MICROSCOPIC SETS
WITH RESPECT TO SEQUENCES OF FUNCTIONS

GRAŻYNA HORBACZEWSKA

ABSTRACT. Consequences of replacing the geometric sequence with another in the definition of microscopic sets are considered.

The notion of a microscopic set on the real line was introduced by J. Appell in [1] at the beginning of the 21st century. Thereafter, some papers were devoted to this topic ([7], [9]–[11]). Lately, a chapter on microscopic sets was published in a monography [8].

DEFINITION 1. A set $E \subset \mathbb{R}$ is microscopic if for each $\epsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n \in \mathbb{N}}$ such that

$$E \subset \bigcup_{n \in \mathbb{N}} I_n \quad \text{and} \quad \lambda(I_n) \leq \epsilon^n \quad \text{for} \quad n \in \mathbb{N}.$$ 

The family of all microscopic sets will be denoted by $\mathcal{M}$.

A special role in this definition is played by a geometric sequence. A question about the consequences of replacing this specific sequence with another one was raised by J. Appell, E. D'Aniello and M. Váth in [3].

If we consider an arbitrary sequence here, we get a definition of a family of strong measure zero sets ([5]), denoted by $\mathcal{S}$.

DEFINITION 2. A set $E \subset \mathbb{R}$ is of strong measure zero if for each sequence of positive reals $\{\epsilon_n\}_{n \in \mathbb{N}}$ there exists a sequence of intervals $\{I_n\}_{n \in \mathbb{N}}$ such that

$$E \subset \bigcup_{n \in \mathbb{N}} I_n \quad \text{and} \quad \lambda(I_n) \leq \epsilon_n \quad \text{for} \quad n \in \mathbb{N}.$$ 

Obviously, $\mathcal{S} \subsetneq \mathcal{M}$. An example of a microscopic set which is not a strong measure zero set is given in [6].
Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of increasing functions \(f_n : (0, 1) \to (0, 1)\) such that \(\lim_{x \to 0^+} f_n(x) = 0\), and there exists \(x_0 \in (0, 1)\) such that for every \(x \in (0, x_0)\) the series \(\sum_{n \in \mathbb{N}} f_n(x)\) is convergent and the sequence \((f_n(x))_{n \in \mathbb{N}}\) is nonincreasing.

**Definition 3.** A set \(E \subseteq \mathbb{R}\) belongs to \(\mathcal{M}(f_n)\) if for each \(x \in (0, 1)\) there exists a sequence of intervals \(\{I_n\}_{n \in \mathbb{N}}\) such that

\[
E \subseteq \bigcup_{n \in \mathbb{N}} I_n \quad \text{and} \quad \lambda(I_n) \leq f_n(x) \quad \text{for} \quad n \in \mathbb{N}.
\]

For \(f_n(x) = x^n, n \in \mathbb{N}\), we have \(\mathcal{M}(f_n) = \mathcal{M}\).

If \(\mathcal{H}\) denotes the family of all sequences of functions with properties described above, then

\[
\bigcap_{(f_n) \in \mathcal{H}} \mathcal{M}(f_n) = S.
\]

Now, we may ask several questions. We would like to know for which sequences we get microscopic sets, when different sequences give different families of sets, under which condition \(\mathcal{M}(f_n)\) is a \(\sigma\)-ideal.

The next theorem gives a sufficient condition for getting microscopic sets.

**Theorem 4.** If there exists \(k \in \mathbb{N}\) such that for every \(n \in \mathbb{N}\) and for every \(x \in (0, 1)\) we have \(x^{n+k} \leq f_n(x) \leq x^n\), then

\[
\mathcal{M}(f_n) = \mathcal{M}.
\]

**Proof.** Since the inclusion \(\mathcal{M}(f_n) \subseteq \mathcal{M}\) is obvious, we have to justify only the inverse one.

