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A NOTE ON THE [0]-LOWER CONTINUOUS FUNCTIONS

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ABSTRACT. We present some properties of [0]-lower continuous functions. We give an equivalent condition of [0]-lower continuity and find maximal additive family and maximal multiplicative family for the class of [0]-lower continuous functions.

1. Preliminaries

In the paper, we apply standard symbols and notations. By $\mathbb{R}$ we denote the set of all real numbers, by $\mathbb{N}$ we denote the set of all positive integers. By $\mathcal{L}$ we denote the family of measurable in sense of Lebesgue subsets of real line. The symbol $|\cdot|$ stands for the Lebesgue measure on $\mathbb{R}$. Throughout the paper, $I = (a, b)$ denotes an open interval (not necessarily bounded) and f is a real-valued function defined on I. By $\mathcal{A}$ we denote the class of approximately continuous functions.

Let E be a measurable subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. According to [2], the numbers

$$\underline{d}^+(E, x) = \liminf_{t \to 0^+} \frac{|E \cap [x, x+t]|}{t} \quad \text{and} \quad \overline{d}^+(E, x) = \limsup_{t \to 0^+} \frac{|E \cap [x, x+t]|}{t}$$

are called the right lower density of E at x and right upper density of E at x, respectively. The left lower and upper densities of E at x are defined analogously. If

$$\underline{d}^+(E, x) = \overline{d}^+(E, x) \qquad \big(\underline{d}^-(E, x) = \overline{d}^-(E, x)\big),$$

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then we call these numbers the right density (left density) of E at x and denote them by $d^+(E,x)$ ($d^-(E,x)$). The numbers

$$\overline{d}(E,x)=\limsup_{\substack{t\to 0^+\\ k\to 0^+}}\frac{|E\cap[x-t,x+k]|}{k+t}\quad\text{and}\quad \underline{d}(E,x)=\liminf_{\substack{t\to 0^+\\ k\to 0^+}}\frac{|E\cap[x-t,x+k]|}{k+t}$$

are called the upper and lower density of E at x, respectively. It is clear that $\overline{d}(E,x)=\max\bigl\{\overline{d}^+(E,x),\overline{d}^-(E,x)\bigr\}$ and $\underline{d}(E,x)=\min\bigl\{\underline{d}^+(E,x),\underline{d}^-(E,x)\bigr\}$.

If $\overline{d}(E,x)=\underline{d}(E,x)$, we call this number the density of E at x and denote it by $d(E,x)$.

Let us recall the definition of $[\lambda,\varrho]$-continuous function.

Definition 1.1 ([7])**.** Let $E\in\mathcal{L}$, $x\in\mathbb{R}$ and $0<\lambda\le\varrho\le 1$, $\lambda<1$. We say that x is a point of $[\lambda,\varrho]$-type density of E, if

$$\underline{d}(E,x)>\lambda\quad\text{and}\quad\overline{d}(E,x)>\varrho\quad\text{when}\quad\lambda<1\quad\text{and}\quad\varrho<1$$

or

$$\underline{d}(E,x)>\lambda\quad\text{and}\quad\overline{d}(E,x)=\varrho\quad\text{when}\quad\lambda<1\quad\text{and}\quad\varrho=1.$$

Definition 1.2 ([7])**.** A real-valued function f defined on an open interval I is called $[\lambda,\varrho]$-continuous at $x\in I$ provided that there is a measurable set $E\subset I$ such that x is a point of $[\lambda,\varrho]$-density of E, $x\in E$ and $f|E$ is continuous at x. If f is $[\lambda,\varrho]$-continuous at every point of I, we simply say that f is $[\lambda,\varrho]$-continuous.

We will denote the class of $[\lambda,\varrho]$-continuous functions by $\mathcal{C}_{[\lambda,\varrho]}$.

In [7], an equivalent condition of $\mathcal{C}_{[\lambda,\varrho]}$-continuity was proved.

Theorem 1.1 ([7])**.** *Let $0<\lambda\le\varrho<1$, $x_0\in I$ and let $f\colon I\to\mathbb{R}$ be a measurable function. Then f is $[\lambda,\varrho]$-continuous at x_0 if and only if*

$$\lim_{\varepsilon\to 0^+}\underline{d}\Bigl(\bigl\{x\in I\colon |f(x)-f(x_0)|<\varepsilon\bigr\},x_0\Bigr)>\lambda$$

and

$$\lim_{\varepsilon\to 0^+}\overline{d}\Bigl(\bigl\{x\in I\colon |f(x)-f(x_0)|<\varepsilon\bigr\},x_0\Bigr)>\varrho.$$

We will need the following technical lemma from [6].

