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NOTES ON MODIFICATIONS OF A wQN-SPACE

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ABSTRACT. We continue to investigate the generalizations of the notion of wQN-space introduced by [L. Bukovský—J. Šupina: *Modifications of sequence selection principles,* Topology Appl. **160** (2013), 2356–2370] and by [J. Šupina: *On Ohta–Sakai's properties of a topological space* (to appear)]. We present covering characterizations, slightly different formulations, and some new relations among them.

1. Introduction

All topological spaces are assumed to be infinite and Hausdorff. By a function we mean a real-valued function, and symbol 0 denotes both the number and the function with constant zero value (defined on appropriate topological space). Basic set-theoretical and topological terminology follows mainly [2] and [8]. Preliminary definitions can be found in [1], [3], [12] or in the introduction here.

We continue with investigation of properties introduced in [6] and [16]. Their definitions are generalizations of the definition of wQN-space. Definitions of properties of [16] were motivated by H. Ohta and M. Sakai [12]. To simplify the notation, we need to denote the following preordering on ${}^{\omega}({}^{X}\mathbb{R})$. Let $\langle f_n; n \in \omega \rangle$, $\langle g_n; n \in \omega \rangle$ be two sequences of real-valued functions on X. Then we write that $\langle f_n; n \in \omega \rangle \leq^* \langle g_n; n \in \omega \rangle$ if for any $x \in X$ the sequence $\{g_n(x)\}_{n=0}^{\infty}$ eventually dominates sequence $\{f_n(x)\}_{n=0}^{\infty}$, i.e.,

$$\langle f_n; n \in \omega \rangle \leq^* \langle g_n; n \in \omega \rangle \equiv (\forall x \in X)(\exists n_0)(\forall n \geq n_0) \qquad f_n(x) \leq g_n(x).$$

To make our results easy to formulate, we introduce the following schema which is more general than the schemas of [6] and [16]. Let X be a set, $\mathcal{F}, \mathcal{G}, \mathcal{H} \subseteq {}^{X}\mathbb{R}$ being families of functions containing the zero constant function, i.e., $0 \in \mathcal{F}, \mathcal{G}, \mathcal{H}$.

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We say that X has **a property $\mathbf{wED}^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}$**, if for any sequence $\langle f_m; m\in\omega\rangle$ of functions from $\mathcal{F}$ converging to a function $f\in\mathcal{H}$, there are sequences $\langle g_m; m\in\omega\rangle$ and $\langle h_m; m\in\omega\rangle$ of functions from $\mathcal{G}$ converging to f, and there is an increasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$ such that

for any $x\in X$, the sequence $\big\{g_m(x)\big\}_{m=0}^{\infty}$ eventually dominates $\big\{f_{n_m}(x)\big\}_{m=0}^{\infty}$, and the sequence $\big\{f_{n_m}(x)\big\}_{m=0}^{\infty}$ eventually dominates $\big\{h_m(x)\big\}_{m=0}^{\infty}$, i.e.,

$$\langle h_m; m\in\omega\rangle \le^* \langle f_{n_m}; m\in\omega\rangle \le^* \langle g_m; m\in\omega\rangle.$$

If $\mathcal{H}=\{0\}$, then we say that X has **a property $\mathbf{wED}_{\mathcal{F},\mathcal{G}}$**. We will use some assumptions about families of functions:

(a) $\{-f;\ f\in\mathcal{F}\}\subseteq\mathcal{F}$, (b) $\{|f|;\ f\in\mathcal{F}\}\subseteq\mathcal{F}$,

(c) $\big\{\min\{f,1\};\ f\in\mathcal{F}\big\}\subseteq\mathcal{F}$, (d) $\big\{\max\{f,0\};\ f\in\mathcal{F}\big\}\subseteq\mathcal{F}$.

