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SEPARATE AND JOINT PROPERTIES OF SOME ANALOGUES OF POINTWISE DISCONTINUITY

Vasyl' Nesterenko

ABSTRACT. We study separate and joint properties of pointwise discontinuity, simple continuity and mild continuity of functions of two variables. In particular, it is shown that for a Baire space X, a Baire space Y which has a countable pseudo-base and for a metric space Z, a function $f\colon X\times Y\to Z$ is pointwise discontinuous if and only if f satisfies (α,β)-condition and condition (C), and $M=\big\{x\in X:\overline{C(f^x)}=Y\big\}$ is a residual subset of X. In addition, a characterization of simple continuity for mappings of one and two variables is given.

1. Introduction

A mapping $f\colon X\to Y$ is called *pointwise discontinuous* if the set $C(f)$ of points of continuity of f is dense in X. There are many analogues of pointwise discontinuity (cliquish [1] quasi-continuity [2], simple continuity [3], mild continuity [4], etc.) that are investigated in the works of many mathematicians.

Among many problems for mappings of several variables, problems related to their separate and joint properties occupy a special place. The investigation of relationships between separate and joint continuity, which started its history in classical works of Baire and Osgood, has spread out on various weakening of continuity. In particular, in [5], the relationship between different separate and joint properties of weakening of continuity (quasi-continuity, almost continuity, somewhat continuity, etc.) is investigated. In [6]–[8], separate properties of functions $f\colon X\times Y\to\mathbb{R}$, for which sections $f_y=f(\cdot,y), y\in Y$ are monotone, lower and upper quasi-continuous or have closed graph, have been studied.

For the mapping $f\colon X\times Y\to Z$ of two variables, it is an interesting problem to find minimum conditions for sections

$$f^x=f(x,\cdot),\ x\in X\quad\text{and}\quad f_y=f(\cdot,y),\ y\in Y$$

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at which the mapping f has a property of some weakened continuity. In [2], S. Kempisty proved that a separately quasi-continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ is jointly quasi-continuous. The Kempisty's result was generalized by many mathematicians. In [9], the following characterization for quasi-continuity mappings of two variables was obtained: if X is a Baire space, Y a space which has a countable base, and Z a separable metrizable space, then a mapping $f\colon X \times Y \to Z$ is quasi-continuous if and only if f is horizontally quasi-continuous, and $\{x \in X\colon f^x \text{ is quasi-continuous}\}$ is a residual subset of X.

In contrast to quasi-continuity, cliquishness or pointwise discontinuity relative to each variable does not guarantee jointly cliquishness or more pointwise discontinuity. For cliquishness, L. Fudali did the first attempt to characterize this concept for functions of two variables. In [10], he showed that if X is a Baire space, Y a space which has a countable pseudo-base, Z a metric space, and if a function $f\colon X \times Y \to Z$ is such that sections f_y are quasi-continuous for each $y \in Y$ and sections f^x are cliquish for each $x \in X$, then f is joint cliquish. In [11], the following characterization of cliquishness for functions of two variables is obtained: if X is a Baire space, Y a space which has a countable pseudo-base, Z a metric space, then the function $f\colon X \times Y \to Z$ is cliquish if and only if f satisfies (α, β)-condition and condition (C), and $M = \{x \in X\colon f^x \text{ is cliquish}\}$ is a residual subset of X.

In [12], for pointwise discontinuity, the following result was obtained: if X is a Baire space, Y a Baire space which has a countable pseudo-base, Z a metrizable space, a mapping $fX \times Y \to Z$ is horizontally quasi-continuous, and f^x is pointwise discontinuous for each $x \in X$, then

$$A = \Big\{x \in X\colon \overline{\big\{y \in Y : (x,y) \in C(f)\big\}} = Y\Big\}$$

is a residual subset of X, and if $X \times Y$ is a Baire space, then f is pointwise discontinuous.

In this article, for functions of two variables, we obtain a characterization of pointwise discontinuity, we establish sufficient conditions for simple continuity and mild continuity as well as we obtain new characterizations of simple continuity for mappings of one and two variables.

2. Basic definitions and concepts

Let (Z, d) be a metric space. Recall that for a non-empty subset A of a set X and for a function $g\colon T \to Z$, the number $\omega_g(A) = \sup\big\{d(g(x'), g(x''))\colon x', x'' \in A\big\}$ is called the *oscillation of g on the set A*. If T is a topological space and $\mathcal{U}_t$ is a system of neighborhoods of t in T, then the number $\omega_g(t) = \inf\big\{\omega_g(U) : U \in \mathcal{U}_t\big\}$ is called the *oscillation of g at t*. Obviously, $t \in C(g)$ if and only if $\omega_g(t) = 0$.