Lemma 1.1 ([6, Lemma 2.3])**.** *Let F be a measurable set and let $x\in\mathbb{R}$. There exists a sequence of closed intervals $\bigl\{I_n=[a_n,b_n]\colon x<\dots<b_{n+1}<a_n<\cdots\bigr\}$ such that*

$$\overline{d}^+\Bigl(F\setminus\bigcup_{n=1}^{\infty}I_n,x\Bigr)=\overline{d}^+\Bigl(\bigcup_{n=1}^{\infty}I_n\setminus F,x\Bigr)=0.$$

Now, we will give a basic definition of the present paper.

Definition 1.3. A real-valued function $f\colon I \to \mathbb{R}$ is called [0]-lower continuous at $x \in I$ if there exists $\lambda_x > 0$ such that f is $[\lambda_x, \lambda_x]$-continuous at x. If f is [0]-lower continuous at every point of I, we simply say that f is [0]-lower continuous.

We will denote the class of [0]-lower continuous functions by $\mathcal{C}_{[0]}$.

Theorem 1.2. *Let $f\colon I \to \mathbb{R}$ be a measurable function and let $x_0 \in I$. The following conditions are equivalent:*

i) *function f is [0]-lower continuous at x_0,*

ii) *there exists measurable set $E \subset I$ such that $x_0 \in E$, $f|E$ is continuous at x_0 and $\underline{d}(E, x_0) > 0$,*

iii) $\lim_{\varepsilon\to 0^+} \underline{d}\big(\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\}, x_0\big) > 0$.

Proof. Assume that f is [0]-lower continuous at x_0. There exists $\lambda > 0$ such that function f is $[\lambda, \lambda]$-continuous at x_0. So, we can find a measurable set $E \subset I$ such that $x_0 \in E$, $f|E$ is continuous at x_0 and $\underline{d}(E, x_0) > \lambda > 0$.

Assume that there exists a measurable set $E \subset I$ such that $x_0 \in E$, $f|E$ is continuous at x_0 and $\underline{d}(E, x_0) > 0$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \cap E \subset \big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}$. Hence,

$$\underline{d}\Big(\big\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) \geq \underline{d}\Big(\big\{x \in E\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) = \underline{d}(E, x_0) \quad \text{for every } \varepsilon > 0.$$

Therefore,

$$\lim_{\varepsilon\to 0^+} \underline{d}\Big(\big\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) \geq \underline{d}(E, x_0) > 0.$$

Now, suppose that

$$\lim_{\varepsilon\to 0^+} \underline{d}\big(\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\}, x_0\big) > 0.$$

There exists $\lambda > 0$ such that

$$\lim_{\varepsilon\to 0^+} \underline{d}\big(\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\}, x_0\big) > \lambda.$$

From Theorem 1.1, we conclude that f is $[\lambda, \lambda]$-continuous at x_0. Hence f is [0]-lower continuous at x_0. □

Example 1.1. We shall show that there exists $f \in \mathcal{C}_{[0]} \setminus \bigcup_{0<\lambda\leq\varrho\leq 1, \lambda<1} \mathcal{C}_{[\lambda,\varrho]}$.

Let $\{x_n\}_{n\geq 1}$ be a sequence of points from I such that $\lim_{n\to\infty} x_n = b$ and $x_{n+1} > x_n$ for every $n \geq 1$. We can find a sequence $\big\{J_n = [p_n, q_n]\big\}_{n\geq 1} \subset (a, b)$ of pairwise disjoint closed intervals, for which $x_n \in (p_n, q_n)$.

For each $n \in \mathbb{N}$ there exists a sequence of closed intervals $\{[a_m^n, b_m^n]\}_{m\geq 1}$ such that $x_n < b_{m+1}^n < a_m^n < b_m^n$ and $[a_m^n, b_m^n] \subset J_n$ for every $m \geq 1$ and

$$\underline{d}^+\left(\bigcup_{m=1}^{\infty}[a_m^n, b_m^n], x_n\right) = \frac{1}{n}.$$

For each $n \geq 1$ there exists a sequence of pairwise disjoint closed intervals $\{[c_m^n, d_m^n]\}_{m\geq 1}$ such that $[c_m^n, d_m^n] \subset J_n$ and $[a_m^n, b_m^n] \subset (c_m^n, d_m^n)$ for every $m \geq 1$ and

$$\overline{d}^+\left(\bigcup_{m=1}^{\infty}\Big([c_m^n, d_m^n] \setminus [a_m^n, b_m^n]\Big), x_n\right) = 0.$$

Let $I_m^n = [a_m^n, b_m^n]$ and $K_m^n = [c_m^n, d_m^n]$ for every $m \geq 1$.

