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PRODUCTS OF INTERNALLY QUASI-CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper we characterize the product of internally quasi-continuous functions and we construct a bounded internally quasi-continuous strong Świątkowski function which cannot be written as a finite product of internally strong Świątkowski functions.

1. Preliminaries

The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. The symbol $\mathrm{I}(a,b)$ denotes an open interval with the endpoints a and b. For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, and $\operatorname{card} A$ to denote the interior, the closure, the boundary, and the cardinality of A, respectively. We say that a set $A \subset \mathbb{R}$ is *simply open* [1], if it can be written as the union of an open set and a nowhere dense set.

The word function denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated. The symbol $\mathcal{C}(f)$ stands for the set of all points of continuity of f. We say that f is a *Darboux function*, if it maps the connected sets onto connected sets. We say that f is *cliquish* [11] ($f \in \mathcal{C}_q$), if the set $\mathcal{C}(f)$ is dense in $\mathbb{R}$. We say that f is *internally cliquish* ($f \in \mathcal{C}_{qi}$), if the set $\operatorname{int}\mathcal{C}(f)$ is dense in $\mathbb{R}$. We say that f is *quasi-continuous* in the sense of Kempisty [5] ($f \in \mathcal{Q}$), if for all $x \in \mathbb{R}$ and open sets $U \ni x$ and $V \ni f(x)$, the set $\operatorname{int}\big(U \cap f^{-1}(V)\big) \neq \emptyset$. We say that f is *internally quasi-continuous* [8] ($f \in \mathcal{Q}_i$), if it is quasi-continuous and its set of points of discontinuity is nowhere dense; equivalently, f is internally quasi-continuous if $f{\restriction}\operatorname{int}\mathcal{C}(f)$ is dense in f. We say that x_0 is a point of internal

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quasi-continuity of f if and only if there is a sequence $(x_n) \subset \operatorname{int} \mathcal{C}(f)$ such that $x_n \to x_0$ and $f(x_n) \to f(x_0)$ (see [8]). We say that f is a *strong Świątkowski function* [6] ($f \in \acute{\mathcal{S}}_s$), if whenever $a, b \in \mathbb{R}$, $a < b$, and $y \in \mathrm{I}\big(f(a), f(b)\big)$, there is $x_0 \in (a, b) \cap \mathcal{C}(f)$ such that $f(x_0) = y$. We say that f is an *internally strong Świątkowski function* [8] ($f \in \acute{\mathcal{S}}_{si}$), if whenever $a, b \in \mathbb{R}$, $a < b$, and $y \in \mathrm{I}\big(f(a), f(b)\big)$, there is

$$x_0 \in (a, b) \cap \operatorname{int} \mathcal{C}(f) \qquad \text{such that} \quad f(x_0) = y.$$

Clearly, each strong Świątkowski function has the Darboux property. Moreover, we can easily see that the following inclusions

$$\acute{\mathcal{S}}_{si} \subset \acute{\mathcal{S}}_s \subset \mathcal{Q} \subset \mathcal{C}_q \quad \text{and} \quad \acute{\mathcal{S}}_{si} \subset \mathcal{Q}_i \subset \mathcal{C}_{qi} \subset \mathcal{C}_q$$

are satisfied.

Finally, the symbol $[f = a]$ stands for the set $\big\{x \in \mathbb{R} : f(x) = a\big\}$.

2. Introduction

In 1960 S. Marcus remarked that not every function is the product of Darboux functions [9]. The problem of characterizing the class of products of Darboux functions was solved by J. G. Ceder [2], [3]. In 1985 Z. Grande constructed a nonnegative Baire one function which cannot be the product of a finite number of quasi-continuous functions, and asked for characterization of such products [4]. The following theorem (see [7, Theorem III.2.1]) gives an answer to this question.