For a subset A of a topological space, let $intA$ and $\overline{A}$ denote the interior and the closure of A, respectively.

Let Y be a metric space. A function $f\colon X \to Y$ is said to be *cliquish at a point* $x \in X$ [1] if for any $\varepsilon > 0$ and a neighborhood U of $x \in X$ there exists a nonempty open subset G of X such that $G \subseteq U$ and $\omega_f(G) < \varepsilon$, and *cliquish*, if it is such at any point. Obviously, each pointwise discontinuous function with values in a metric space is cliquish. It is well-known ([1]) that if X is a Baire space, then each cliquish function $f\colon X \to Y$ is pointwise discontinuous.

Let Y be now a topological space. A map $f\colon X \to Y$ is said to be *quasi--continuous at* $x \in X$ [13] if for each neighborhood V of $y = f(x)$ in Y and each neighborhood U of x in X, there is a nonempty open subset G of X such that $G \subseteq U$ and $f(G) \subseteq V$. A map $f\colon X \to Y$ is said to be *quasicontinuous* if it is such at each point of the space X. It is easy to see that every quasi-continuous mapping with values in a metric space is cliquish. The converse is not true.

A mapping $f\colon X \to Y$ is *simply continuous* [3] if, for each open subset V of Y, the set $f^{-1}(V)$ is the union of an open subset U of X and a nowhere dense subset N of X. The last condition is equivalent to the fact that the set $fr\big(f^{-1}(V)\big)$ is nowhere dense, i.e., $int\overline{fr\big(f^{-1}(V)\big)} = \varnothing$ where $fr(A)$ is a boundary of the set A.

Put $S_f = \big\{x \in X$ there is a base $\mathcal{A}$ of neighbourhoods of $f(x)$ such that for each $A \in \mathcal{A}$ and each neighbourhood U of $x, f^{-1}(A) \setminus intf^{-1}(A)$ is not dense in $U\big\}$. If the set S_f is dense in X, then f is called *mildly continuous* [4]. A function f is called *almost continuous (in the sense of Husain) at* $x \in X$ [14] if, for each neighborhood V of $y = f(x)$ in Y, there is a subset A of X such that $x \in int\overline{A}$ and $f(A) \subseteq V$. Let $Q(f)$ and $P(f)$ denote the sets of all points at which the mapping f is quasi-continuous and almost continuous, respectively. In [15, Theorem 2.3], it is shown that $S_f = \big(X \setminus P(f)\big) \cup Q(f)$. It is easy to see that a quasi-continuous mapping is mildly continuous. In [4, Lemma 2], it is shown that a cliquish function is mildly continuous.

3. Some properties of simple continuity

The following characterization of quasi-continuity is well-known (see [16, Proposition 3.1.1]).

Theorem 3.1. *Let X and Y be topological spaces. A mapping $f\colon X \to Y$ is quasicontinuous if and only if for any nonempty open subset U of X and for any subset A of X such that $U \subseteq \overline{A}$, we have $f(U) \subseteq \overline{f(A)}$.*

Simple continuity has a similar characterization.

Theorem 3.2. *Let X and Y be topological spaces. A mapping $f\colon X \to Y$ is simply continuous if and only if the following condition holds*

($\star$) *for any nonempty open subset U of X and any subset A of X with $U \subseteq \overline{A}$, there exists a nonempty open subset G of X such that*

$$G \subseteq U \quad \textit{and} \quad f(G) \subseteq \overline{f(A)}.$$

P r o o f. N e c e s s i t y. Let the mapping f be simply continuous but do not let it satisfy condition ($\star$). Then, there exist a nonempty open subset U of X and a set E dense in U such that for each nonempty open subset $G \subseteq U$ there is a point $x_G \in G$ such that $f(x_G) \notin \overline{f(E)}$. Consider

the open set $V = Y \setminus \overline{f(E)}$ and the set $A = \{x_G : G \text{ is an open subset of } U\}$.