Finally, for every $n \in \mathbb{N}$ take any $y_n \in (p_n, x_n)$.

Define $f\colon (a,b) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{for } x \in \bigcup_{n=1}^{\infty}\big([y_n, x_n] \cup \bigcup_{m=1}^{\infty} I_m^n\big), \\ 1 & \text{for } x \in \big((a,b) \setminus \bigcup_{n=1}^{\infty} J_n\big) \cup \Big(\bigcup_{n=1}^{\infty}\big((x_n, d_1^n] \setminus \bigcup_{m=1}^{\infty} K_m^n\big)\Big), \\ \multicolumn{2}{l}{\text{linear on the intervals } [c_m^n, a_m^n], [b_m^n, d_m^n], [p_n, y_n], [d_1^n, q_n], n, m \geq 1.} \end{cases}$$

Then, f is continuous at every point except at $x_1, x_2, \ldots$ and constant on every set

$$E_n = \left([y_n, x_n] \cup \bigcup_{m=1}^{\infty} I_m^n\right).$$

Since $\underline{d}(E_n, x_n) = \frac{1}{n} > 0$, f is $C_{[0]}$-continuous at $x_1, x_2, \ldots$ Hence, $f \in C_{[0]}$.

Let λ, ϱ be any real numbers such that $0 < \lambda \leq \varrho \leq 1$ and $\lambda < 1$. There exists n_0 such that $\frac{1}{n_0} < \lambda$. Then

$$\underline{d}\Big(\big\{x \in J_{n_0} : |f(x) - f(x_{n_0})| < 1\big\}, x_{n_0}\Big) \leq \underline{d}^+\left(\bigcup_{m=1}^{\infty} K_m^{n_0}, x_{n_0}\right) \leq$$

$$\underline{d}^+\left(\bigcup_{m=1}^{\infty} I_m^{n_0}, x_{n_0}\right) + \overline{d}^+\left(\bigcup_{m=1}^{\infty}\left(K_m^{n_0} \setminus I_m^{n_0}\right), x_{n_0}\right) = \tfrac{1}{n_0} + 0 < \lambda.$$

Hence $f \notin \mathcal{C}_{[\lambda,\varrho]}$ and $f \notin \bigcup_{0<\lambda\leq\varrho\leq 1, \lambda<1} \mathcal{C}_{[\lambda,\varrho]}$.

Corollary 1.1. $\mathcal{C}_{[0]} \supsetneq \bigcup_{0<\lambda\leq\varrho<1} \mathcal{C}_{[\lambda,\varrho]}$.

Remark 1.1. It seems that, in the same way as in [1, Theorem 4], one can prove that the set $\bigcup_{0<\lambda\leq\varrho<1} \mathcal{C}_{[\lambda,\varrho]}$ is even nowhere dense in $\mathcal{C}_{[0]}$.

2. Basic results

Theorem 2.1. *If $f \in \mathcal{C}_{[0]}$, then f is measurable.*

Proof. Let $f\colon I \to \mathbb{R}$, $f \in \mathcal{C}_{[0]}$ and suppose that f is not measurable. There exists a number $a \in \mathbb{R}$ for which at least one of the sets $\{x \in I\colon f(x) < a\}$, $\{x \in I\colon f(x) > a\}$ is non-measurable. We may assume that the $\{x \in I\colon f(x) < a\}$ is non-measurable. Let $A = \{x \in I\colon f(x) < a\}$ and $B = \{x \in I\colon f(x) \geq a\}$. Then $B = I \setminus A$ is also non-measurable. There exist measurable sets $A_1 \subset A$, $B_1 \subset B$ such that $A \setminus A_1$ and $B \setminus B_1$ do not contain any measurable set of positive measure. Therefore $A \setminus A_1$ and $B \setminus B_1$ are non-measurable. Moreover,

$$F = (A \setminus A_1) \cup (B \setminus B_1) = I \setminus (A_1 \cup B_1)$$

is measurable.Let $L(F)$ be a set of all density points of a set F. Since $|F \setminus L(F)| = 0$, there exists $x_0 \in (A \setminus A_1) \cap L(F)$.