Theorem 2.1. *For each function f the following conditions are equivalent*:

i) *there are quasi-continuous functions g_1 and g_2 with $f = g_1 g_2$,*

ii) *f is a finite product of quasi-continuous functions,*

iii) *f is cliquish and the set $[f = 0]$ is simply open.*

In 1996 A. Maliszewski characterized the product of Darboux quasi-continuous functions [7, Theorem III.3.1] and proved that there exists a bounded Darboux quasi-continuous function which cannot be written as the finite product of strong Świątkowski functions [7, Proposition III.4.1]. Ten years later P. Szczuka characterized the product of four and more strong Świątkowski functions [10, Theorem 4.2].

In this paper we characterize the product of internally quasi-continuous functions (Theorem 4.2) and we construct a bounded internally quasi-continuous strong Świątkowski function which cannot be written as the finite product of internally strong Świątkowski functions (Proposition 4.3).

3. Auxiliary lemmas

Lemma 3.1 is due to A. Maliszewski [7, Lemma III.1.10].

Lemma 3.1. *Let $I=(a,b)$, $\Gamma>0$ be an extended real number, and $k>1$. There are functions $g_1,\dots,g_k$ such that $g_1\dots g_k=0$ on $\mathbb{R}$ and for $i\in\{1,\dots,k\}$: $\mathbb{R}\setminus\mathcal{C}(g_i)=\operatorname{bd}I$ and $g_i\big[(a,c)\big]=g_i\big[(c,b)\big]=(-\Gamma,\Gamma)$ for each $c\in I$.*

The proof of Lemma 3.2 can be found in [10, Lemma 3.4].

Lemma 3.2. *Assume that $F\subset C$ are closed and $\mathcal{J}$ is a family of components of $\mathbb{R}\setminus C$ such that $C\subset\operatorname{cl}\bigcup\mathcal{J}$. There is a family $\mathcal{J}'\subset\mathcal{J}$ such that:*

i) *for each $J\in\mathcal{J}$, if $F\cap\operatorname{bd}J\neq\emptyset$, then $J\in\mathcal{J}'$;*

ii) *for each $c\in F$, if c is a right-hand (left-hand) limit point of C, then c is a right-hand (respectively left-hand) limit point of the union $\bigcup\mathcal{J}'$;*

iii) *$\operatorname{cl}\bigcup\mathcal{J}'\subset F\cup\bigcup_{J\in\mathcal{J}'}\operatorname{cl}J$.*

Lemma 3.3. *Let $I=(a,b)$ and let the function $f\colon \operatorname{cl}I\to(0,+\infty)$ be continuous. There are continuous functions $\psi_1,\psi_2\colon I\to(0,+\infty)$ such that $f=\psi_1\psi_2$ on I and $\psi_i\big[(a,c)\big]=\psi_i\big[(c,b)\big]=(0,+\infty)$ for each $i\in\{1,2\}$ and $c\in I$.*

Proof. Define the function $\bar\psi\colon\mathbb{R}\to(0,+\infty)$ by

$$\bar\psi(x)=\begin{cases}\max\left\{\frac{\sin x^{-1}+1}{|x|},|x|\right\} & \text{if } x\neq0,\\ 1 & \text{if } x=0.\end{cases}$$

Then, clearly, $\mathcal{C}(\bar\psi)=\mathbb{R}\setminus\{0\}$ and $\bar\psi\big[(-\delta,0)\big]=\bar\psi\big[(0,\delta)\big]=(0,+\infty)$ for each $\delta>0$. Choose elements $x_1,x_2\in(a,b)$ and assume that $x_1<x_2$. Define the function $\psi_1\colon I\to(0,+\infty)$ by the formula

$$\psi_1(x)=\begin{cases}\bar\psi(x-a) & \text{if } x\in(a,x_1],\\ \bar\psi(x-b) & \text{if } x\in[x_2,b),\\ \text{linear,} & \text{on the interval } [x_1,x_2].\end{cases}$$

