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MICROSCOPIC SETS WITH RESPECT TO SEQUENCES OF FUNCTIONS

Grażyna Horbaczewska

ABSTRACT. Consequences of replacing the geometric sequence with another in the definition of microscopic sets are considered.

The notion of a microscopic set on the real line was introduced by J. Appell in [1] at the beginning of the 21st century. Thereafter, some papers were devoted to this topic ([7], [9]–[11]). Lately, a chapter on microscopic sets was published in a monography [8].

Definition 1. A set $E \subset \mathbb{R}$ is microscopic if for each $\epsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \bigcup_{n\in\mathbb{N}} I_n \quad \text{and} \quad \lambda(I_n) \le \epsilon^n \qquad \text{for} \quad n \in \mathbb{N}.$$

The family of all microscopic sets will be denoted by $\mathcal{M}$.

A special role in this definition is played by a geometric sequence. A question about the consequences of replacing this specific sequence with another one was raised by J. Appell, E. D'Aniello and M. Väth in [3].

If we consider an arbitrary sequence here, we get a definition of a family of strong measure zero sets ([5]), denoted by $\mathcal{S}$.

Definition 2. A set $E \subset \mathbb{R}$ is of strong measure zero if for each sequence of positive reals $\{\epsilon_n\}_{n\in\mathbb{N}}$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \bigcup_{n\in\mathbb{N}} I_n \quad \text{and} \quad \lambda(I_n) \le \epsilon_n \qquad \text{for} \quad n \in \mathbb{N}.$$

Obviously, $\mathcal{S} \subsetneq \mathcal{M}$. An example of a microscopic set which is not a strong measure zero set is given in [6].

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Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of increasing functions $f_n\colon (0,1)\to(0,1)$ such that $\lim_{x\to 0^+} f_n(x)=0$, and there exists $x_0\in(0,1)$ such that for every $x\in(0,x_0)$ the series $\sum_{n\in\mathbb{N}} f_n(x)$ is convergent and the sequence $\big(f_n(x)\big)_{n\in\mathbb{N}}$ is nonincreasing.

Definition 3. A set $E\subset\mathbb{R}$ belongs to $\mathcal{M}_{(f_n)}$ if for each $x\in(0,1)$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E\subset\bigcup_{n\in\mathbb{N}} I_n \quad\text{and}\quad \lambda(I_n)\le f_n(x) \qquad\text{for}\quad n\in\mathbb{N}\,.$$

For $f_n(x)=x^n$, $n\in\mathbb{N}$, we have $\mathcal{M}_{(f_n)}=\mathcal{M}$.

If $\mathcal{H}$ denotes the family of all sequences of functions with properties described above, then

$$\bigcap_{(f_n)\in\mathcal{H}} \mathcal{M}_{(f_n)}=\mathcal{S}\,.$$

Now, we may ask several questions. We would like to know for which sequences we get microscopic sets, when different sequences give different families of sets, under which condition $\mathcal{M}_{(f_n)}$ is a σ-ideal.

The next theorem gives a sufficient condition for getting microscopic sets.

Theorem 4. *If there exists $k\in\mathbb{N}$ such that for every $n\in\mathbb{N}$ and for every $x\in(0,1)$ we have $x^{n+k}\le f_n(x)\le x^n$, then*

$$\mathcal{M}_{(f_n)}=\mathcal{M}\,.$$

Proof. Since the inclusion $\mathcal{M}_{(f_n)}\subset\mathcal{M}$ is obvious, we have to justify only the inverse one.

Let $E\in\mathcal{M}$. Fix $x\in(0,1)$. Since $E\in\mathcal{M}$ for $x_0=x^{k+1}$, there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E\subset\bigcup_{n\in\mathbb{N}} I_n \quad\text{and}\quad \lambda(I_n)<(x_0)^n \qquad\text{for}\ \ n\in\mathbb{N}\,.$$

Therefore

$$\lambda(I_n)<(x^{k+1})^n=x^{nk+n}\le x^{n+k}\le f_n(x),\quad\text{so}\quad E\in\mathcal{M}_{(f_n)}\,. \qquad\square$$

The above condition is not necessary, for example, for a sequence $f_n(x)=x^{2n}$, $n\in\mathbb{N}$, the assumption of the above theorem is not satisfied, but $\mathcal{M}_{(f_n)}=\mathcal{M}$.