Let \(E \in \mathcal{M}\). Fix \(x \in (0, 1)\). Since \(E \in \mathcal{M}\) for \(x_0 = x^{k+1}\), there exists a sequence of intervals \(\{I_n\}_{n \in \mathbb{N}}\) such that

\[
E \subseteq \bigcup_{n \in \mathbb{N}} I_n \quad \text{and} \quad \lambda(I_n) < (x_0)^n \quad \text{for} \quad n \in \mathbb{N}.
\]

Therefore

\[
\lambda(I_n) < (x^{k+1})^n = x^{nk+n} \leq x^{n+k} \leq f_n(x), \quad \text{so} \quad E \in \mathcal{M}(f_n). \quad \square
\]

The above condition is not necessary, for example, for a sequence \(f_n(x) = x^{2n}, n \in \mathbb{N}\), the assumption of the above theorem is not satisfied, but \(\mathcal{M}(f_n) = \mathcal{M}\).

More examples of sequences of functions leading to microscopic sets can be given for:

\[
f_n(x) = \left(\frac{x}{a}\right)^n, \quad \text{where} \quad a > 0,
\]

\[
f_n(x) = x^{an}, \quad \text{where} \quad a > 0,
\]

\[
f_n(x) = \frac{x^n}{n^a}, \quad \text{where} \quad a \geq 1,
\]

the family \(\mathcal{M}(f_n)\) is exactly the family of microscopic sets.
MICROSCOPIC SETS WITH RESPECT TO SEQUENCES OF FUNCTIONS

For reader’s convenience we show that in the last case every microscopic set belongs to $M(f_n)$. Let $A \in M$. Fix $x \in (0, 1)$ and find $n_0 \in \mathbb{N}$ such that $1/n_0 < x$. For $\varepsilon := \frac{1}{n_0^a}$ there exists a sequence of intervals $\{I_n\}_{n \in \mathbb{N}}$ such that

$$E \subset \bigcup_{n \in \mathbb{N}} I_n \quad \text{and} \quad \lambda(I_n) \leq \varepsilon^n \quad \text{for} \quad n \in \mathbb{N}.$$ 

Then

$$\lambda(I_n) \leq \varepsilon^n = \frac{1}{(n_0^a+1)^n} = \frac{1}{(n_0^a)^n n_0^n} < \frac{1}{(n_0^n a^n x^n < \frac{x^n}{n^n}}$$

since $n_0^n > n$ for $n_0 > 1$. Therefore, $a \in M(f_n)$.

One can observe an obvious sufficient condition for an inclusion between the families $M(g_n)$ and $M(f_n)$ for sequences $(f_n)_{n \in \mathbb{N}}$, $(g_n)_{n \in \mathbb{N}}$ from the family $H$:

$$\forall x \in (0, 1) \quad \forall n \in \mathbb{N} \quad g_n(x) \leq f_n(x) \Rightarrow M(g_n) \subset M(f_n).$$

The next theorem gives a not so obvious sufficient condition.

**Theorem 5.** If for every $x \in (0, 1)$ there exists $y \in (0, 1)$ such that there exists a sequence $(P_m)_{m \in \mathbb{N}}$ of pairwise disjoint, nonempty subsets of $\mathbb{N}$ such that

$$g_m(y) \leq \sum_{i \in P_m} f_i(x) \quad \text{for every} \quad m \in \mathbb{N},$$

then

$$M(g_n) \subset M(f_n).$$

**Proof.** Let $E \in M(g_n)$. Let $x \in (0, 1)$. By our assumption, there exists $y \in (0, 1)$ and a sequence $(P_m)$ of pairwise disjoint nonempty subsets of $\mathbb{N}$ such that $g_m(y) \leq \sum_{i \in P_m} f_i(x)$. Since $E \in M(g_n)$, for $y$ there exists a sequence of intervals $(I_m)_{m \in \mathbb{N}}$ such that

$$E \subset \bigcup_{m \in \mathbb{N}} I_m \quad \text{and} \quad \lambda(I_m) \leq g_m(y).$$