Observe that ψ_1 is continuous on I and it is easy to see that $\psi_1\big[(a,c)\big]=\psi_1\big[(c,b)\big]=(0,+\infty)$ for each $c\in I$. Now, define the function $\psi_2\colon I\to(0,+\infty)$ as follows

$$\psi_2=\frac{f}{\psi_1}.$$

Since f is positive, bounded and continuous on $\operatorname{cl}I$, the function ψ_2 is continuous on I and $\psi_2\big[(a,c)\big]=\psi_2\big[(c,b)\big]=(0,+\infty)$ for each $c\in I$. Finally, $f=\psi_1\psi_2$ on I, which completes the proof. □

Lemma 3.4. *Let $I=(a,b)$ and $y\in(0,1]$. There is a strong Świątkowski function $\psi\colon\operatorname{cl}I\to(0,1]$ such that $\psi[I]=\psi\big[I\cap\mathcal{C}(\psi)\big]=(0,1]$, $\psi(a)=\psi(b)=y$, $\operatorname{bd}I\subset\mathcal{C}(\psi)$, and $\operatorname{card}\big(I\setminus\mathcal{C}(\psi)\big)=1$.*

P r o o f. Define the function $\bar{\psi}\colon \mathbb{R} \to (0,1]$ by

$$\bar{\psi}(x) = \begin{cases} \min\{1, \sin x^{-1} + |x| + 1\} & \text{if } x \neq 0, \\ 2^{-1} & \text{if } x = 0. \end{cases}$$

Then, clearly, $\bar{\psi} \in \acute{\mathcal{S}}_s$. Choose elements $x_1, x_2, x_3 \in (a,b)$ and assume that $x_1 < x_2 < x_3$. Define the function $\psi\colon \operatorname{cl} I \to (0,1]$ by the formula

$$\psi(x) = \begin{cases} \bar{\psi}(x - x_2) & \text{if } x \in [x_1, x_3], \\ y & \text{if } x \in \{a, b\}, \\ \text{linear}, & \text{on intervals } [a, x_1] \text{ and } [x_3, b]. \end{cases}$$

One can easily show that the function ψ has all required properties. □

4. Main results

Remark 4.1. Product of two internally cliquish functions is internally cliquish.

P r o o f. If the functions f and g are internally cliquish, then the sets $\operatorname{int} \mathcal{C}(f)$ and $\operatorname{int} \mathcal{C}(g)$ are dense in $\mathbb{R}$. Hence, the set $\operatorname{int} \mathcal{C}(f) \cap \operatorname{int} \mathcal{C}(g)$ is dense in $\mathbb{R}$, too. Moreover, $\operatorname{int} \mathcal{C}(f) \cap \operatorname{int} \mathcal{C}(g) \subset \operatorname{int} \mathcal{C}(fg)$, which proves that the function fg is internally cliquish. □

Theorem 4.2. *For each function f the following conditions are equivalent*:

i) *there are internally quasi-continuous functions g_1 and g_2 with $f = g_1 g_2$,*

ii) *f is a finite product of internally quasi-continuous functions,*

iii) *f is internally cliquish and the set $[f = 0]$ is simply open.*

P r o o f. The implication i) ⇒ ii) is evident.

ii) ⇒ iii). Assume that there is $k \in \mathbb{N}$ and there are internally quasi-continuous functions $g_1, \dots, g_k$ such that $f = g_1 \dots g_k$. Since each internally quasi-continuous function is internally cliquish, using Remark 4.1 we obtain that the function f is internally cliquish. Moreover, since each internally quasi-continuous function is quasi-continuous, by Theorem 2.1, $[f = 0]$ is simply open.