More examples of sequences of functions leading to microscopic sets can be given for:

$$f_n(x)=\Big(\frac{x}{a}\Big)^n,\quad\text{where}\quad a>0,$$

$$f_n(x)=x^{an},\qquad\text{where}\quad a>0,$$

$$f_n(x)=\frac{x^n}{n^a}\,,\qquad\text{where}\quad a\ge 1,$$

the family $\mathcal{M}_{(f_n)}$ is exactly the family of microscopic sets.

For reader's convenience we show that in the last case every microscopic set belongs to $\mathcal{M}_{(f_n)}$. Let $A \in \mathcal{M}$. Fix $x \in (0,1)$ and find $n_0 \in \mathbb{N}$ such that $1/n_0 < x$. For $\epsilon := \frac{1}{n_0^{a+1}}$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \bigcup_{n\in\mathbb{N}} I_n \quad \text{and} \quad \lambda(I_n) \le \epsilon^n \qquad \text{for} \quad n \in \mathbb{N}\,.$$

Then

$$\lambda(I_n) \le \epsilon^n = \frac{1}{(n_0^{a+1})^n} = \frac{1}{(n_0^a)^n n_0^n} < \frac{1}{(n_0^n)^a} x^n < \frac{x^n}{n^a}$$

since $n_0^n > n$ for $n_0 > 1$. Therefore, $a \in \mathcal{M}_{(f_n)}$.

One can observe an obvious sufficient condition for an inclusion between the families $\mathcal{M}_{(g_n)}$ and $\mathcal{M}_{(f_n)}$ for sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ from the family $\mathcal{H}$:

$$\forall_{x\in(0,1)} \ \forall_{n\in\mathbb{N}} \ g_n(x) \le f_n(x) \ \Rightarrow \ \mathcal{M}_{(g_n)} \subset \mathcal{M}_{(f_n)}\,.$$

The next theorem gives a not so obvious sufficient condition.

Theorem 5. *If for every $x \in (0,1)$ there exists $y \in (0,1)$ such that there exists a sequence $(P_m)_{m\in\mathbb{N}}$ of pairwise disjoint, nonempty subsets of $\mathbb{N}$ such that*

$$g_m(y) \le \sum_{i\in P_m} f_i(x) \qquad \textit{for every} \quad m \in \mathbb{N}\,,$$

then

$$\mathcal{M}_{(g_n)} \subset \mathcal{M}_{(f_n)}\,.$$

Proof. Let $E \in \mathcal{M}_{(g_n)}$. Let $x \in (0,1)$. By our assumption, there exists $y \in (0,1)$ and a sequence $\{P_m\}$ of pairwise disjoint nonempty subsets of $\mathbb{N}$ such that $g_m(y) \le \sum_{i\in P_m} f_i(x)$. Since $E \in \mathcal{M}_{(g_n)}$, for y there exists a sequence of intervals $(I_m)_{m\in\mathbb{N}}$ such that

$$E \subset \bigcup_{m\in\mathbb{N}} I_m \quad \text{and} \quad \lambda(I_m) \le g_m(y)\,.$$

Since $\lambda(I_m) \le g_m(y) \le \sum_{i\in P_m} f_i(x)$, we may divide the interval I_m into nonoverlapping intervals $J_i^{(m)}$, $i \in P_m$, such that

$$\lambda(J_i^{(m)}) \le f_i(x) \qquad \text{for} \quad i \in P_m\,.$$

Let $n \in \mathbb{N}$. If n belongs to none of P_m, $m \in \mathbb{N}$, then $J_n := \emptyset$. If $n \in P_m$, then $J_n := J_n^{(m)}$. Therefore,

$$E \subset \bigcup_{m\in\mathbb{N}} I_m = \bigcup_{n\in\mathbb{N}} J_n \quad \text{and} \quad \lambda(J_n) \le f_n(x), \quad \text{so} \quad E \in \mathcal{M}_{(f_n)}\,.$$

□

Theorem 6.
$$\mathcal{M}_{(f_n)} \setminus \mathcal{S} \neq \emptyset \quad \textit{for every} \quad (f_n)_{n\in\mathbb{N}} \in \mathcal{H}.$$

Proof.