It follows that there exists a measurable set $E \subset I$ such that $x_0 \in E$, $\underline{d}(E, x_0) > 0$ and $f|E$ is continuous at x_0, because f is 0-lower continuous at x_0. As $x_0 \in A$, we have $f(x_0) < a$. Therefore it is possible to find $\delta > 0$ such that $E \cap (x_0 - \delta, x_0 + \delta) \subset A$. Let $E' = E \cap (x_0 - \delta, x_0 + \delta)$. Hence $x_0 \in E'$, $f|E'$ is continuous at x_0, $E' \subset A$ and

$$\underline{d}(E', x_0) = \underline{d}(E, x_0) > 0. \tag{$\mathcal{R}$}$$

We have

$$E' = (E' \cap A_1) \cup \big(E' \cap (A \setminus A_1)\big).$$

Since E' and $E' \cap A_1$ are measurable, $E' \cap (A \setminus A_1)$ is also measurable. Hence, $|E' \cap (A \setminus A_1)| = 0$. Moreover,

$$\underline{d}(E' \cap A_1, x_0) = 1 - \overline{d}\big(I \setminus (E' \cap A_1), x_0\big) \leq 1 - \overline{d}(F, x_0) = 1 - 1 = 0.$$

Therefore,

$$\begin{aligned} \underline{d}(E', x_0) &= \underline{d}\Big(\big(E' \cap A\big) \cup \big(E' \cap (A \setminus A_1), x_0\big)\Big) \\ &\leq \underline{d}(E' \cap A, x_0) + \overline{d}\big(E' \cap (A \setminus A_1), x_0\big) \\ &= 0 + 0 = 0, \end{aligned}$$

contradicting to $(\mathcal{R})$. □

Applying Proposition 7 from [1], we see that $C_{[0]}$ is not closed under the uniform limit.

Theorem 2.2. *Let a sequence $\{f_n\}_{n \geq 1}$ of measurable functions $f_n\colon I \to \mathbb{R}$ be uniformly convergent to f, $f\colon I \to \mathbb{R}$ and let $x_0 \in I$. Then f is $[0]$-lower continuous at x_0 if and only if*

$$\inf_{\delta > 0} \liminf_{k \to \infty} \underline{d}\Big(\big\{x \in I\colon |f_k(x) - f_k(x_0)| < \delta\big\}, x_0\Big) > 0. \tag{1}$$

P r o o f. Let
$$\alpha = \inf_{\delta>0} \liminf_{k\to\infty} \underline{d}\Big(\big\{x \in I\colon |f_k(x) - f_k(x_0)| < \delta\big\}, x_0\Big) > 0.$$
Take any $\varepsilon > 0$. There exists $n_0 \geq 1$ such that for every $k > n_0$ and every $x \in I$, the inequality
$$|f_k(x) - f(x)| < \frac{\varepsilon}{3}$$
holds. In particular,
$$|f_k(x_0) - f(x_0)| < \frac{\varepsilon}{3}$$
for $n \geq n_1$. By (1), we can find $n > n_0$ such that
$$\underline{d}\Big(\Big\{x \in I\colon |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}\Big\}, x_0\Big) > \frac{\alpha}{2}.$$
Notice that
$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon$$
$$\text{for } x \in \Big\{t \in I\colon |f_n(t) - f_n(x_0)| < \frac{\varepsilon}{3}\Big\}.$$
Therefore,
$$\Big\{x \in I\colon |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}\Big\} \subset \big\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\big\}.$$
Hence,
$$\underline{d}\Big(\big\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) \geq \underline{d}\Big(\Big\{x \in I\colon |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}\Big\}, x_0\Big) > \frac{\alpha}{2}.$$
Since $\varepsilon > 0$ was taken arbitrarily,
$$\lim_{\varepsilon\to 0^+} \underline{d}\Big(\big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) \geq \frac{\alpha}{2} > 0.$$
It follows that f is [0]-lower continuous at x_0.