iii) ⇒ i). Now, assume that the set $[f = 0]$ is simply open and the function f is internally cliquish. Hence, $\operatorname{int} \mathcal{C}(f)$ is dense in $\mathbb{R}$. Let $U = \operatorname{int} \mathcal{C}(f) \setminus \operatorname{bd}[f = 0]$. Observe that the set $\mathbb{R} \setminus U$ is closed. Moreover,

$$\mathbb{R} \setminus U = \big(\mathbb{R} \setminus \operatorname{int} \mathcal{C}(f)\big) \cup \operatorname{bd}[f = 0].$$

Since $\mathbb{R} \setminus \operatorname{int} \mathcal{C}(f)$ is boundary and closed, and $[f = 0]$ is simply open, the set $\mathbb{R} \setminus U$ is nowhere dense. Let $\mathcal{J}$ be the family of all components of U. Since $[f = 0]$ is simply open, $J \subset [f = 0]$ or $J \cap [f = 0] = \emptyset$ for each $J \in \mathcal{J}$. So, if there is

$J \in \mathcal{J}$ such that $J \cap [f = 0] = \emptyset$, then $f > 0$ on J or $f < 0$ on J. Write the set U as the union of families $\mathcal{I}_1$ and $\mathcal{I}_2$ consisting of the pairwise disjoint compact intervals, such that for each $x \in U$, there are $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ with $x \in \operatorname{int}(I_1 \cup I_2)$.

Fix an interval $J \in \mathcal{J}$ and let $J = (a, b)$. If $J \subset [f = 0]$, then by Lemma 3.1 applied for $\Gamma = +\infty$ and $k = 2$, there are continuous functions $g_{1J}, g_{2J} \colon J \to \mathbb{R}$ such that $0 = f \restriction J = g_{1J} g_{2J}$ and for $i \in \{1, 2\}$

$$g_{iJ}\big[(a, c)\big] = g_{iJ}\big[(c, b)\big] = \mathbb{R} \qquad \text{for each} \quad c \in J. \tag{1}$$

If $J \cap [f = 0] = \emptyset$, then $|f| > 0$ on J. Fix an interval $I \in \mathcal{I}_1 \cup \mathcal{I}_2$ such that $I \subset J$ and let $I = [\alpha, \beta]$. Since $|f \restriction I| > 0$ and f is continuous on I, by Lemma 3.3 there are continuous functions $\psi_{1I}, \psi_{2I} \colon (\alpha, \beta) \to (0, +\infty)$ such that $|f| = \psi_{1I} \psi_{2I}$ on (α, β) and for $i \in \{1, 2\}$

$$\psi_{iI}\big[(\alpha, c)\big] = \psi_{iI}\big[(c, \beta)\big] = (0, +\infty) \qquad \text{for each} \quad c \in (\alpha, \beta). \tag{2}$$

Now, define functions $\psi_{1J}, \psi_{2J} \colon J \to \mathbb{R}$ as follows:

$$\psi_{1J}(x) = \begin{cases} \;\;\psi_{1I}(x) & \text{if } x \in \operatorname{int} I \text{ and } I \in \mathcal{I}_1, \\ -\psi_{1I}(x) & \text{if } x \in \operatorname{int} I \text{ and } I \in \mathcal{I}_2, \\ \;\;1 & \text{otherwise,} \end{cases}$$

$$\psi_{2J}(x) = \begin{cases} \;\;\psi_{2I}(x) \cdot \operatorname{sgn} f(x) & \text{if } x \in \operatorname{int} I \text{ and } I \in \mathcal{I}_1, \\ -\psi_{2I}(x) \cdot \operatorname{sgn} f(x) & \text{if } x \in \operatorname{int} I \text{ and } I \in \mathcal{I}_2, \\ \;\;f(x) & \text{otherwise.} \end{cases}$$

Then, clearly, $f \restriction J = \psi_{1J} \psi_{2J}$. By condition (2) and since $|f| > 0$ on J, for $i \in \{1, 2\}$

$$\psi_{iJ}\big[(a, c)\big] = \psi_{iJ}\big[(c, b)\big] = \mathbb{R} \setminus \{0\} \qquad \text{for each} \quad c \in J. \tag{3}$$