Since $\lambda(I_m) \leq g_m(y) \leq \sum_{i \in P_m} f_i(x)$, we may divide the interval $I_m$ into nonoverlapping intervals $J_i^{(m)}$, $i \in P_m$, such that

$$\lambda(J_i^{(m)}) \leq f_i(x) \quad \text{for} \quad i \in P_m.$$

Let $n \in \mathbb{N}$. If $n$ belongs to none of $P_m$, $m \in \mathbb{N}$, then $J_n := \emptyset$. If $n \in P_m$, then $J_n := J_n^{(m)}$. Therefore,

$$E \subset \bigcup_{m \in \mathbb{N}} I_m = \bigcup_{n \in \mathbb{N}} J_n \quad \text{and} \quad \lambda(J_n) \leq f_n(x), \quad \text{so} \quad E \in M(f_n).$$

□
Theorem 6. \( \mathcal{M}(f_n) \setminus S \neq \emptyset \) for every \((f_n)_{n \in \mathbb{N}} \in \mathcal{H}\).

Proof.

Let \( I := [0,1] \). We will define by induction the sequence of open intervals \( \{J_{n,i}\}, i \in \{1, \ldots, 2^{n-1}\}, n \in \mathbb{N} \), in the following way.

Let \( k_1 := \min \{k : f_{2^k}(\frac{1}{k+1}) < \frac{1}{3}\} \). Put \( J_{1,1} := \left( f_{2^{k_1}}(\frac{1}{k_1+1}), 1 - f_{2^{k_1}}(\frac{1}{k_1+1}) \right) \).

Then, of course, \( \lambda(J_{1,1}) > \frac{1}{3} \).

Let \( K_{1,1}, K_{1,2} \) denote successive components of the set \( I \setminus J_{1,1} \). Obviously, \( \lambda(K_{1,i}) = f_{2^{k_1}}(\frac{1}{k_1+1}) \) for \( i \in \{1, 2\} \).

Let \( k_2 := \min \{k : f_{2^k}(\frac{1}{k+1}) < \frac{1}{3}f_{2^{k_1}}(\frac{1}{k_1+1})\} \).

Let \( J_{2,1}, J_{2,2} \) be two open intervals concentric with \( K_{1,1} \) and \( K_{1,2} \), respectively, such that \( \lambda(J_{2,1}) = \lambda(J_{2,2}) = \lambda(K_{1,1}) - 2f_{2^{k_2}}(\frac{1}{k_2+1}) \).

Let \( K_{2,1}, K_{2,2}, K_{2,3}, \) and \( K_{2,4} \) denote successive components of the set \( I \setminus \left( J_{1,1} \cup J_{2,1} \cup J_{2,2} \right) \). Notice that \( \lambda(K_{2,i}) = f_{2^{k_2}}(\frac{1}{k_2+1}) \) for \( i \in \{1, 2, 3, 4\} \).

Let \( m \geq 2 \). Assume that we have constructed the open, nonempty intervals \( J_{l,1}, \ldots, J_{l,2^{l-1}} \) concentric with \( K_{l-1,1}, \ldots, K_{l-1,2^{l-1}} \), respectively, such that \( \lambda(J_{l,i}) = \lambda(K_{l-1,1}) - 2f_{2^{k_l}}(\frac{1}{k_l+1}) \) for \( l \in \{2, \ldots, m\}, i \in \{1, \ldots, 2^{l-1}\} \), where \( k_l := \min \{k : f_{2^k}(\frac{1}{k+1}) < \frac{1}{3}f_{2^{k_{l-1}}}(\frac{1}{k_{l-1}+1})\} \), for \( l \in \{2, \ldots, m\} \).

Let \( K_{m,1}, \ldots, K_{m,2^m} \) be successive components of the set \( I \setminus \bigcup_{l=1}^{m} \bigcup_{i=1}^{2^{l-1}} J_{l,i} \).

Notice that \( \lambda(K_{m,i}) = f_{2^{k_m}}(\frac{1}{k_m+1}) \) for \( i \in \{1, \ldots, 2^m\} \).