For particular families $\mathcal{F},\mathcal{G},\mathcal{H}\subseteq{}^X\mathbb{R}$, we obtain some modifications of wQN-space considered earlier. If $\mathcal{F}\subseteq{}^X[0,1]$ and $\mathcal{G}$ satisfies (c), (d), then property $\mathrm{wED}_{\mathcal{F},\mathcal{G}}$ is equivalent to the property with the same name introduced in [16]. In fact, many pairs $\mathcal{F},\mathcal{G}$ of families of functions in $\mathrm{wED}_{\mathcal{F},\mathcal{G}}$ will satisfy the former condition, and therefore, property $\mathrm{wED}_{\mathcal{F},\mathcal{G}}$ of this paper often corresponds to property $\mathrm{wED}_{\mathcal{F},\mathcal{G}}$ of [16]. Let Const denote the family of all constant functions on the considered set. For property $\mathrm{wQN}_{\mathcal{F}}$ from [6], we have

$$\mathrm{wQN}_{\mathcal{F}}\equiv\mathrm{wED}_{\mathcal{F},\mathrm{Const}}.$$

$\mathrm{C}_p(X)$ denotes the family of all continuous functions from X to $\mathbb{R}$. L. Bukovský, I. Recław and M. Repický introduced wQN-space in [4] and $\mathrm{w}\overline{\mathcal{F}}\mathrm{QN}$-space in [5],

$$\mathrm{wQN}=\mathrm{wED}_{\mathrm{C}_p(X),\mathrm{Const}},\quad \mathrm{w}\overline{\mathcal{F}}\mathrm{QN}=\mathrm{wED}^{{}^X\mathbb{R}}_{\mathcal{F},\mathrm{Const}}.$$

L. Bukovský [1] introduced wQN^*-space and wQN_*-space.[1] $\mathcal{U}$ and $\mathcal{L}$ are families of all upper and lower semicontinuous functions on X, respectively. For a family $\mathcal{F}\subseteq{}^X\mathbb{R}$ we denote $\widetilde{\mathcal{F}}=\mathcal{F}\cap{}^X[0,1]$. Then,

$$\mathrm{wQN}^*\equiv\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{Const}},\qquad \mathrm{wQN}_*\equiv\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{Const}}.$$

We begin with simple properties of $\mathrm{wED}^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}$. If X is any set and $\mathcal{G}\subseteq\mathcal{F}\subseteq{}^X\mathbb{R}$, $\mathcal{H}\subseteq{}^X\mathbb{R}$, then property $\mathrm{wED}^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}$ is trivially satisfied in X. If $\mathcal{F}_1\subseteq\mathcal{F}_2$, $\mathcal{G}_1\subseteq\mathcal{G}_2$ and $\mathcal{H}_1\subseteq\mathcal{H}_2$, then

$$\mathrm{wED}^{\mathcal{H}_2}_{\mathcal{F}_2,\mathcal{G}_1}\to\mathrm{wED}^{\mathcal{H}_1}_{\mathcal{F}_1,\mathcal{G}_2}.$$

By $\mathcal{B}$ we denote the family of all Borel functions on X. If X is a perfectly normal space, $\mathcal{F},\mathcal{H}\subseteq\mathcal{B}$ and Const $\subseteq\mathcal{G}$ then, in accordance with Tsaban–Zdomskyy Theorem [17], we have

$$\mathrm{QN}\to\mathrm{wED}^{\mathcal{B}}_{\mathcal{B},\mathrm{Const}}\to\mathrm{wED}^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}.$$

[1]For more information about values of functions in their definitions see [15].

If a family $\mathcal{F}$ satisfies (b) and $\mathcal{G}$ satisfies (a), (d), then

$$\mathrm{wED}_{\mathcal{F},\mathcal{G}} \equiv \mathrm{wED}_{\mathcal{F}\cap {}^X[0,\infty),\mathcal{G}\cap {}^X[0,\infty)}.$$

If a family $\mathcal{F}$ satisfies (b), (c) and $\mathcal{G}$ satisfies (a), (c), (d), then

$$\mathrm{wED}_{\mathcal{F},\mathcal{G}} \equiv \mathrm{wED}_{\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}}.$$

Finally, if $\mathcal{F} \subseteq {}^X[0,1]$ and $\mathcal{G}$ satisfies (c), (d), then

$$\mathrm{wED}_{\mathcal{F},\mathcal{G}} \equiv \mathrm{wED}_{\mathcal{F},\widetilde{\mathcal{G}}}.$$

We will use these relations without any comment.

2. Coverings

There are already known covering characterizations of some properties $\mathrm{wED}_{\mathcal{F},\mathcal{G}}$. Such a characterization for wQN-space was found in [3]. By [1], [3], [13], property $\alpha_1(\Gamma,\Gamma)$ in the sense of Lj. D. R. Kočinac [10] is a characterization of $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{Const}}$. In [16], we found covering characterizations of $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$ and $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{C}_p(X)}$.