It is clear that

$$\overline{A} \supseteq U \quad \text{and} \quad E \cap A = \varnothing.$$

Since the mapping f is simply continuous, $fr\big(f^{-1}(V)\big) = \overline{f^{-1}(V)} \cap \overline{X \setminus f^{-1}(V)}$ is a nowhere dense set. However,

$$f^{-1}(V) \supseteq A \quad \text{and therefore,} \quad \overline{f^{-1}(V)} \supseteq \overline{A} \supseteq U.$$

In addition,

$$X \setminus f^{-1}(V) \supseteq E \quad \text{and} \quad \overline{X \setminus f^{-1}(V)} \supseteq \overline{E} \supseteq U.$$

Hence,

$$fr\big(f^{-1}(V)\big) \supseteq U \neq \varnothing,$$

what is impossible because the set $fr\big(f^{-1}(V)\big)$ is nowhere dense. Thus, the assumption is not true. □

S u f f i c i e n c y. Let the mapping f satisfy condition ($\star$) but do not let it be simply continuous. Since f is not simply continuous, there is a nonempty open subset V of Y such that the set $fr\big(f^{-1}(V)\big)$ is dense in some nonempty open subset U of X. Then,

$$fr\big(f^{-1}(V)\big) = \overline{fr\big(f^{-1}(V)\big)} \supseteq U.$$

Since,

$$fr\big(f^{-1}(V)\big) = \overline{f^{-1}(V)} \cap \overline{X \setminus f^{-1}(V)}, \quad \overline{f^{-1}(V)} \supseteq U \quad \text{and} \quad \overline{X \setminus f^{-1}(V)} \supseteq U.$$

From condition ($\star$), it follows that there is a nonempty open subset G of X such that

$$G \subseteq U \quad \text{and} \quad f(G) \subseteq \overline{f\big(X \setminus f^{-1}(V)\big)}.$$

Then,

$$f(G) \subseteq \overline{f\big(X \setminus f^{-1}(V)\big)} \subseteq \overline{Y \setminus V} = Y \setminus V.$$

On the other hand,

$$f(G) \cap V \neq \varnothing, \quad \text{because} \quad \overline{f^{-1}(V)} \supseteq U \quad \textit{and} \quad G \subseteq U.$$

The obtained contradiction completes the proof of sufficiency. □

□

The next result follows easily from Theorem 3.1 and Theorem 3.2.

Corollary 3.2.1. *Let X and Y be topological spaces and $f\colon X \to Y$ a quasi--continuous mapping. Then, f is simply continuous.*

It is possible to get a more accurate result.

Theorem 3.3. *Let X and Y be topological spaces, $f\colon X \to Y$ a mapping, and $X \setminus Q(f)$ a nowhere dense subset of X. Then, f is simply continuous.*

Proof. Consider a nonempty open subset U of X and a subset E of X such that $U \subseteq \overline{E}$. Since $X \setminus Q(f)$ is a nowhere dense set, there is a nonempty open subset G of X such that $G \subseteq U \cap Q(f)$. Put $g = f|_G$. The mapping g is quasi-continuous at each point of G. Since E is dense in U, $\overline{E \cap G} \supseteq G$. From Theorem 3.1, it follows that

$$f(G) = g(G) \subseteq \overline{g(E \cap G)} = \overline{f(E \cap G)} \subseteq \overline{f(E)}.$$

Hence, f is simply continuous. □

Note that Theorem 3.3 follows immediately from [17, Lemma 1.1].

The converse is not true. It is shown in the following example from [18, Example 1].

Example 1. Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ be a set of rational numbers. Let the function $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x + \frac{1}{n}, & x = r_n, \\ x, & \text{otherwise.} \end{cases}$$

The function f is simply continuous, and $\mathbb{R} \setminus Q(f) = \mathbb{Q}$.

4. Jointly pointwise discontinuity

Let Z be a metric space. We say that a function $f\colon X \times Y \to Z$ *satisfies (α, β)-condition* if for any α, β with $0 < \alpha < \beta$, any nonempty open subsets U of X and V of Y and for any set $E \subseteq X$ dense in U with $\omega_f(E \times V) < \alpha$, there exist nonempty open subsets G of X and H of Y such that $G \subseteq U$, $H \subseteq V$ and $\omega_f(G \times H) < \beta$. A function $f\colon X \times Y \to Z$ *satisfies condition (C)* if for any $\varepsilon > 0$, any non-meager subset E of X and any nonempty open subset V of Y, there exist a somewhere dense subset E_1 of X and a mapping $g\colon E_1 \to V$ such that $E_1 \subseteq E$ and $\omega_f\bigl(Gr(g)\bigr) < \varepsilon$, where $Gr(g) = \bigl\{(x, g(x)) : x \in E_1\bigr\}$ is the graph of g.

Note that if Z is a separable metric space, then an arbitrary function $f\colon X \times Y \to Z$ satisfies condition (C). Also, notice that if a function $f\colon X \times Y \to Z$ is simply continuous, then it satisfies (α, β)-condition.