Now, let \( k_{m+1} := \min \{k : f_{2^k}(\frac{1}{k+1}) < \frac{1}{3}f_{2^{k_m}}(\frac{1}{k_m+1})\} \) and \( J_{m+1,1}, \ldots, J_{m+1,2^m} \) be open intervals concentric with \( K_{m,1}, \ldots, K_{m,2^m} \), respectively, such that \( \lambda(J_{m+1,i}) = \lambda(K_{m,1}) - 2f_{2^{k_{m+1}}}(\frac{1}{k_{m+1}+1}) \) for \( i \in \{1, \ldots, 2^m\} \).

Let \( K_{m+1,1}, \ldots, K_{m+1,2^m} \) be successive components of the set

\[
I \setminus \bigcup_{l=1}^{m+1} \bigcup_{i=1}^{2^{l-1}} J_{l,i}.
\]

Obviously, \( \lambda(K_{m+1,i}) = f_{2^{k_{m+1}}}(\frac{1}{k_{m+1}+1}) \) for \( i \in \{1, \ldots, 2^{m+1}\} \).

Let us put \( M := \bigcap_{m \in \mathbb{N}} \bigcup_{i=1}^{2^m} K_{m,i} \).

Now, let \( x \in (0,1) \). There exists \( m_0 \in \mathbb{N} \) such that \( \frac{1}{k_{m_0}} < x \).

Let \( I_i := K_{m_0,i} \) for \( i \in \{1, \ldots, 2^{m_0}\} \) and \( I_i := \emptyset \) for \( i > 2^{m_0} \). Obviously, \( M \subset \bigcup_{i=1}^{2^{m_0}} K_{m_0,i} = \bigcup_{i \in \mathbb{N}} I_i \) and \( \lambda(I_i) = \lambda(K_{m_0,i}) = f_{2^{k_{m_0}}}(\frac{1}{k_{m_0}+1}) \leq f_{2^{m_0}}(\frac{1}{k_{m_0}+1}) < f_{2^{m_0}}(x) \leq f_i(x) \) for \( i \in \{1, \ldots, 2^{m_0}\} \). Hence, \( M \) is a Cantor-type set from \( \mathcal{M}(f_n) \).

As a perfect set, \( M \) cannot be a strong measure zero set (compare [5] Corollary 8.1.5), so \( M \in \mathcal{M}(f_n) \setminus S \).
Theorem 7. Let \((f_n), (g_n) \in \mathcal{H}\) and let \(k_n\) be a sequence of natural numbers chosen as in the proof of the previous theorem. Suppose that there exists \(\delta > 0\) such that \(g_n(x) < f_n(x)\) for every \(n \in \mathbb{N}\) and for every \(x \in (0, \delta)\). If there exists a point \(x_0 \in (0, 1)\) such that \(g_n(x_0) < f_{2^{k_n}}(\frac{1}{k_n + 1})\), for every \(n \in \mathbb{N}\), then
\[
\mathcal{M}_{(g_n)} \not\subset \mathcal{M}_{(f_n)}. 
\]

Proof. Since the inclusion \(\mathcal{M}_{(g_n)} \subset \mathcal{M}_{(f_n)}\) is obvious, we only need to show that these families are different.

Consider the set \(M\) from the proof of the previous theorem.

Let \(\{I_n\}_{n \in \mathbb{N}}\) be any sequence of intervals such that \(\lambda(I_n) \leq g_n(x_0)\).

As \(\lambda(I_1) \leq g_1(x_0) < f_{2^{k_1}}(\frac{1}{2})\), \(I_1\) cannot have common points with both sets \(K_{1,1}\) and \(K_{1,2}\), since \(f_{2^{k_1}}(\frac{1}{2}) < \frac{1}{3} < \lambda(J_{1,1}) = \text{dist}(K_{1,1}, K_{1,2})\).

Put \(P_0 := [0, 1]\).

Let \(P_1 := K_{1,i_1}\), where \(i_1 \in \{1, 2\}\) and \(I_1 \cap K_{1,i_1} = \emptyset\).