ψ_{1J} and ψ_{2J} are internally quasi-continuous on J. Fix $x \in J$. If there is $I \in \mathcal{I}_1 \cup \mathcal{I}_2$ such that $x \in \operatorname{int} I$, then, since ψ_{1I} and ψ_{2I} are continuous on I, the functions ψ_{1J} and ψ_{2J} are internally quasi-continuous at x. In another case, there are $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ such that $x \in \operatorname{bd} I_1 \cap \operatorname{bd} I_2$. Since $\psi_{1J}(x) = 1$ and $\psi_{1J} = \psi_{1I}$ is positive and continuous on $\operatorname{int} I_1$, using (2), we clearly obtain that ψ_{1J} is internally quasi-continuous at x. Moreover, $\psi_{2J}(x) = f(x)$ and ψ_{2J} is continuous on $\operatorname{int} I_1$, it has the same sign as the function f on I_1. (Recall that f does not change its sign on J.) So, by (2), ψ_{2J} is internally quasi-continuous at x, too.

Further, we define functions $g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$g_1(x) = \begin{cases} g_{1J}(x) & \text{if } x \in J,\ J \in \mathcal{J} \text{ and } J \subset [f = 0], \\ \psi_{1J}(x) & \text{if } x \in J,\ J \in \mathcal{J} \text{ and } J \cap [f = 0] = \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

$$g_2(x) = \begin{cases} g_{2J}(x) & \text{if } x \in J,\ J \in \mathcal{J} \text{ and } J \subset [f = 0], \\ \psi_{2J}(x) & \text{if } x \in J,\ J \in \mathcal{J} \text{ and } J \cap [f = 0] = \emptyset, \\ f(x) & \text{otherwise.} \end{cases}$$

Then, clearly, $f = g_1 g_2$. Finally we will show that functions g_1 and g_2 are internally quasi-continuous.

Fix $i \in \{1, 2\}$ and let $x \in \mathbb{R}$. First, assume that there is $J \in \mathcal{J}$ such that $x \in \operatorname{cl} J$. If $x \in \operatorname{int} J$ then, since g_{iJ} is continuous on J and ψ_{iJ} is internally quasi-continuous on J, the function g_i is internally quasi-continuous at x. So, let $x \in \operatorname{bd} J$. Since $g_i \restriction J$ is internally quasi-continuous, by (1) or (3), we clearly obtain that g_i is internally quasi-continuous at x.

Assume now that $x \in \mathbb{R} \setminus \bigcup_{J \in \mathcal{J}} \operatorname{cl} J$. In this case, $x \in \mathbb{R} \setminus U$. Since $\mathbb{R} \setminus U$ is nowhere dense and conditions (1) and (3) hold, for each $n \in \mathbb{N}$, there is $J_n \in \mathcal{J}$ such that $J_n \subset \left(x, x + \frac{1}{n}\right)$ and there is

$$x_n \in \mathcal{J}_n \cap \operatorname{int} \mathcal{C}(g_i) \quad \text{with} \quad |g_i(x_n) - g_i(x)| < \frac{1}{n}.$$

Hence there is a sequence $(x_n) \subset \operatorname{int} \mathcal{C}(g_i)$ such that $x_n \to x$. Consequently, the function g_i is internally quasi-continuous at x. This completes the proof. □

Proposition 4.3. *There is a bounded internally quasi-continuous strong Świątkowski function which cannot be written as the finite product of internally strong Świątkowski functions.*

Proof. Let $C \subset [0, 1]$ be the Cantor ternary set, and let $\mathcal{I}_1$ and $\mathcal{I}_2$ be disjoint families of all components of the set $[0, 1] \setminus C$ such that $C \cup \bigcup \mathcal{I}_1 \cup \bigcup \mathcal{I}_2 = [0, 1]$ and $C = \left(\operatorname{cl} \bigcup \mathcal{I}_1\right) \cap \left(\operatorname{cl} \bigcup \mathcal{I}_1\right)$. Put $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and define

$$A = C \setminus \bigcup\nolimits_{I \in \mathcal{I}} \operatorname{bd} I. \tag{4}$$