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a cover of a topological space X if $X = \bigcup \mathcal{A}$ and $X \notin \mathcal{A}$.[2] An infinite cover $\mathcal{A}$ is a γ-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{A}$. Γ denotes the family of all open γ-covers of X. L. Bukovský [1] showed that any $\mathrm{S}_1(\Gamma,\Gamma)$-space has $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{Const}}$, and M. Sakai [14] proved that if a topological space X has $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{Const}}$, then X is an $\mathrm{S}_1(\Gamma,\Gamma)$-space. We prove generalizations of these results using ideas of their proofs.

Let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$. $\mathcal{A}^c$ denotes the family $\{X \setminus A;\ A \in \mathcal{A}\}$. By $\Gamma_{\mathcal{A}}$ we mean the family of all γ-covers of X by sets from $\mathcal{A}$. A function f on X is called lower, upper $\mathcal{A}$-measurable if $f^{-1}((r,\infty)) \in \mathcal{A}$, $f^{-1}((-\infty,r)) \in \mathcal{A}$ for any $r \in \mathbb{R}$, respectively. The family of all lower, upper $\mathcal{A}$-measurable functions on X with values in $[0,1]$ is denoted by $\mathcal{L}(\mathcal{A})$, $\mathcal{U}(\mathcal{A})$, respectively.

Theorem 2.1. *Let X be a topological space, $\mathcal{A} \subseteq \mathcal{P}(X)$ being closed under finite unions and intersections.*[3] *Then the following are equivalent.*

(1) *X is an $\mathrm{S}_1(\Gamma_{\mathcal{A}},\Gamma_{\mathcal{A}})$-space.*

(2) *X has $\mathrm{wED}_{\mathcal{U}(\mathcal{A}),\mathrm{Const}}$.*

(3) *X has $\mathrm{wED}_{\mathcal{L}(\mathcal{A}^c),\mathrm{Const}}$.*

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[2] Similarly to [3], the empty set $\emptyset$ can be an element of a cover. If we consider the enumeration of a cover, then we always assume that the set is repeated only finitely many times in the enumeration, i.e., the enumeration is adequate in the sense of [2].

[3] Thus $\emptyset, X \in \mathcal{A}$.

P r o o f. (1) → (2) Let $\langle f_m; m \in \omega\rangle$ be a sequence of upper $\mathcal{A}$-measurable functions on X with values in $[0,1]$ such that $f_m \to 0$. We define the sets $A_{n,m}$, n, $m \in \omega$ by

$$A_{n,m} = \{x \in X;\ f_m(x) < 2^{-n}\}.$$

If there are increasing sequences $\{n_k\}_{k=0}^{\infty}, \{m_k\}_{k=0}^{\infty}$ such that $A_{n_k,m_k} = X$ for all $k \in \omega$, then $\langle f_{m_k}; k \in \omega\rangle$ converges uniformly. Thus, we may assume that $\langle\{A_{n,m};\ m \in \omega\}; n \in \omega\rangle$ is a sequence of γ-covers by sets from $\mathcal{A}$. By $\mathrm{S}_1(\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{A}})$ there is an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $\{A_{n,m_n};\ n \in \omega\}$ is a γ-cover enumerated bijectively. Then, $\langle f_{m_n}; n \in \omega\rangle$ converges quasi-normally to zero with the control $\{2^{-n}\}_{n=0}^{\infty}$.

(2) → (1) Let $\langle\{A_{n,m}; m \in \omega\}; n \in \omega\rangle$ be a sequence of γ-covers by sets from $\mathcal{A}$. Since $\mathcal{A}$ is closed under finite intersections, we may assume that $A_{n+1,m} \subseteq A_{n,m}$ for any $n, m \in \omega$. We define the upper $\mathcal{A}$-measurable functions $f_m, m \in \omega$ by

$$f_m(x) = \begin{cases} 1, & x \in X \setminus A_{0,m}, \\ \frac{1}{2^{n+1}}, & x \in A_{n,m} \setminus A_{n+1,m}, n \in \omega, \\ 0, & \text{otherwise.} \end{cases}$$

Sequence $\langle f_m; m \in \omega\rangle$ converges to zero. We have $f_m(x) < \frac{1}{2^n}$ if and only if $x \in A_{n,m}$. By $\mathrm{wED}_{\mathcal{U}(\mathcal{A}),\mathrm{Const}}$ there is an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $\langle f_{m_n}; n \in \omega\rangle$ converges quasi-normally to zero with the control $\{2^{-n}\}_{n=0}^{\infty}$. Thus, $\{A_{n,m_n}; n \in \omega\}$ is a γ-cover (by respective reselection we may assume that the enumeration is bijective).