Let $I := [0,1]$. We will define by induction the sequence of open intervals $\{J_{n,i}\}$, $i \in \{1,\dots,2^{n-1}\}$, $n \in \mathbb{N}$, in the following way.

Let $k_1 := \min\{k : f_{2^k}\left(\frac{1}{k+1}\right) < \frac{1}{3}\}$. Put $J_{1,1} := \left(f_{2^{k_1}}\left(\frac{1}{k_1+1}\right), 1 - f_{2^{k_1}}\left(\frac{1}{k_1+1}\right)\right)$. Then, of course, $\lambda(J_{1,1}) > \frac{1}{3}$.

Let $K_{1,1}$, $K_{1,2}$ denote successive components of the set $I \setminus J_{1,1}$. Obviously, $\lambda(K_{1,i}) = f_{2^{k_1}}\left(\frac{1}{k_1+1}\right)$ for $i \in \{1,2\}$.

Let $k_2 := \min\{k : f_{2^k}\left(\frac{1}{k+1}\right) < \frac{1}{3} f_{2^{k_1}}\left(\frac{1}{k_1+1}\right)\}$.

Let $J_{2,1}$, $J_{2,2}$ be two open intervals concentric with $K_{1,1}$ and $K_{1,2}$, respectively, such that $\lambda(J_{2,1}) = \lambda(J_{2,2}) = \lambda(K_{1,1}) - 2f_{2^{k_2}}\left(\frac{1}{k_2+1}\right)$.

Let $K_{2,1}$, $K_{2,2}$, $K_{2,3}$, and $K_{2,4}$ denote successive components of the set $I \setminus (J_{1,1} \cup J_{2,1} \cup J_{2,2})$. Notice that $\lambda(K_{2,i}) = f_{2^{k_2}}\left(\frac{1}{k_2+1}\right)$ for $i \in \{1,2,3,4\}$.

Let $m \geq 2$. Assume that we have constructed the open, nonempty intervals $J_{l,1},\dots,J_{l,2^{l-1}}$ concentric with $K_{l-1,1},\dots,K_{l-1,2^{l-1}}$, respectively, such that $\lambda(J_{l,i}) = \lambda(K_{l-1,1}) - 2f_{2^{k_l}}\left(\frac{1}{k_l+1}\right)$ for $l \in \{2,\dots,m\}$, $i \in \{1,\dots,2^{l-1}\}$, where $k_l := \min\{k : f_{2^k}\left(\frac{1}{k+1}\right) < \frac{1}{3} f_{2^{k_{l-1}}}\left(\frac{1}{k_{l-1}+1}\right)\}$, for $l \in \{2,\dots,m\}$.

Let $K_{m,1},\dots,K_{m,2^m}$ be successive components of the set $I \setminus \bigcup_{l=1}^{m} \bigcup_{i=1}^{2^{l-1}} J_{l,i}$. Notice that $\lambda(K_{m,i}) = f_{2^{k_m}}\left(\frac{1}{k_m+1}\right)$ for $i \in \{1,\dots,2^m\}$.

Now, let $k_{m+1} := \min\{k : f_{2^k}\left(\frac{1}{k+1}\right) < \frac{1}{3} f_{2^{k_m}}\left(\frac{1}{k_m+1}\right)\}$ and $J_{m+1,1},\dots,J_{m+1,2^m}$ be open intervals concentric with $K_{m,1},\dots,K_{m,2^m}$, respectively, such that $\lambda(J_{m+1,i}) = \lambda(K_{m,1}) - 2f_{2^{k_{m+1}}}\left(\frac{1}{k_{m+1}+1}\right)$ for $i \in \{1,\dots,2^m\}$.

Let $K_{m+1,1},\dots,K_{m+1,2^{m+1}}$ be successive components of the set
$$I \setminus \bigcup_{l=1}^{m+1} \bigcup_{i=1}^{2^{l-1}} J_{l,i}.$$
Obviously, $\lambda(K_{m+1,i}) = f_{2^{k_{m+1}}}\left(\frac{1}{k_{m+1}+1}\right)$ for $i \in \{1,\dots,2^{m+1}\}$.