Since A is a G_δ-set, then $C \setminus A$ is an F_σ-set, whence there is a sequence (F_n) consisting of closed sets such that

$$C \setminus A = \bigcup\nolimits_{n \in \mathbb{N}} F_n. \tag{5}$$

Define $F'_0 = \emptyset$. For each $n \in \mathbb{N}$, use Lemma 3.2 two times to construct a sequence of sets (F'_n) and a sequence of families of intervals $(\mathcal{J}'_n)$ such that

$$\mathcal{J}'_n = \mathcal{J}'_{1,n} \cup \mathcal{J}'_{2,n}, \tag{6}$$

$$F'_n = F_n \cup \bigcup\nolimits_{k<n} \Big(F'_k \cup \bigcup\nolimits_{I \in \mathcal{J}'_k} \operatorname{bd} I\Big) \tag{7}$$

and for $j \in \{1, 2\}$,

$$\mathcal{J}'_{j,n} \subset \mathcal{I}_j, \tag{8}$$

$$\text{for each } I \in \mathcal{I}_j, \text{ if } F'_n \cap \operatorname{bd} I \neq \emptyset, \text{ then } I \in \mathcal{J}'_{j,n}, \tag{9}$$

$$\begin{gathered}\text{for each } c \in F'_n, \text{ if } c \text{ is a right-hand (left-hand) limit point of } C,\\ \text{then } c \text{ is a right-hand (left-hand) limit point of the union } \bigcup \mathcal{J}'_{j,n},\end{gathered} \tag{10}$$

$$\operatorname{cl} \bigcup \mathcal{J}'_{j,n} \subset F'_n \cup \bigcup_{J \in \mathcal{J}'_{j,n}} \operatorname{cl} J. \tag{11}$$

Observe that by (11), for each $k < n$, the set $F'_k \cup \bigcup_{I \in \mathcal{J}'_k} \operatorname{bd} I$ is closed. So, by (7), the set F'_n is also closed and $F'_n \subset C \setminus A$. Fix an interval $I \in \mathcal{I}$. Using Lemma 3.4, we construct a strong Świątkowski function $f_I[I] = f_I\big[I \cap \mathcal{C}(f_I)\big] = (0, 1]$, $f_I(\inf I) = f_I(\sup I) = 2^{-1}$, $\operatorname{bd} I \subset \mathcal{C}(f_I)$, and $\operatorname{card}\big(I \setminus \mathcal{C}(f_I)\big) = 1$. Put

$$n_I = \min\left\{n \in \mathbb{N} : I \in \mathcal{J}'_n\right\},$$

and observe that by (9), $\bigcup_{n \in \mathbb{N}} \bigcup \mathcal{J}'_n = [0, 1] \setminus C$, whence n_I is well defined.

Now, define the function $f\colon \mathbb{R} \to [-2^{-1}, 2^{-1}]$ by the formula:

$$f(x) = \begin{cases} (-1)^j 2^{-n_I} f_I(x) & \text{if } x \in \operatorname{cl} I \text{ and } I \in \mathcal{I}_j,\ j \in \{1, 2\},\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, f is bounded and $A \subset [f = 0]$. First, we will show that $A \subset \mathcal{C}(f)$.

