b\}$),
(5) bilaterally quasicontinuous at x if

$$\mathcal{A} = \big\{V; V = (a, b) \cup (c, d) \text{ and } a < b < x < c < d\big\}.$$

Now, we can see that our last theorem implies the validity of the following assertions:

Corollary 3.9. *Let X be a topological space and let (Y, d) be a Fréchet space. Let $S \subset X$, $S \neq \emptyset$. Then the set of all functions from X to Y that are quasicontinuous, (α-continuous, left (right) hand sided quasicontinuous, bilaterally quasicontinuous) at all points of S is a union of a system of nonempty linear spaces of functions such that each of these spaces is containing all continuous functions from X to Y and it is closed with respect to the uniform convergence. More concretely, each of these spaces can be of the form $\mathcal{C}_f$, where $f\colon X \to Y$, is a function that is quasicontinuous (α-continuous, left (right) hand sided quasicontinuous, bilaterally quasicontinuous) at all points of S.*

Open question: Does there exist a noncontinuous, quasicontinuous function f from $\mathbb{R}$ to $\mathbb{R}$ such that the linear space $\mathcal{C}_f$ only consists of the elements of the form: $c_1 f + c_2 h$, where h (an arbitrary continuous function from $\mathbb{R}$ to $\mathbb{R}$) and $c_1, c_2 \in \mathbb{R}$ are variable?

4. Optimization applications

We say that a topological space X is locally arcwise connected at a point x if every neighbourhood U of x contains a neighbourhood V of x such that any two points a, b from V can be joined by an arc in V, i.e., there exists a function $h\colon <0,1> \to V$ such that $h\colon <0,1> \to h(<0,1>)$ is a homeomorphism and $h(0) = a$, $h(1) = b$ holds.

The following optimization result was proved in [6].

Theorem 4.1. *Let X be a topological space, let x be from X. Let X be locally arcwise connected at x. Let $f\colon X \to \mathbb{R}$, $g\colon X \to \mathbb{R}$ be continuous functions. Let $f \approx g$. Then*

(j) *x is a point of a local extremum of f if and only if x is a point of a local extremum of g,*

(jj) *x is a point of a strict local extremum of f if and only if x is a point of a strict local extremum of g.*

The following theorem will enable us to prove two optimization theorems.

Theorem 4.2. *Let X be a nonempty set and Y, Z be T_2 topological spaces. Let $f\colon X \to Y$, $g : X \to Z$ be functions. Let $f \approx g$. Then, for any subset A of X, the following is true:*

$f(A)$ is closed (compact) in Y if and only if $g(A)$ is closed (compact) in Z.

P r o o f. First, we will prove the "closedness" part of our assertion. It suffices to show that if $f(A)$ is closed in Y, then $g(A)$ is closed in Z.

Let $\{z_\gamma\}_{\gamma\in\Gamma}$ be a net of points from $g(A)$ which is convergent in Z. Denote its limit by z. We have to prove that there exists a point a in A such that $g(a) = z$ is true.

Since each point z_γ is from $g(A)$, for every γ from Γ, there exists x_γ from A such that $g(x_\gamma) = z_\gamma$. We see that $\big\{g(x_\gamma)\big\}_{\gamma\in\Gamma} = \{z_\gamma\}_{\gamma\in\Gamma}$ converges in Z. Together with $f \approx g$, this implies that the net $\big\{f(x_\gamma)\big\}_{\gamma\in\Gamma}$ converges in Y. We will denote its limit by y. The set $f(A)$ is closed, so $y \in f(A)$. Consider a point $a \in A$ such that $f(a) = y$.

Now, let us consider the net $\{p_{\gamma'}\}_{\gamma'\in\Gamma'} := \{x_\gamma, a\}$. We can see that

$$\lim_{\gamma'\in\Gamma'} f(p_{\gamma'}) = y = f(a).$$

Since $f \approx g$, this means there exists $m \in Z$ such that $m = \lim_{\gamma'\in\Gamma'} g(p_{\gamma'})$. However, the nets $\{x_\gamma\}_{\gamma\in\Gamma}$ and $\{a\}_{\gamma\in\Gamma}$ (by this we mean the net $\{a_\gamma\}_{\gamma\in\Gamma}$ where for all γ from Γ, $a_\gamma = a$) are both subnets of the net $\{p_{\gamma'}\}_{\gamma'\in\Gamma'}$. This implies $\lim_{\gamma'\in\Gamma'} g(p_{\gamma'}) = \lim_{\gamma\in\Gamma} g(x_\gamma) = z$ and $\lim_{\gamma'\in\Gamma'} g(p_{\gamma'}) = \lim_{\gamma\in\Gamma} g(a) = g(a)$. So, $z = g(a)$, and this also means $z \in g(A)$. The closedness of $g(A)$ is proven.