Let us put $M := \bigcap_{m\in\mathbb{N}} \bigcup_{i=1}^{2^m} K_{m,i}$.

Now, let $x \in (0,1)$. There exists $m_0 \in \mathbb{N}$ such that $\frac{1}{k_{m_0}} < x$.

Let $I_i := K_{m_0,i}$ for $i \in \{1,\dots,2^{m_0}\}$ and $I_i := \emptyset$ for $i > 2^{m_0}$. Obviously, $M \subset \bigcup_{i=1}^{2^{m_0}} K_{m_0,i} = \bigcup_{i\in\mathbb{N}} I_i$ and $\lambda(I_i) = \lambda(K_{m_0,i}) = f_{2^{k_{m_0}}}\left(\frac{1}{k_{m_0}+1}\right) \leq f_{2^{m_0}}\left(\frac{1}{k_{m_0}+1}\right) < f_{2^{m_0}}(x) \leq f_i(x)$ for $i \in \{1,\dots,2^{m_0}\}$. Hence, M is a Cantor-type set from $\mathcal{M}_{(f_n)}$. As a perfect set, M cannot be a strong measure zero set (compare [5, Corollary 8.1.5]), so $M \in \mathcal{M}_{(f_n)} \setminus \mathcal{S}$.

□

Theorem 7. *Let (f_n), $(g_n) \in \mathcal{H}$ and let k_n be a sequence of natural numbers chosen as in the proof of the previous theorem. Suppose that there exists $\delta > 0$ such that $g_n(x) < f_n(x)$ for every $n \in \mathbb{N}$ and for every $x \in (0, \delta)$. If there exists a point $x_0 \in (0,1)$ such that $g_n(x_0) < f_{2^{k_n}}\left(\frac{1}{k_n+1}\right)$, for every $n \in \mathbb{N}$, then*

$$\mathcal{M}_{(g_n)} \subsetneq \mathcal{M}_{(f_n)}.$$

P r o o f. Since the inclusion $\mathcal{M}_{(g_n)} \subset \mathcal{M}_{(f_n)}$ is obvious, we only need to show that these families are different.

Consider the set M from the proof of the previous theorem.

Let $\{I_n\}_{n\in\mathbb{N}}$ be any sequence of intervals such that $\lambda(I_n) \le g_n(x_0)$.

As $\lambda(I_1) \le g_1(x_0) < f_{2^{k_1}}\left(\frac{1}{2}\right)$, I_1 cannot have common points with both sets $K_{1,1}$ and $K_{1,2}$, since $f_{2^{k_1}}\left(\frac{1}{2}\right) < \frac{1}{3} < \lambda(J_{1,1}) = \operatorname{dist}(K_{1,1}, K_{1,2})$.

Put $P_0 := [0,1]$.

Let $P_1 := K_{1,i_1}$, where $i_1 \in \{1,2\}$ and $I_1 \cap K_{1,i_1} = \emptyset$.

Let $n \ge 2$. We assume that for $l \in \{1, \dots, n-1\}$ we have already chosen an interval P_l, such that $P_l := K_{l,i_l}$, where $i_l \in \{1, \dots, 2^l\}$, $I_l \cap K_{l,i_l} = \emptyset$ and $P_l \subset P_{l-1}$.

The interval I_n satisfies a condition

$$\lambda(I_n) \le g_n(x_0) < f_{2^{k_n}}\left(\frac{1}{k_n+1}\right) = \lambda(K_{n,i})$$

for $i \in \{1, \dots, 2^n\}$ and by the construction of M, precisely by the way of defining k_n, we have $\lambda(K_{n,i}) < \lambda(J_{l,j})$, where $J_{l,j}$ is a gap between the intervals $K_{n,i}$ and $K_{n,i'}$ contained in P_{n-1}, so one of them has no common points with I_n. We denote it by P_n.