Put
$$C^x(f) = \{y \in Y : (x, y) \in C(f)\}.$$

Theorem 4.1. *Let X be a topological space, Y a topological space which has a countable pseudo-base, (Z, d) a metric space, $f : X \times Y \to Z$ a function such that (α, β)-condition and condition (C) hold, and let $M = \{x \in X : \overline{C(f^x)} = Y\}$ be a residual subset of X. Then, $A = \{x \in X : C^x(f)$ is a residual subset of $Y\}$ is a residual subset of X.*

P r o o f. Let $\{V_n : n \in \mathbb{N}\}$ be a pseudo-base of Y. Assume the contrary. Let $X \setminus A$ be a non-meager subset of X. Put $B = (X \setminus A) \cap M$. It is clear that B is also a non-meager subset of X. For each $x \in B$, there is a non-meager subset W_x of Y such that f is discontinuous at each point of $\{x\} \times W_x$. Then, $W_x = \bigcup_{k=1}^{\infty} W_{x,k}$, where $W_{x,k} = \{y \in W_x : \omega_f(x, y) \geq \frac{1}{k}\}$. Since W_x is a non-meager subset of Y for each $x \in B$, there are positive integers j_x and k_x such that $\overline{W_{x,k_x}} \supseteq V_{j_x}$. Consider sets $A_{j,k} = \{x \in B : j_x = j,\ k_x = k\}$. Since $B = \bigcup_{j,k=1}^{\infty} A_{j,k}$, there are positive integers m and n such that $A_{m,n}$ is a non-meager subset of X.

Let $\{V_{m,i} : i \in \mathbb{N}\}$ be a pseudo-base subspace of V_m of Y. Consider the sets $B_i = \{x \in A_{m,n} : \omega_f(\{x\} \times V_{m,i}) < \frac{1}{8n}\}$. Since for each $x \in A_{m,n} \subseteq B$ the function f^x is pointwise discontinuous, $A_{m,n} = \bigcup_{i=1}^{\infty} B_i$. Hence, there is a positive integer l such that B_l is a non-meager subset of X.

Put $V = V_{m,l}$. Since f satisfies condition (C), there are a somewhere dense subset E of X and the mapping $g : E \to V$ such that $E \subseteq B_l$ and $\omega_f(Gr(g)) < \frac{1}{8n}$. The open set $U = int\overline{E}$ is nonempty because the set E is somewhere dense. We show that $\omega_f(E \times V) \leq \frac{3}{8n}$. Take the points $p_i = (u_i, v_i) \in E \times V$, $i = 1, 2$. Then,
$$\begin{aligned} d\big(f(p_1), f(p_2)\big) \leq\ & d\Big(f(p_1), f\big(u_1, g(u_1)\big)\Big) \\ & + d\Big(f\big(u_1, g(u_1)\big), f\big(u_2, g(u_2)\big)\Big) \\ & + d\Big(f\big(u_2, g(u_2)\big), f(p_2)\Big) \\ \leq\ & \frac{1}{8n} + \frac{1}{8n} + \frac{1}{8n} = \frac{3}{8n}. \end{aligned}$$

Hence, $\omega_f(E \times V) \leq \frac{3}{8n} < \frac{1}{2n}$.

Since f satisfies (α, β)-condition, for $\alpha = \frac{1}{2n}$ and $\beta = \frac{1}{n}$, there are nonempty open subsets G of X and H of Y such that $G \subseteq U$, $H \subseteq V$ and $\omega_f(G \times H) < \frac{1}{n}$. Take any point $a \in G \cap B_l$. Since $B_l \subseteq A_{m,n}$ and for each $x \in A_{m,n}$, we have that
$$H \subseteq V = V_{m,l} \subseteq V_m \subseteq \overline{W_{x,n}},$$

the set $H \cap W_{a,n}$ is nonempty. Take any point $b \in H \cap W_{a,n}$. The set $G \times H$ is a neighborhood of (a, b). Then, both $\omega_f(a, b) < \frac{1}{n}$ and $\omega_f(a, b) \geq \frac{1}{n}$. This contradiction proves the theorem. $\square$

Corollary 4.1.1. *Let X be a Baire space, Y a Baire space which has a countable pseudo-base, Z a metric space, $f : X \times Y \to Z$ a function such that (α, β)-condition and condition (C) hold, and let $M = \{x \in X : \overline{C(f^x)} = Y\}$ be a residual subset of X. Then, f is pointwise discontinuous.*

Proof. Since a residual subset of a Baire space is everywhere dense, by Theorem 4.1, $A = \{x \in X : C^x(f) \text{ is a residual subset of } Y\}$ is a dense subset of X. In addition,

$$A \subseteq \{x \in X : \overline{C^x(f)} = Y\} \quad \text{and} \quad \bigcup_{x \in A} \big(\{x\} \times C^x(f)\big) \subseteq C(f).$$