The equivalence of (1) and (3) can be proved similarly. □

Let us denote by $\mathbf{F}$ the family of all closed subsets of X. Then, we have Corollary 2.2. Note that according to B. T s a b a n and L. Z d o m s k y y [17] and L. B u k o v s k ý [1] the result is known for perfectly normal space.

Corollary 2.2. *A topological space X has $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{Const}}$ if and only if X is an $\mathrm{S}_1(\Gamma_{\mathbf{F}}, \Gamma_{\mathbf{F}})$-space.*

Note that the paper [6] contains characterizations of properties of Theorem 2.1 by so-called sequence selection properties.

3. Various families of functions

The paper [6] contains relations among properties $\mathrm{wED}_{\mathcal{F},\mathrm{Const}}$ for various interesting families $\mathcal{F}$, e.g., if $\widetilde{\mathcal{L}} \subseteq \mathcal{F} \subseteq \mathcal{B}$, then we have

$$\mathrm{QN} \equiv \mathrm{wED}_{\mathcal{F},\mathrm{Const}}, \qquad \mathrm{S}_1(\Gamma, \Gamma) \equiv \mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{Const}}.$$

We accomplish the similar investigations of properties $\mathrm{wED}_{\widetilde{\mathcal{F}},\mathrm{C}_p(X)}$ and $\mathrm{wED}_{\widetilde{\mathcal{F}},\mathcal{U}}$. For interesting families, these properties can be divided into two groups of equivalent properties. By $\mathcal{B}_1$ we denote the family of all pointwise limits of continuous functions on X. If X is a perfectly normal space, then $\mathcal{B}_1$ is the family of all F_σ-measurable functions on X.

Theorem 3.1. *Let X be a perfectly normal space.*

(1) *X has $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$ if and only if X has $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathcal{U}}$.*

(2) *If $\widetilde{\mathcal{B}}_1 \subseteq \mathcal{F} \subseteq \widetilde{\mathcal{B}}$, then*

$$\mathrm{wED}_{\mathcal{B},\mathrm{C}_p(X)} \equiv \mathrm{wED}_{\mathcal{F},\mathrm{C}_p(X)} \equiv \mathrm{wED}_{\mathcal{F},\mathcal{U}}.$$

Proof.

(1) Let us assume that X has $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathcal{U}}$. By [16, Corollary 5.2] we have that X has $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{C}_p(X)}$. One can easily see that if X has $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathcal{U}}$ and $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{C}_p(X)}$, then X has $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$ [16, Lemma 2.2].

(2) Similarly to (1), one can show that if X has $\mathrm{wED}_{\mathcal{F},\mathcal{U}}$, then X has $\mathrm{wED}_{\mathcal{F},\mathrm{C}_p(X)}$. If X has $\mathrm{wED}_{\mathcal{F},\mathrm{C}_p(X)}$, then X is a σ-set according to [16, Corollary 5.2]. Therefore, any Borel function is F_σ-measurable and belongs to family $\mathcal{F}$. Hence, X has $\mathrm{wED}_{\widetilde{\mathcal{B}},\mathrm{C}_p(X)}$. □

However, following in [16, Theorem 1.2], we have

Proposition 3.2. *Let X be a perfectly normal space with Hurewicz property*[4]*, $\widetilde{\mathcal{L}} \subseteq \mathcal{F} \subseteq \widetilde{\mathcal{B}}$. Then*

$$\mathrm{QN} \equiv \mathrm{wED}_{\mathcal{F},\mathrm{C}_p(X)} \equiv \mathrm{wED}_{\mathcal{F},\mathcal{U}}.$$

In [16], we showed that, for any perfectly normal space, the property $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$ is hereditary. The same is true for $\mathrm{wED}_{\mathcal{B},\mathrm{C}_p(X)}$.