Therefore, we have inductively constructed a descending sequence of closed intervals P_n, such that $\lambda(P_n) = f_{2^{k_n}}\left(\frac{1}{k_n+1}\right)$ and $P_n \cap I_n = \emptyset$ for $n \in \mathbb{N}$. By Cantor Theorem, there exists a point $x \in \bigcap_{n\in\mathbb{N}} P_n$. Of course $x \in M$ and $x \notin \bigcup_{n\in\mathbb{N}} I_n$, so $M \notin \mathcal{M}_{(g_n)}$. □

Theorem 8. *For every $(f_n) \in \mathcal{H}$, there exists $(g_n) \in \mathcal{H}$ such that*

$$\mathcal{S} \subsetneq \mathcal{M}_{(g_n)} \subsetneq \mathcal{M}_{(f_n)}.$$

P r o o f. Suffice it to put $g_n(x) := f_n(x) f_{2^{k_n}}\left(\frac{1}{k_n+1}\right)$, for $n \in \mathbb{N}$, where k_n is chosen as in the proof of Theorem 6. Then $(g_n) \in \mathcal{H}$ and (g_n) satisfies the condition from Theorem 7, since for every $x \in (0,1)$

$$g_n(x) = f_n(x) f_{2^{k_n}}\left(\frac{1}{k_n+1}\right) < f_{2^{k_n}}\left(\frac{1}{k_n+1}\right).$$

Therefore, by Theorem 6 and Theorem 7, we are done. □

Remark 9. If $f_2\left(\frac{1}{2}\right) < \frac{1}{3}$ and $\frac{f_{2^l}\left(\frac{1}{l+1}\right)}{f_{2^{l-1}}\left(\frac{1}{l}\right)} < \frac{1}{3}$ for $l \geq 2$, then $k_n = n$ and for such a sequence (f_n) the condition from the Theorem 7 is as follows

$$\exists_{x_0\in(0,1)}\forall_{n\in\mathbb{N}}\ g_n(x_0) < f_{2^n}\left(\frac{1}{n+1}\right). \tag{$*$}$$

The sequence $f_n(x) = x^n$ satisfies assumptions from the above remark, so for the sequence $g_n(x) = \frac{x^n}{(n+1)^{2^n}}$ (see the proof of Theorem 8), we have

$$\mathcal{M}_{(g_n)} \subsetneq \mathcal{M}\,.$$

Another sequence satisfying condition $(*)$ for $f_n(x) = x^n$ is the sequence $g_n(x) = x^{n^n}$, more precisely, for $x_0 = \frac{1}{4}$ and for every $n > 2$, we have

$$g_n(x_0) < \left(\frac{1}{n+1}\right)^{2^n}, \quad \text{so} \quad \mathcal{M}_{(x^{n^n})} \subsetneq \mathcal{M}\,.$$

The next theorem gives a sufficient condition for a family $\mathcal{M}_{(f_n)}$ to be a σ-ideal.

Theorem 10. *Let $f_n \in \mathcal{H}$ for $n \in \mathbb{N}$. If there exists a bijection $\Phi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that for a fixed $k \in \mathbb{N}$ a function $\Phi_k(n) := \Phi(n,k)$ is increasing and for every $x > 0$ and for every $k \in \mathbb{N}$ $\lim_{n\to+\infty} f_n^{-1}\big(f_{\Phi(n,k)}(x)\big) > 0$, then $\mathcal{M}_{(f_n)}$ is a σ-ideal.*

Proof. Suffice it to show that a countable sum of sets from $\mathcal{M}_{(f_n)}$ belongs to $\mathcal{M}_{(f_n)}$. Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of sets, such that $A_k \in \mathcal{M}_{(f_n)}$ for $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and $x > 0$. A sequence $\left\{f_n^{-1}(f_{\Phi(n,k)}(x))\right\}_{n\in\mathbb{N}}$ is monotonous, so denoting $a_k := \lim_{n\to+\infty} f_n^{-1}\big(f_{\Phi(n,k)}(x)\big)$, we have $f_n^{-1}\big(f_{\Phi(n,k)}(x)\big) \geq a_k > \frac{a_k}{2}$ for every $n \in \mathbb{N}$. By monotonocity of f_n, we get