Now, for "compactness" part of our assertion, again, it suffices to prove that if $f(A)$ is compact, then $g(A)$ is compact, too. Suppose $f(A)$ to be compact. Then, it is closed, so $g(A)$ is closed, too. Considering an arbitrary net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ in $g(A)$, we now have to prove that it has a convergent subnet. However, the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ in $f(A)$ has a convergent subnet, say $\{f(x_\delta)\}_{\delta\in\Delta}$, and because of strong similarity of f and g, the net $\{g(x_\delta)\}_{\delta\in\Delta}$ is convergent (in $g(A)$), too. □

Now, the following theorems will be easy corollaries of the preceding theorem. Let us observe that we do not need any topological structure on the domain set X in the following theorem.

Theorem 4.3. *Let X be a set. Let $f\colon X \to \mathbb{R}$, $g\colon X \to \mathbb{R}$ be functions. Let A be a subset of X such that the set $f(A)$ is closed. Let f achieve its maximum and minimum on A. Let $f \approx g$. Then g achieves its maximum and minimum on A, too.*

Proof. Under the conditions of our theorem, the set $f(A)$ must be compact. So the set $g(A)$ is compact, too. □

Now, we present an optimisation result concerning functions that are strongly similar to Darboux functions.

Theorem 4.4. *Let X be a topological space. Let $f\colon X \to \mathbb{R}$, $g : X \to \mathbb{R}$ be functions. Let f be a Darboux function (an image of each connected set under f is connected). Let f achieve its maximum and minimum on a connected subset A of X. Let $f \approx g$. Then g also achieves its maximum and minimum on A.*

Proof. Under the conditions of our theorem, the set $f(A)$ must be connected and bounded. Moreover, it contains its supremum and infimum. So, $f(A)$ is a compact interval. This means, the set $g(A)$ is compact, too. Therefore g achieves a global maximum and a global minimum on A. □

To conclude this section, let us remark that all the results presented here show some possibilies how to investigate a non-differentiable function for extrema. Namely, some non-differentiable functions are strongly similar to differentiable ones and these can be investigated in a classical way.

REFERENCES

[1] BORSÍK, J.: *Bilateral quasicontinuity in topological spaces*, Tatra Mt. Math. Publ. **28** (2004), 159–168.

[2] BORSÍK, J.: *On the points of bilateral quasicontinuity of functions*, Real Anal. Exchange **19** (1993/1994), 529–536.

[3] BORSÍK, J.—HOLÁ, Ľ.—HOLÝ, D.: *Baire spaces and quasicontinuous mappings*, Filomat **25** (2011), 69–83.
[4] DEIMLING, K.: *Nonlinear Functional Analysis*. Springer-Verlag, Berlin, 1985.
[5] ENGELKING, R.: *General Topology*. PWN, Warszaw, 1977.
[6] KUPKA, I.: *On similarity of functions*, Topology Proc. **36** (2010), 173–187.
[7] LEVINE, N.: *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.
[8] MATEJDES, M.: *Quasicontinuous selections of upper Baire continuous mappings*, Mat. Vesnik **62** (2010), 69–76.
[9] MATEJDES, M.: *Selection theorems and minimal mappings in a cluster setting*, Rocky Mountain J. Math. **41** (2011) 851–867.
[10] NEUBRUNN, T.: *Quasi-continuity*, Real Anal. Exchange **14** (1989), 259–306.
[11] NJÅSTAD, O.: *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.
[12] PIOTROWSKI, Z.: *A survey of results concerning generalized continuity on topological spaces*, Acta Math. Univ. Comenian. **52-53** (1987), 91–110.

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