Let \(n \geq 2\). We assume that for \(l \in \{1, \ldots, n-1\}\) we have already chosen an interval \(P_l\), such that \(P_l := K_{l,i_l}\), where \(i_l \in \{1, \ldots, 2^l\}\), \(I_l \cap K_{l,i_l} = \emptyset\) and \(P_l \subset P_{l-1}\).

The interval \(I_n\) satisfies a condition
\[
\lambda(I_n) \leq g_n(x_0) < f_{2^{k_n}}\left(\frac{1}{k_n + 1}\right) = \lambda(K_{n,i})
\]
for \(i \in \{1, \ldots, 2^n\}\) and by the construction of \(M\), precisely by the way of defining \(k_n\), we have \(\lambda(K_{n,i}) < \lambda(J_{l,j})\), where \(J_{l,j}\) is a gap between the intervals \(K_{n,i}\) and \(K_{n,i'}\) contained in \(P_{n-1}\), so one of them has no common points with \(I_n\). We denote it by \(P_n\).

Therefore, we have inductively constructed a descending sequence of closed intervals \(P_n\), such that \(\lambda(P_n) = f_{2^{k_n}}(\frac{1}{k_n + 1})\) and \(P_n \cap I_n = \emptyset\) for \(n \in \mathbb{N}\). By Cantor Theorem, there exists a point \(x \in \bigcap_{n \in \mathbb{N}} P_n\). Of course \(x \in M\) and \(x \notin \bigcup_{n \in \mathbb{N}} I_n\), so \(M \not\subset \mathcal{M}_{(g_n)}\).

Theorem 8. For every \((f_n) \in \mathcal{H}\), there exists \((g_n) \in \mathcal{H}\) such that
\[
S \subset \mathcal{M}_{(g_n)} \subset \mathcal{M}_{(f_n)}. 
\]

Proof. Suffice it to put \(g_n(x) := f_n(x)f_{2^{k_n}}(\frac{1}{k_n + 1})\), for \(n \in \mathbb{N}\), where \(k_n\) is chosen as in the proof of Theorem 6. Then \((g_n) \in \mathcal{H}\) and \((g_n)\) satisfies the condition from Theorem 7 since for every \(x \in (0, 1)\)
\[
g_n(x) = f_n(x)f_{2^{k_n}}\left(\frac{1}{k_n + 1}\right) < f_{2^{k_n}}\left(\frac{1}{k_n + 1}\right). 
\]

Therefore, by Theorem 6 and Theorem 7 we are done.
Remark 9. If \( f_2 \left( \frac{1}{2} \right) < \frac{1}{3} \) and \( \frac{f_2^{(l+1)}}{f_2^{l}} < \frac{1}{3} \) for \( l \geq 2 \), then \( k_n = n \) and for such a sequence \( (f_n) \) the condition from the Theorem 7 is as follows

\[
\exists x_0 \in (0,1) \forall n \in \mathbb{N} \; g_n(x_0) < f_2^n \left( \frac{1}{n + 1} \right). \tag{\ast}
\]

The sequence \( f_n(x) = x^n \) satisfies assumptions from the above remark, so for the sequence \( g_n(x) = \frac{x^n}{(n+1)^2} \) (see the proof of Theorem 8), we have

\[
\mathcal{M}(g_n) \not\subseteq \mathcal{M}.
\]

Another sequence satisfying condition (\ast) for \( f_n(x) = x^n \) is the sequence \( g_n(x) = x^{n^2} \), more precisely, for \( x_0 = \frac{1}{4} \) and for every \( n > 2 \), we have

\[
g_n(x_0) < \left( \frac{1}{n + 1} \right)^{2^n}, \quad \text{so} \quad \mathcal{M}(x^{n^2}) \not\subseteq \mathcal{M}.
\]

The next theorem gives a sufficient condition for a family \( \mathcal{M}(f_n) \) to be a \( \sigma \)-ideal.