Therefore,

$$\overline{\bigcup_{x \in A} \big(\{x\} \times C^x(f)\big)} = X \times Y \quad \text{and} \quad \overline{C(f)} = X \times Y.$$

$\square$

Theorem 4.2. *Let X be a topological space, Y a Baire space which has a countable pseudo-base, Z a metric space and $f \colon X \times Y \to Z$ a pointwise discontinuous function. Then, f satisfies (α, β)-condition and condition (C), and $M = \{x \in X : \overline{C(f^x)} = Y\}$ is a residual subset of X.*

Proof. First, we show that f satisfies (α, β)-condition. Take arbitrary numbers $0 < \alpha < \beta$, nonempty open subsets U of X and V of Y, and a subset E of X with $U \subseteq \overline{E}$ such that $\omega_f(E \times V) < \alpha$. Since f is pointwise discontinuous, $C(f) \cap (U \times V) \neq \varnothing$. Take the point $p \in C(f) \cap (U \times V)$. By continuity of f at p, it follows that there are nonempty open subset G of X and H of Y such that $G \subseteq U$, $H \subseteq V$ and $\omega_f(G \times H) < \beta$.

We show that f satisfies condition (C). Take any $\varepsilon > 0$, any non-meager subset E of X and any nonempty open subset V of Y. Since E is a non-meager set, $int\overline{E} \neq \varnothing$. Put $U = int\overline{E}$. Since f is pointwise discontinuous, $C(f) \cap (U \times V) \neq \varnothing$. Take the point $p = (a, b) \in C(f) \cap (U \times V)$. By continuity of f at p, it follows that there are nonempty open subset G of X and H of Y such that $a \in G \subseteq U$, $b \in H \subseteq V$ and $\omega_f(G \times H) < \varepsilon$. Put $E_1 = E \cap G$. Then, $int\overline{E_1} \neq \varnothing$. Consider the mapping $g \colon E_1 \to V$, $g(x) = b$ for all $x \in E_1$. Then, $\omega_f\big(Gr(g)\big) \leq \omega_f(G \times H) < \varepsilon$.

Now, we show that $M = \{x \in X : \overline{C(f^x)} = Y\}$ is residual subset of X. Let $\{V_n : n \in \mathbb{N}\}$ be a pseudo-base of Y. Assume the contrary. Let $E = X \setminus M$ be a non-meager subset of X. For each $x \in E$, there is a nonempty open subset W_x of Y such that $W_x \cap C(f^x) = \varnothing$. The sets $W_{x,k} = \{y \in W_x : \omega_{f^x}(y) \geq \frac{1}{k}\}$ are closed in W_x and $W_x = \bigcup_{k=1}^{\infty} W_{x,k}$. Since W_x is non-meager subset of Y, for each $x \in E$, there are positive integers j_x and k_x such that $W_{x,k_x} \supseteq V_{j_x}$.

Consider the sets $A_{j,k} = \{x \in E : j_x = j,\, k_x = k\}$. Since $E = \bigcup_{j,k=1}^{\infty} A_{j,k}$, there are positive integers m and n such that the set $A_{m,n}$ is dense in some nonempty open subset U of X. By pointwise discontinuity of f, it follows that $C(f) \cap (U \times V_m) \neq \varnothing$. Take any point $p \in C(f) \cap (U \times V_m)$. Then, there are nonempty subsets G of X and H of Y such that $G \subseteq U$, $H \subseteq V_m$ and $\omega_f(G \times H) < \frac{1}{n}$. Take any points $a \in G \cap A_{m,n}$ and $b \in H \cap W_{a,n}$. Then, both $\omega_{f^a}(b) \geq \frac{1}{n}$ and $\omega_{f^a}(b) \leq \omega_f(G \times H) < \frac{1}{n}$. This contradiction proves the theorem. □

Corollary 4.2.1. *Let X be a Baire space, Y a Baire space which has a countable pseudo-base, Z a metric space. A function $f : X \times Y \to Z$ is pointwise discontinuous if and only if f satisfies (α, β)-condition and condition (C), and $M = \left\{x \in X : \overline{C(f^x)} = Y\right\}$ is a residual subset of X.*

5. Jointly simple continuity

Theorem 5.1. *Let X be a Baire space, Y a Baire space which has a countable pseudo-base and Z a separable metrizable space. A mapping $f\colon X \times Y \to Z$ is simply continuous if and only if the following condition holds:*

($\star\star$) *for each nonempty open subsets U of X and V of Y, a non-meager subset M of X with $M \subseteq U$ and subsets $B_x, x \in M$ of Y with $V \subseteq \overline{B_x}$, there are nonempty open subsets G of X and H of Y such that*

$$G \times H \subseteq int\overline{M} \times V \quad \text{and} \quad f(G \times H) \subseteq \overline{f\left(\bigcup_{x \in M} (\{x\} \times B_x)\right)}.$$

P r o o f. We will use Theorem 3.2 to characterize the notion of simple continuity. Since Z is a separable metrizable space, Z is a regular second countable space. Let $\{V_n : n \in \mathbb{N}\}$ be a pseudo-base of Y and $\{W_k : k \in \mathbb{N}\}$ a base of Z.