Take an $x_0 \in A$ and let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $2^{-n_0} < \varepsilon$ and put $\delta = \operatorname{dist}\big(\operatorname{cl} \bigcup \mathcal{J}'_{n_0}, x_0\big)$. Since by (11), (7), (4), and (5),

$$\begin{aligned} A \cap \operatorname{cl} \bigcup \mathcal{J}'_{j,n_0} &\subset \big(A \cap F'_{n_0}\big) \cup \Big(A \cap \bigcup_{J \in \mathcal{J}'_{j,n_0}} \operatorname{cl} J\Big)\\ &\subset \Big(A \cap \bigcup_{n \le n_0} F_n\Big) \cup \Big(\big(C \setminus \bigcup_{I \in \mathcal{I}} \operatorname{bd} I\big) \cap \bigcup_{J \in \mathcal{I}} \operatorname{cl} J\Big) = \emptyset,\end{aligned}$$

we have $x_0 \notin \operatorname{cl} \bigcup \mathcal{J}'_{j,n_0}$ and $\delta > 0$.

Observe that by (10), $F'_{n_0} \subset \operatorname{cl} \bigcup \mathcal{J}'_{n_0}$. If $|x - x_0| < \delta$, then $x \notin \operatorname{cl} \bigcup \mathcal{J}'_{n_0}$, whence

$$|f(x) - f(x_0)| = |f(x)| \le 2^{-n_0} < \varepsilon.$$

So, $x_0 \in \mathcal{C}(f)$.

Now, we will prove that

$$\forall_{n \in \mathbb{N}}\ \forall_{\delta > 0}\ \Big(x \in F'_n \setminus \{\sup I : I \in \mathcal{I}\} \Rightarrow f\big[(x - \delta, x) \cap \mathcal{C}(f)\big] \supset \big[-2^{-n}, 2^{-n}\big]\Big). \tag{12}$$

Let $n \in \mathbb{N}$, $\delta > 0$ and $x \in F'_n \setminus \{\sup I : I \in \mathcal{I}\}$. Then for $j \in \{1, 2\}$, by (10), there is $I_j \in \mathcal{J}'_{j,n}$ with $I_j \subset (x - \delta, x)$. Notice that $\max\big\{n_{I_1}, n_{I_2}\big\} \le n$. So,

$$f\big[(x - \delta, x) \cap \mathcal{C}(f)\big] \supset f\big[I_1 \cap \mathcal{C}(f)\big] \cup f\big[I_2 \cap \mathcal{C}(f)\big] \supset [-2^{-n}, 2^{-n}] \setminus \{0\}.$$

Since $x \notin \{\sup I : I \in \mathcal{I}\}$, we have $(x - \delta, x) \cap A \neq \emptyset$ and finally

$$f\big[(x - \delta, x) \cap \mathcal{C}(f)\big] \supset [-2^{-n}, 2^{-n}].$$

Similarly, we can prove that

$$\underset{n\in\mathbb{N}}{\forall}\ \underset{\delta>0}{\forall}\ \Big(x \in F'_n \setminus \{\inf I : I \in \mathcal{I}\} \Rightarrow f\big[(x, x+\delta) \cap \mathcal{C}(f)\big] \supset \big[-2^{-n}, 2^{-n}\big]\Big).$$

Now, we will show that $f \in \acute{\mathcal{S}}_s$. Let $c, d \in \mathbb{R}$, $c < d$, and $y \in \mathrm{I}\big(f(c), f(d)\big)$. Without loss of generality, we can assume that $c, d \in [0, 1]$ and $f(c) < f(d)$. If $c, d \in \operatorname{cl} I$ for some $I \in \mathcal{I}$, then since $f_I \in \acute{\mathcal{S}}_s$, there is $x_0 \in (c, d) \cap \mathcal{C}(f)$ with $f(x_0) = y$. So, assume that the opposite case holds.

Assume that $y \geq 0$. (The case $y < 0$ is analogous.) Then $f(d) > 0$, whence $d \notin A$. We consider two cases.

Case 1. $d \notin \bigcup_{n\in\mathbb{N}} F'_n$ or $d \in \{\sup I : I \in \mathcal{I}\}$.