$$f_{\Phi(n,k)}(x) \geq f_n\left(\frac{a_k}{2}\right).$$

For $\frac{a_k}{2}$ there exists a cover $\{I_n^{(k)}\}_{n\in\mathbb{N}}$ of the set A_k such that $\lambda(I_n^{(k)}) < f_n\left(\frac{a_k}{2}\right)$. Arrange intervals $\{I_n^{(k)}\}_{n\in\mathbb{N},\,k\in\mathbb{N}}$ in a sequence $\{I_m\}_{m\in\mathbb{N}}$, where $m = \Phi(n,k)$. Then $\bigcup_{k\in\mathbb{N}} A_k \subset \bigcup_{m\in\mathbb{N}} I_m$ and

$$\lambda(I_m) = \lambda\big(I_{\phi(n,k)}\big) = \lambda\big(I_n^{(k)}\big) < f_n\left(\frac{a_k}{2}\right) \leq f_{\Phi(n,k)}(x) = f_m(x)\,.$$

Therefore, $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{M}_{(f_n)}$. □

It is known that if $\mathcal{I}$ is a σ-ideal containing all singletons and $\mathcal{G}_\delta$-generated then a family $\mathcal{I}^* := \{A \subset \mathbb{R} : \exists_{B\in\mathcal{F}_\sigma\cap\mathcal{I}} A \subset B\}$ is also a σ-ideal ([4]). We do not know if $\mathcal{M}_{(x^{2^n})}$ is a σ-ideal, but we can proove that $\mathcal{M}^*_{(x^{2^n})}$—a family of sets which can be covered by F_σ sets from $\mathcal{M}_{(x^{2^n})}$ is a σ-ideal.

Lemma 11. *Let $A_k \in \mathcal{M}_{(x^{2^n})}$ and A_k be compact for every $k \in \mathbb{N}$. Then,*

$$\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{M}_{(x^{2^n})}.$$

Proof. Let $x \in (0,1)$. For $x_1 := x$ there exists a cover $\{I_n^{(1)}\}$ of A_1 such that $\lambda(I_n^{(1)}) < (x_1)^{2^n}$ for every $n \in \mathbb{N}$. By compactness of A_1, we can choose a finite subcover $\{I_n^{(1)}\}_{n=1}^{n_1}$. For $x_2 := x^{2^{n_1}}$ there exists a cover $\{I_n^{(2)}\}$ of A_2 such that $\lambda(I_n^{(2)}) < (x_2)^{2^n}$ for every $n \in \mathbb{N}$. By compactness of A_2, we can choose a finite subcover $\{I_n^{(2)}\}_{n=1}^{n_2}$. And so on, for $x_k := (x_{k-1})^{2^{n_{k-1}}}$ we find a finite cover $\{I_n^{(k)}\}_{n=1}^{n_k}$ of A_k such that $\lambda(I_n^{(k)}) < (x_k)^{2^n}$ for every $n \in \mathbb{N}$.

Now, for every $k \in \mathbb{N}$, there exists $t \in \mathbb{N}$ such that $\Sigma_{i=0}^{t} n_i < k \le \Sigma_{i=0}^{t+1} n_i$, where $n_0 := 0$, and there exists $l \in \{1, \dots, n_{t+1}\}$ such that $k = n_0 + \dots + n_t + l$. Let $J_k = I_l^{(t+1)}$. Then, $\bigcup_{k\in\mathbb{N}} A_k \subset \bigcup_{k\in\mathbb{N}} J_k$ and

$$\lambda(J_k) = \lambda\big(I_l^{(t+1)}\big) < (x_{t+1})^{2^l} = \big(x^{2^{n_1+n_2+\dots+n_t}}\big)^{2^l} = x^{2^{n_1+n_2+\dots+n_t+l}} = x^{2^k},$$

so $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{M}_{(x^{2^n})}$. □

Lemma 12. *If $A_n \in \mathcal{M}^*_{(x^{2^n})}$ for every $n \in \mathbb{N}$, then there exists a sequence of compact sets $\{D_k\}_{k\in\mathbb{N}} \subset \mathcal{M}_{(x^{2^n})}$ such that*

$$\bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{k\in\mathbb{N}} D_k\,.$$