**Theorem 10.** Let \( f_n \in \mathcal{H} \) for \( n \in \mathbb{N} \). If there exists a bijection \( \Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that for a fixed \( k \in \mathbb{N} \) a function \( \Phi_k(n) := \Phi(n,k) \) is increasing and for every \( x > 0 \) and for every \( k \in \mathbb{N} \) \( \lim_{n \to +\infty} f_n^{-1} \left( f_{\Phi(n,k)}(x) \right) > 0 \), then \( \mathcal{M}(f_n) \) is a \( \sigma \)-ideal.

**Proof.** Suffice it to show that a countable sum of sets from \( \mathcal{M}(f_n) \) belongs to \( \mathcal{M}(f_n) \). Let \( \{A_k\}_{k \in \mathbb{N}} \) be a sequence of sets, such that \( A_k \in \mathcal{M}(f_n) \) for \( k \in \mathbb{N} \). Fix \( k \in \mathbb{N} \) and \( x > 0 \). A sequence \( \{f_n^{-1}(f_{\Phi(n,k)}(x))\}_{n \in \mathbb{N}} \) is monotonous, so denoting \( a_k := \lim_{n \to +\infty} f_n^{-1}(f_{\Phi(n,k)}(x)) \), we have \( f_n^{-1}(f_{\Phi(n,k)}(x)) \geq a_k > \frac{a_k}{2} \) for every \( n \in \mathbb{N} \). By monotonocity of \( f_n \), we get

\[
f_{\Phi(n,k)}(x) \geq f_n \left( \frac{a_k}{2} \right).
\]

For \( \frac{a_k}{2} \) there exists a cover \( \{I_n^{(k)}\}_{n \in \mathbb{N}} \) of the set \( A_k \) such that \( \lambda(I_n^{(k)}) < f_n \left( \frac{a_k}{2} \right) \). Arrange intervals \( \{I_n^{(k)}\}_{n \in \mathbb{N}, k \in \mathbb{N}} \) in a sequence \( \{I_m\}_{m \in \mathbb{N}} \), where \( m = \Phi(n,k) \). Then \( \bigcup_{k \in \mathbb{N}} A_k \subset \bigcup_{m \in \mathbb{N}} I_m \) and

\[
\lambda(I_m) = \lambda(I_{\Phi(n,k)}) = \lambda(I_n^{(k)}) < f_n \left( \frac{a_k}{2} \right) \leq f_{\Phi(n,k)}(x) = f_m(x).
\]

Therefore, \( \bigcup_{k \in \mathbb{N}} A_k \in \mathcal{M}(f_n) \). \( \square \)

It is known that if \( \mathcal{I} \) is a \( \sigma \)-ideal containing all singletons and \( \mathcal{G}_\delta \)-generated then a family \( \mathcal{I}^* := \{ A \subset \mathbb{R} : \exists B \in \mathcal{F} \cap \mathcal{I} A \subset B \} \) is also a \( \sigma \)-ideal ([4]). We do not know if \( \mathcal{M}(x^{2^n}) \) is a \( \sigma \)-ideal, but we can proove that \( \mathcal{M}^*(x^{2^n}) \)—a family of sets which can be covered by \( F_\sigma \) sets from \( \mathcal{M}(x^{2^n}) \) is a \( \sigma \)-ideal.
**Lemma 11.** Let $A_k \in \mathcal{M}_{(x^{2^n})}$ and $A_k$ be compact for every $k \in \mathbb{N}$. Then,