The proof of necessity is obvious because the set $\bigcup_{x \in M}\big(\{x\} \times B_x\big)$ is dense in $int\overline{M} \times V$.

Sufficiency will be proved by contradiction. Suppose f is not simply continuous. Then, there are nonempty open subsets U of X and V of Y and a subset E of $X \times Y$ with $U \times V \subseteq \overline{E}$ such that $f(G \times H) \not\subseteq \overline{f(E)}$ for any nonempty open subsets G of X and H of Y with $G \times H \subseteq U \times V$.

Consider the set

$$A = \Big\{x \in U : \big(\exists B_x \subseteq V, V \setminus B_x \text{ is a meager set}\big)\big(\forall y \in B_x\big)\big(f(x, y) \notin \overline{f(E)}\big)\Big\}.$$

We show that A is a non-meager set. Suppose not. Let A be a meager set. Since X is a Baire space, $U \setminus A$ is a non-meager set. For each $n \in \mathbb{N}$, we consider the sets

$$A_n = \Big\{x \in U \setminus A : (\exists D_x \subseteq V, \overline{D_x} \supseteq V_n)\,(\forall y \in D_x)\big(f(x,y) \in \overline{f(E)}\big)\Big\}.$$

We show that $U \setminus A \subseteq \bigcup_{n=1}^{\infty} A_n$. Let $x \in U \setminus A$. Since $x \notin A$, for each set $B_x \subseteq V$ such that $V \setminus B_x$ is a meager set, there is a point $y \in B_x$ such that $f(x,y) \in \overline{f(E)}$. Put

$$D_x = \Big\{y \in V : (\exists B_x \subseteq V, V \setminus B_x \text{ is a meager set})(y \in B_x)\big(f(x,y) \in \overline{f(E)}\big)\Big\}.$$

It is clear that the set D_x is non-meager. Then, there is a positive integer n such that $\overline{D_x} \supseteq V_n$. Hence, $x \in A_n$ and $U \setminus A \subseteq \bigcup_{n=1}^{\infty} A_n$.

Since $U \setminus A$ is a non-meager set, there is a positive integer k such that A_k is a non-meager set. By condition ($\star\star$), there are nonempty open subsets G of X and H of Y such that

$$G \times H \subseteq int\overline{A_k} \times V_k \subseteq U \times V \quad \text{and} \quad f(G \times H) \subseteq \overline{f\left(\bigcup_{x \in A_k} (\{x\} \times D_x)\right)}.$$

Then,

$$f(G \times H) \subseteq \overline{f\left(\bigcup_{x \in A_k} (\{x\} \times D_x)\right)} \subseteq \overline{\overline{f(E)}} = \overline{f(E)}.$$

This contradiction proves that A is a non-meager set.

Since Y is a Baire space, for each $x \in A$, the set B_x is non-meager. For any $x \in A$, there are positive integers n_x and m_x such that for each $y \in B_x \cap V_{n_x}$ we have $f(x,y) \in W_{m_x}$ and $\overline{W_{m_x}} \cap f(E) = \varnothing$. Consider the sets $A_{n,m} = \{x \in A : n_x = n, m_x = m\}$. It is clear that $A = \bigcup_{m,n=1}^{\infty} A_{n,m}$. Since A is a non-meager set, there are positive integers n_0 and m_0 such that A_{n_0,m_0} is a non-meager set.

Now again, we apply condition ($\star\star$). Then, there are nonempty open subsets G of X and H of Y such that

$$G \times H \subseteq int\overline{A_{n_0,m_0}} \times V_{n_0} \subseteq U \times V$$

and

$$f(G \times H) \subseteq \overline{f\left(\bigcup_{x \in A_{n_0,m_0}} (\{x\} \times (B_x \cap V_{n_0}))\right)}.$$

Therefore,

$$f(G \times H) \subseteq \overline{f\left(\bigcup_{x \in A_{n_0,m_0}} (\{x\} \times (B_x \cap V_{n_0}))\right)} \subseteq \overline{W_{m_0}}.$$

Since $\overline{W_{m_0}} \cap f(E) = \varnothing$, $f(G \times H) \cap f(E) = \varnothing$. But it contradicts the fact that $U \times V \subseteq \overline{E}$. This contradiction proves that f is simply continuous. $\square$

A mapping $f\colon X\times Y\to Z$ *satisfies condition (B)* if for each nonempty open subsets U of X and V of Y and a subset A of X with $U\subseteq\overline{A}$, there are nonempty open subsets G of X and H of Y such that $G\times H\subseteq U\times V$ and $f(G\times H)\subseteq\overline{f(A\times V)}$.