Lemma 3.3. *Let X be a topological space, and let $\mathcal{G} \in \{\mathrm{C}_p(X), \mathcal{U}, \mathcal{L}, \mathcal{B}_1\}$. Then, any Borel subset of X with property $\mathrm{wED}_{\widetilde{\mathcal{B}},\mathcal{G}}$ has $\mathrm{wED}_{\widetilde{\mathcal{B}},\mathcal{G}}$ as well.*

Proof. Let $B \subseteq X$. For a sequence $\langle f_n; n \in \omega\rangle$ of Borel functions on B, one can define a sequence $\langle h_n; n \in \omega\rangle$ of Borel functions on X by $h_n(x) = f_n(x)$ for $x \in B$ and $h_n(x) = 0$ for $x \in X \setminus B$. □

If X is perfectly normal space, then by Kuratowski Extension Theorem for Borel measurable functions (see, e.g., [11, §31, VI, Théorème] or [7, Theorem 2.4]) we obtain

Proposition 3.4. *For any perfectly normal space X, the property $\mathrm{wED}_{\mathcal{B},\mathrm{C}_p(X)}$ is hereditary.*

[4] We say that a topological space X possesses Hurewicz property if for any sequence $\langle \mathcal{U}_n; n \in \omega\rangle$ of countable open covers not containing a finite subcover, there exist finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n, n \in \omega$ such that $\{\bigcup \mathcal{V}_n; n \in \omega\}$ is a γ-cover. Note that this definition corresponds to property E^{**}_ω rather than to original property E^{**} by W. Hurewicz [9], see, e.g., [3].

4. Different formulations

In [15], we showed that the range of functions in definitions of properties $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{Const}}$ and $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{Const}}$ is essential, e.g., if X is a normal space, then

$$\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{Const}} \equiv \mathrm{wED}_{\mathcal{L},\mathrm{Const}} \equiv \mathrm{wED}_{\mathcal{U},\mathrm{Const}}.$$

In this section, we present similar, but not the same, results on main objects of investigation in [16], i.e., on properties $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{C}_p(X)}$ and $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$.

For a perfectly normal space X, we show that the limit function in the definition of property $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{C}_p(X)}$ can be any F_σ-measurable function, and the range of functions can be $\mathbb{R}$.

Proposition 4.1. *Let X be a perfectly normal space. The following are equivalent.*

(1) *X possesses $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{C}_p(X)}$.*

(2) *For any sequence $\langle f_m; m \in \omega\rangle$ of upper semicontinuous functions on X with values in $\mathbb{R}$ converging to F_σ-measurable function f, there exists a sequence $\langle g_m; m \in \omega\rangle$ of continuous functions converging to f, and there is an increasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$ such that*

$$\langle f_{n_m}; m \in \omega\rangle \le^* \langle g_m; m \in \omega\rangle.$$

Proof. Since there is an increasing homeomorphism between $(0,1)$ and $\mathbb{R}$, we will restrict our proof to functions with $(0,1)$ range. Thus, let $\langle f_m; m \in \omega\rangle$ be a sequence of upper semicontinuous functions on X with values in $(0,1)$ converging to an F_σ-measurable function f, and let $\langle h_n; n \in \omega\rangle$ be continuous functions such that $h_n \to f$. Then, $\max\{f_n - h_n; 0\} \to 0$. In accordance with $\mathrm{wED}_{\widetilde{\mathcal{U}},\mathrm{C}_p(X)}$, there exist a sequence $\langle g'_m; m \in \omega\rangle$ of continuous functions converging to zero and an increasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$ such that for any $x \in X$ there is $m_0 \in \omega$ with $\max\big\{f_{n_m}(x) - h_{n_m}(x); 0\big\} \le g'_m(x)$ for any $m \ge m_0$. Then, $f_{n_m}(x) \le g'_m(x) + h_{n_m}(x)$ for any $m \ge m_0$.

Finally, we define a sequence $\langle g_m; m \in \omega\rangle$ by

$$g_m = \min\big\{1 - 2^{-m-1}; \max\{2^{-m-1}; g'_m + h_{n_m}\}\big\}$$

and we obtain functions with ranges in $(0,1)$ such that

$$\langle f_{n_m}; m \in \omega\rangle \le^* \langle g_m; m \in \omega\rangle. \qquad \square$$

Although the condition (2) of Proposition 4.1 resembles property $\mathrm{wED}^{\mathcal{B}_1}_{\mathcal{U},\mathrm{C}_p(X)}$, it cannot be replaced with this property, as Theorem 4.2 shows.

Let property $\mathrm{ED}^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}$ be defined as property $\mathrm{wED}^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}$, except for the condition asking $\langle h_m; m \in \omega\rangle \le^* \langle f_m; m \in \omega\rangle \le^* \langle g_m; m \in \omega\rangle$. In [16], we showed that a topological space X has $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$ if and only if X has $\mathrm{ED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$.