$$
\bigcup_{k \in \mathbb{N}} A_k \in \mathcal{M}_{(x^{2^n})}.
$$

**Proof.** Let $x \in (0, 1)$. For $x_1 := x$ there exists a cover $\{I_n^{(1)}\}$ of $A_1$ such that $\lambda(I_n^{(1)}) < (x_1)^{2^n}$ for every $n \in \mathbb{N}$. By compactness of $A_1$, we can choose a finite subcover $\{I_n^{(1)}\}_{n=1}^{n_1}$. For $x_2 := x^{2^{n_1}}$ there exists a cover $\{I_n^{(2)}\}$ of $A_2$ such that $\lambda(I_n^{(2)}) < (x_2)^{2^n}$ for every $n \in \mathbb{N}$. By compactness of $A_2$, we can choose a finite subcover $\{I_n^{(2)}\}_{n=1}^{n_2}$. And so on, for $x_k := (x_{k-1})^{2^{n_{k-1}}}$ we find a finite cover $\{I_n^{(k)}\}_{n=1}^{n_k}$ of $A_k$ such that $\lambda(I_n^{(k)}) < (x_k)^{2^n}$ for every $n \in \mathbb{N}$.

Now, for every $k \in \mathbb{N}$, there exists $t \in \mathbb{N}$ such that $\sum_{i=0}^{t} n_i < k \leq \sum_{i=0}^{t+1} n_i$, where $n_0 := 0$, and there exists $l \in \{1, \ldots, n_{t+1}\}$ such that $k = n_0 + \cdots + n_t + l$. Let $J_k = I_l^{(t+1)}$. Then, $\bigcup_{k \in \mathbb{N}} A_k \subset \bigcup_{k \in \mathbb{N}} J_k$ and

$$
\lambda(J_k) = \lambda(I_l^{(t+1)}) < (x_{t+1})^{2^l} = (x^{2^{n_1+n_2+\cdots+n_l}})^{2^l} = x^{2^{n_1+n_2+\cdots+n_l+t}} = x^k,
$$

so $\bigcup_{k \in \mathbb{N}} A_k \in \mathcal{M}_{(x^{2^n})}$. □

**Lemma 12.** If $A_n \in \mathcal{M}^*_{(x^{2^n})}$ for every $n \in \mathbb{N}$, then there exists a sequence of compact sets $\{D_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^*_{(x^{2^n})}$ such that

$$
\bigcup_{k \in \mathbb{N}} A_n \subset \bigcup_{k \in \mathbb{N}} D_k.
$$

**Proof.** Since $A_n \in \mathcal{M}^*_{(x^{2^n})}$ for every $n \in \mathbb{N}$ then, for every $n \in \mathbb{N}$, there exists $B_n \in \mathcal{F}_\sigma \cap \mathcal{M}_{(x^{2^n})}$ such that $A_n \subset B_n$. Then, $B_n = \bigcup_{l \in \mathbb{N}} B_l^{(n)}$, where $B_l^{(n)}$ is closed, hence it is a countable union of compact sets $D_t^{(l, n)}$ for $t \in \mathbb{N}$, so

$$
\bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{n \in \mathbb{N}} \bigcup_{l \in \mathbb{N}} D_t^{(l, n)} = \bigcup_{k \in \mathbb{N}} D_k.
$$

Obviously, $\{D_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^*_{(x^{2^n})}$. □

**Theorem 13.** $\mathcal{M}^*_{(x^{2^n})}$ is a $\sigma$-ideal.

**Proof.** It suffices to show that $\mathcal{M}^*_{(x^{2^n})}$ is closed under a countable union. Let $A_n \in \mathcal{M}^*_{(x^{2^n})}$ for $n \in \mathbb{N}$. By Lemma 12 there exists a sequence $\{D_k\}_{k \in \mathbb{N}}$ such that $D_k \in \mathcal{M}_{(x^{2^n})}$ and $D_k$ is compact for $\bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{k \in \mathbb{N}} D_k$ and for every $k \in \mathbb{N}$. By Lemma 11 we have $\bigcup_{k \in \mathbb{N}} D_k \in \mathcal{M}_{(x^{2^n})}$ and, of course, $\bigcup_{k \in \mathbb{N}} D_k \in \mathcal{F}_\sigma$, so $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}^*_{(x^{2^n})}$. □
REFERENCES


Received January 2, 2014

Department of Mathematics and Computer Science
University of Łódź
Banacha 22
PL–90-238 Łódź
POLAND
E-mail: grhorb@math.uni.lodz.pl