We note that horizontally quasi-continuous ([19]) or simply continuous mappings satisfy condition (B). For a metric space Z, if a function $f\colon X\times Y\to Z$ satisfies condition (B) then f satisfies (α,β)-condition.

Theorem 5.2. *Let X be a Baire space, Y a Baire space which has a countable pseudo-base and Z a separable metrizable space, $f\colon X\times Y\to Z$ a function such that condition (B) holds, and $E=\big\{x\in X\colon f^x$ is simply continuous$\big\}$ a residual subset of X. Then, f is simply continuous.*

P r o o f. We will use Theorem 5.1. Take arbitrary nonempty open subsets U of X and V of Y and a non-meager subset M of X such that $M\subseteq U$. For each $x\in M$, we consider a subset B_x of Y such that $V\subseteq\overline{B_x}$. Let $\{V_n\colon n\in\mathbb{N}\}$ be a pseudo-base of Y. For each $n\in\mathbb{N}$, we consider the set

$$A_n=\big\{x\in M\cap E: f^x(V_n)\subseteq\overline{f^x(B_x)}\big\}.$$

Note that $M\cap E$ is a non-meager set. For each $x\in M$, the mapping f^x is simply continuous. Therefore, $\bigcup_{n=1}^{\infty}A_n=M\cap E$. Since $M\cap E$ is a non-meager set, there is a positive integer m such that A_m is dense in some nonempty open subset U_0 of X, i.e., $U_0\subseteq\overline{A_m}$. By condition (B), it follows that there are nonempty open subsets G of X and H of Y such that $G\times H\subseteq U_0\times V_m$ and $f(G\times H)\subseteq\overline{f(A_m\times V_m)}$. Then, $G\times H\subseteq U_0\times V_m\subseteq U\times V$ and

$$\begin{aligned}
f(G\times H)&\subseteq\overline{f(A_m\times V_m)}\subseteq\overline{f\Bigg(\bigcup_{x\in A_m}\big(\{x\}\times V_m\big)\Bigg)}\\
&=\overline{\bigcup_{x\in A_m}f\big(\{x\}\times V_m\big)}=\overline{\bigcup_{x\in A_m}f^x(V_m)}\\
&\subseteq\overline{\bigcup_{x\in A_m}\overline{f^x(B_x)}}\subseteq\overline{\overline{\bigcup_{x\in A_m}f^x(B_x)}}\\
&=\overline{\bigcup_{x\in A_m}f^x(B_x)}=\overline{f\Bigg(\bigcup_{x\in A_m}\big(\{x\}\times B_x\big)\Bigg)}\\
&\subseteq\overline{f\Bigg(\bigcup_{x\in M}\big(\{x\}\times B_x\big)\Bigg)}.
\end{aligned}$$

Hence, by Theorem 5.1, the mapping f is simply continuous. $\square$

6. Jointly mildly continuity

Theorem 6.1. *Let X be a Baire space, Y a space which has a countable pseudo--base, Z a separable metrizable space, $f\colon X \times Y \to Z$ a function such that condition (B) holds, and $M = \{x \in X : f^x$ is mildly continuous$\}$ a residual subset of X. Then, f is mildly continuous.*

Proof. Assume the contrary. Do not let f be mildly continuous. This means that $\overline{S(f)} \neq X \times Y$. Then, there are nonempty open subset U of X and V of Y such that $S(f) \cap (U \times V) = \varnothing$. Hence, $U \times V \subseteq \big((X \times Y) \setminus Q(f)\big) \cap P(f)$, i.e., f is not quasi-continuous but almost continuous at each point of $U \times V$.

Since f^x is mildly continuous for each $x \in U \cap M$, $\overline{S(f^x)} \supseteq V$. Then, $S(f^x) \cap V \neq \varnothing$ for all $x \in U \cap M$. It means that for each $x \in U \cap M$ there is a point $y_x \in V$ such that f^x is quasi-continuous at y_x, or f^x is not almost continuous at y_x.