Theorem 4.2. *Let X be a perfectly normal space. Then*

$$\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)} \equiv \mathrm{wED}^{\mathcal{B}}_{\mathcal{L},\mathrm{C}_p(X)} \equiv \mathrm{wED}^{\mathcal{B}}_{\mathcal{U},\mathrm{C}_p(X)}.$$

Proof. If X has $\mathrm{wED}^{\mathcal{B}}_{\mathcal{L},\mathrm{C}_p(X)}$, then X has $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$ as well. Since there is an increasing homeomorphism between $(0,1)$ and $\mathbb{R}$, to prove the reversed implication, we will restrict to the functions with $(0,1)$ range. Let X possess $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$, and let $\langle f_m; m\in\omega\rangle$ be a sequence of upper semicontinuous functions on X with values in $(0,1)$ converging to $f\in\mathcal{B}$. By [16, Corollary 5.2], the function f is $\mathbf{\Delta}^0_2$-measurable, thus $f \in \mathcal{B}_1$. Moreover, X has USC by the same corollary and [12, Corollary 2.4]. Thus, by [16, Theorem 6.1], there is a sequence $\langle \varphi_m; m \in \omega\rangle$ of continuous functions converging to f such that $f_m \le \varphi_m$. Taking the lower semicontinuous functions $\varphi_m - f_m$, we have $\varphi_m - f_m \to 0$. Due to $\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)}$, there is a sequence $\langle \psi_m; m\in\omega\rangle$ of continuous functions converging to zero such that $\langle \varphi_m - f_m; m\in\omega\rangle \le^* \langle \psi_m; m\in\omega\rangle$. The sequences $\langle h_m; m\in\omega\rangle$ and $\langle g_m; m\in\omega\rangle$ will be defined by

$$h_m = \max\{2^{-m-1}; \varphi_m - \psi_m\},\ g_m = \min\{1-2^{-m-1}; \varphi_m\}.$$

To prove the equivalence $\mathrm{wED}^{\mathcal{B}}_{\mathcal{L},\mathrm{C}_p(X)} \equiv \mathrm{wED}^{\mathcal{B}}_{\mathcal{U},\mathrm{C}_p(X)}$ for a sequence $\langle f_m; m\in\omega\rangle$ of upper/lower semicontinuous functions converging to a Borel function f, we can consider the lower/upper semicontinuous functions $-f_m, m\in\omega$ and the Borel function $-f$. □

Let us remark that

$$\mathrm{wED}^{\mathcal{B}}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)} \equiv \mathrm{wED}^{\mathcal{B}}_{\widetilde{\mathcal{U}},\mathrm{C}_p(X)}.$$

To prove this, one can use functions $1-f_m$, $m\in\omega$ and $1-f$ instead of $-f_m, m\in\omega$ and $-f$ in the second part of the proof of Theorem 4.2.

In fact, notice that slightly more is proved in Theorem 4.2, i.e.,

$$\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)} \equiv \mathrm{ED}^{\mathcal{B}}_{\mathcal{L},\mathrm{C}_p(X)} \equiv \mathrm{ED}^{\mathcal{B}}_{\mathcal{U},\mathrm{C}_p(X)}.$$

Consequently, for any $\{0\}\subseteq\mathcal{F}\subseteq\mathcal{B}$, we obtain

$$\mathrm{wED}_{\widetilde{\mathcal{L}},\mathrm{C}_p(X)} \equiv \mathrm{wED}^{\mathcal{F}}_{\mathcal{L},\mathrm{C}_p(X)} \equiv \mathrm{wED}^{\mathcal{F}}_{\mathcal{U},\mathrm{C}_p(X)} \equiv \mathrm{ED}^{\mathcal{F}}_{\mathcal{L},\mathrm{C}_p(X)} \equiv \mathrm{ED}^{\mathcal{F}}_{\mathcal{U},\mathrm{C}_p(X)}.$$

Finally, note that, for perfectly normal space X and $\widetilde{\mathcal{B}}_1 \subseteq \mathcal{F} \subseteq \mathcal{B}$, we have

$$\mathrm{wED}_{\mathcal{B},\mathrm{C}_p(X)} \equiv \mathrm{wED}^{\mathcal{B}}_{\mathcal{F},\mathrm{C}_p(X)}.$$

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