Proof. Since $A_n \in \mathcal{M}^*_{(x^{2^n})}$ for every $n \in \mathbb{N}$ then, for every $n \in \mathbb{N}$, there exists $B_n \in \mathcal{F}_\sigma \cap \mathcal{M}_{(x^{2^n})}$ such that $A_n \subset B_n$. Then, $B_n = \bigcup_{l\in\mathbb{N}} B_l^{(n)}$, where $B_l^{(n)}$ is closed, hence it is a countable union of compact sets $D_t^{(l,n)}$ for $t \in \mathbb{N}$, so

$$\bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{n\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} \bigcup_{t\in\mathbb{N}} D_t^{(l,n)} = \bigcup_{k\in\mathbb{N}} D_k\,.$$

Obviously, $\{D_k\}_{k\in\mathbb{N}} \subset \mathcal{M}_{(x^{2^n})}$. □

Theorem 13. *$\mathcal{M}^*_{(x^{2^n})}$ is a σ-ideal.*

Proof. It suffices to show that $\mathcal{M}^*_{(x^{2^n})}$ is closed under a countable union. Let $A_n \in \mathcal{M}^*_{(x^{2^n})}$ for $n \in \mathbb{N}$. By Lemma 12, there exists a sequence $\{D_k\}_{k\in\mathbb{N}}$ such that $D_k \in \mathcal{M}_{(x^{2^n})}$ and D_k is compact for $\bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{k\in\mathbb{N}} D_k$ and for every $k \in \mathbb{N}$. By Lemma 11, we have $\bigcup_{k\in\mathbb{N}} D_k \in \mathcal{M}_{(x^{2^n})}$ and, of course, $\bigcup_{k\in\mathbb{N}} D_k \in \mathcal{F}_\sigma$, so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}^*_{(x^{2^n})}$. □

REFERENCES

[1] APPELL, J.: *Insiemi ed operatori "piccoli" in analisi funzionale,* Rend. Instit. Mat. Univ. Trieste **33** (2001), 127–199.

[2] APPELL, J.: *A short story on microscopic sets,* Atti. Sem. Mat. Fis. Univ. Modena Reggio Emilia **52** (2004), 229–233.

[3] APPELL, J.—D'ANIELLO, E.—VÄTH, M.: *Some remarks on small sets,* Ric. Mat. **50** (2001), 255–274.

[4] BALCERZAK, M.—BAUMGARTNER, J. E.—HEJDUK, J.: *On certain σ-ideals of sets,* Real Anal. Exchange **14** (1988–89), 447–453.

[5] BARTOSZYŃSKI, T.—JUDAH, H.: *Set Theory: On the Structure of the Real Line.* A. K. Peters, Ltd., Wellesley, MA, 1995.

[6] FILIPCZAK, M.—WAGNER-BOJAKOWSKA, E.: *Remarks on small sets on the real line,* Tatra Mt. Math. Publ. **42** (2009), 73–80.

[7] HORBACZEWSKA, G.—WAGNER-BOJAKOWSKA, E.: *Some kinds of convergence with respect to small sets,* Reports on Real Analysis, Conference at Rowy (2003), 88–97.

[8] HORBACZEWSKA, G.—KARASIŃSKA, A.—WAGNER-BOJAKOWSKA, E.: *Properties of the σ-ideal of microscopic sets*, in: Traditional and Present-Day Topics in Real Analysis, Łódź University Press, Łódź, 2013, pp. 325–343.

[9] KARASIŃSKA, A.—POREDA, W.—WAGNER-BOJAKOWSKA, E.: *Duality principle for microscopic sets,* in: Monograph Real Functions, Density Topology and Related Topics, Łódź University Press, Łódź, 2011, pp. 83–87.

[10] KARASIŃSKA, A.—WAGNER-BOJAKOWSKA, E.: *Nowhere monotone functions and microscopic sets*, Acta Math. Hungar. **120** (2008), 235–248.

[11] KARASIŃSKA, A.—WAGNER-BOJAKOWSKA, E.: *Homeomorphisms of linear and planar sets of the first category into microscopic sets,* Topology Appl. **159** (2012), 1894–1898.

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Department of Mathematics and Computer Science
University of Łódź
Banacha 22
PL–90-238 Łódź
POLAND

E-mail: grhorb@math.uni.lodz.pl