Then there is $I \in \mathcal{I}$ such that $d \in \operatorname{cl} I$ and $c \notin \operatorname{cl} I$. If $y \in \mathrm{I}\big(f(\inf I), f(d)\big)$, then, since $f_I \in \acute{\mathcal{S}}_s$, there is $x_0 \in (\inf I, d) \cap \mathcal{C}(f) \subset (c, d) \cap \mathcal{C}(f)$ with $f(x_0) = y$.

Now, let $y \in \big[0, f(\inf I)\big]$. By (7), since $I \in \mathcal{J}'_{n_I}$, we have $\inf I \in F'_{n_I+1}$. By (12),

$$y \in \big[0, f(\inf I)\big] \subset \big[-2^{-n_I-1}, 2^{-n_I-1}\big] \subset f\big[(c, \inf I) \cap \mathcal{C}(f)\big].$$

So, there is $x_0 \in (c, \inf I) \cap \mathcal{C}(f) \subset (c, d) \cap \mathcal{C}(f)$ with $f(x_0) = y$.

Case 2. $d \in \bigcup_{n\in\mathbb{N}} F'_n \setminus \{\sup I : I \in \mathcal{I}\}$.

Then, $d \in F'_n \setminus F'_{n-1}$ for some $n \in \mathbb{N}$. By (12),

$$y \in \big[0, f(d)\big) \subset [-2^{-n}, 2^{-n}] \subset f\big[(c, d) \cap \mathcal{C}(f)\big].$$

Consequently, there is $x_0 \in (c, d) \cap \mathcal{C}(f)$ with $f(x_0) = y$. So, $f \in \acute{\mathcal{S}}_s$.

Observe now that $\mathbb{R} \setminus \mathcal{C}(f) = (C \setminus A) \cup \bigcup_{I\in\mathcal{I}} \big(I \setminus \mathcal{C}(f_I)\big)$. However, $\operatorname{card}\big(I \setminus \mathcal{C}(f_I)\big) = 1$ for each $I \in \mathcal{I}$. So, the set $\mathbb{R} \setminus \mathcal{C}(f)$ is nowhere dense. Hence and since each strong Świątkowski function is quasi-continuous, the function f is internally quasi-continuous.

To complete the proof, suppose that there is $k \in \mathbb{N}$ and there are functions $g_1, \ldots, g_k \in \acute{\mathcal{S}}_{si}$ such that $f = g_1 \ldots g_k$ on $\mathbb{R}$. Then, $\operatorname{sgn} \circ f = \operatorname{sgn} \circ (g_1 \ldots g_k)$. Let $(a, b) \subset [0, 1]$ be an interval in which the function f changes its sign. Then, at least one of the functions $g_1, \ldots, g_k$, say g_1, has the same property. Since $g_1 \in \acute{\mathcal{S}}_{si}$, there is $x_0 \in (a, b) \cap \operatorname{int} \mathcal{C}(g_1)$ such that $g_1(x_0) = 0$. Note that $[f = 0] \cap [0, 1] = A$, whence $g(x_1) \neq 0$ for some $x_1 \in (a, b) \cap \operatorname{int} \mathcal{C}(g_1) \cap (C \setminus A)$. Since g_1 is continuous at x_1, there is an open interval (a_1, b_1) such that $g_1\big[(a_1, b_1)\big] \cap [g_1 = 0] = \emptyset$ and f changes its sign in (a_1, b_1). Hence at least one of the functions $g_2, \ldots, g_k$, say g_2, changes its sign in (a_1, b_1), too. Observe that $g_2 \in \acute{\mathcal{S}}_{si}$. Proceeding as above, after k steps, we obtain that there is an open interval J in which the function f changes its sign and $g_i[J] \cap [g_i = 0] = \emptyset$ for each $i \in \{1, \ldots, k\}$, a contradiction.

Consequently, the function f cannot be written as the finite product of internally strong Świątkowski functions. □

Finally, we would like to present the problem.

Problem 4.4. Characterize the products of internally strong Świątkowski functions.

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