Considers the sets

$$A = \big\{x \in U \cap M : y_x \in Q(f^x)\big\} \quad \text{and} \quad B = \big\{x \in U \cap M : y_x \notin P(f^x)\big\}.$$

Then, $U \cap M = A \cup B$. Since X is a Baire space, $U \cap M$ is a non-meager subset of X. Therefore, at least one of the sets A or B is non-meager. Since Z is a separable metrizable space, Z is a regular second countable space. Let $\{V_n : n \in \mathbb{N}\}$ be a pseudo-base of Y and $\{W_m : m \in \mathbb{N}\}$ a base of Z.

First, assume that A is a non-meager subset of X. Since for each $x \in A$ the mapping f is not quasi-continuous at $(x, y_x) \in U \times V$, there are open neighborhoods $U(x)$ of x in X, $V(x)$ of y_x in Y and a closed neighborhood $W(x)$ of $f(x, y_x)$ in Z such that $U(x) \subseteq U$, $V(x) \subseteq V$ and $f(G \times H) \nsubseteq W(x)$ for arbitrary nonempty open subsets G of X and H of Y such that $G \times H \subseteq U(x) \times V(x)$.

For positive integers m and n, we consider the sets

$$A_{n,m} = \big\{x \in A : f(x, y_x) \in W_m \subseteq W(x), V_n \subseteq V(x), f^x(V_n) \subseteq W_m\big\}.$$

Since, f^x is quasi-continuous at y_x for each $x \in A$, $A = \bigcup_{n,m=1}^{\infty} A_{n,m}$. From the fact that A is a meager set, it follows that there are positive integers n_0 and m_0 such that A_{n_0,m_0} is a somewhere dense subset of X. Since f satisfies condition (B), there are nonempty open subsets G of X and H of Y such that

$$G \subseteq int\overline{A_{n_0,m_0}}, \quad H \subseteq V_{n_0} \quad \text{and} \quad f(G \times H) \subseteq \overline{f(A_{n_0,m_0} \times V_n)}.$$

Take any point $a \in A_{n_0,m_0} \cap G$. Then,

$$G \cap U(a) \subseteq U(a), \quad H \subseteq V_{n_0} \subseteq V(a),$$

and

$$f\Big(\big(G \cap U(a)\big) \times H\Big) \subseteq f(G \times H) \subseteq \overline{f(A_{n_0,m_0} \times V_{n_0})} \subseteq \overline{W_{m_0}} \subseteq \overline{W(a)} = W(a).$$

Since $a \in A_{n_0,m_0} \subseteq A$, we obtained a contradiction.

Suppose now that B is a non-meager subset of X. Since f is almost continuous at a point $(x, y_x) \in U \times V$ for each $x \in B$, for any neighborhood W of $f(x, y_x)$ in Z, there are open neighborhoods $U(x, W)$ of x in X, $V(x, W)$ of y_x in Y, and a subset O of $X \times Y$ with $\overline{O} \supseteq U(x, W) \times V(x, W)$ such that $f(O) \subseteq W$. For positive integers m and n, we consider the sets

$$B_{n,m} = \bigl\{x \in B : f(x, y_x) \in W_m, V_n \subseteq V(x, W_m), f^x(V_n) \subseteq Z \setminus W_m\bigr\}.$$

Since, f^x is not almost continuous at y_x for each $x \in B$, $B = \bigcup_{n,m=1}^{\infty} B_{n,m}$.

From the fact that B is a meager set, it follows that there are positive integers n_0 and m_0 such that B_{n_0,m_0} is a somewhere dense subset of X. Since f satisfies condition (B), there are nonempty open subsets G of X and H of Y such that

$$G \subseteq int\overline{B_{n_0,m_0}}, \quad H \subseteq V_{n_0} \quad \text{and} \quad f(G \times H) \subseteq \overline{f(B_{n_0,m_0} \times V_n)}.$$

Take any point $a \in B_{n_0,m_0} \cap G$. Then, for the neighborhood $U(a, W_{m_0}) \times V(a, W_{m_0})$ of (a, y_a), there exists a nonempty open set $\bigl(G \cap U(a, W_{m_0})\bigr) \times H \subseteq U(a, W_{m_0}) \times V(a, W_{m_0})$ such that

$$f\Bigl(\bigl(G \cap U(a, W_{m_0})\bigr) \times H\Bigr) \subseteq f(G \times H) \subseteq \overline{f(B_{n_0,m_0} \times V_n)} \subseteq \overline{Z \setminus W_{m_0}} = Z \setminus W_{m_0}.$$

It means that f is not almost continuous at $(a, y_a) \in U \times V$. This contradiction proves that B is a meager subset of X.

Thus, f is mildly continuous. □

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Chernivtsi National University
Department of Mathematical
Kotsubinsky Str. 2
UA–58012-Chernivtsi
UKRAINE
E-mail: math.analysis.chnu@gmail.com