Now, suppose that f is [0]-lower continuous at x_0. Let
$$\beta = \lim_{\varepsilon\to 0^+} \underline{d}\Big(\big\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) > 0.$$
Then, $\underline{d}\big(\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\}, x_0\big) \geq \beta$ for $\varepsilon > 0$. Fix any $\delta > 0$. There exists $n_0 \geq 1$ such that for every $k > n_0$ and every $x \in I$ the inequality
$$|f_k(x) - f(x)| < \frac{\delta}{3}$$
holds. Similarly as earlier, we can easily check that
$$\Big\{x \in I\colon |f(x) - f(x_0)| < \frac{\delta}{3}\Big\} \subset \big\{x \in I\colon |f_n(x) - f_n(x_0)| < \delta\big\} \quad \text{for } n > n_0.$$
Therefore,
$$\underline{d}\big(\{x \in I\colon |f_k(x) - f_k(x_0)| < \delta\}, x_0\big) \geq \beta \quad \text{for } n \geq n_0$$
and
$$\liminf_{k\to\infty} \underline{d}\Big(\big\{x \in I\colon |f_k(x) - f_k(x_0)| < \delta\big\}, x_0\Big) \geq \beta.$$

Since $\delta > 0$ was taken arbitrarily,

$$\inf_{\delta>0} \liminf_{k\to\infty} \underline{d}\Big(\big\{x \in I\colon |f_k(x) - f_k(x_0)| < \delta\big\}, x_0\Big) \geq \beta > 0,$$

and (1) holds. □

Corollary 2.1. *Assume that every function $f_n\colon I \to \mathbb{R}$ is measurable and there exists $\lambda > 0$ such that every f_n is $[\lambda, \lambda]$-continuous at some $x_0 \in I$. If the sequence $\{f_n\}_{n\geq 1}$ is uniformly convergent to f, $f\colon I \to \mathbb{R}$, then f is also $[0]$-lower continuous at x_0.*

3. Maximal additive class

Definition 3.1. Let $\mathcal{F}$ be any family of real valued functions defined on I. The set

$$\mathcal{M}_a(\mathcal{F}) = \{g\colon \forall_{f\in\mathcal{F}}\ f + g \in \mathcal{F}\}$$

is called the maximal additive family for $\mathcal{F}$.

Remark 3.1. Let f be a constant function, $f(x) = 0$ for each x. If $f \in \mathcal{F}$, then $\mathcal{M}_a(\mathcal{F}) \subset \mathcal{F}$.

Now, we will find a maximal additive family for the family of [0]-lower continuous functions.

Theorem 3.1. *A measurable function $f\colon I \to \mathbb{R}$ belongs to $\mathcal{M}_a\big(\mathcal{C}_{[0]}\big)$ if and only if at every $x_0 \in I$ the following condition*

$$\forall_{\substack{E\in\mathcal{L},\\ E\subset I}} \left(\underline{d}(E, x_0) > 0 \Rightarrow \lim_{\varepsilon\to 0^+} \underline{d}\Big(E \cap \big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) > 0\right) \tag{A}$$

is fulfilled.

P r o o f. Assume that a measurable function f fulfills condition (A). Let $x_0 \in I$ and let g be a lower [0]-continuous at x_0. There exists a measurable set E such that $x_0 \in E$, $g|E$ is continuous at x_0 and $\underline{d}(E, x_0) > 0$. Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$E \cap (x_0 - \delta, x_0 + \delta) \subset \Big\{x\colon |g(x) - g(x_0)| < \frac{\varepsilon}{2}\Big\}.$$

Therefore,

$$\big\{x \in I\colon |(f + g)(x) - (f + g)(x_0)| < \varepsilon\big\} \supset \Big\{x \in E \cap (x_0 - \delta, x_0 + \delta)\colon |f(x) - f(x_0)| < \frac{\varepsilon}{2}\Big\}.$$

Hence,

$$\underline{d}\Big(\big\{x \in I\colon |(f+g)(x) - (f+g)(x_0)| < \varepsilon\big\}, x_0\Big) \geq \underline{d}\Big(\Big\{x \in E\colon |f(x) - f(x_0)| < \frac{\varepsilon}{2}\Big\}, x_0\Big)$$

and

$$\lim_{\varepsilon \to 0^+} \underline{d}\Big(\big\{x \in I\colon |(f+g)(x) - (f+g)(x_0)| < \varepsilon\big\}, x_0\Big) \geq \lim_{\varepsilon \to 0^+} \underline{d}\Big(\Big\{x \in E\colon |f(x) - f(x_0)| < \frac{\varepsilon}{2}\Big\}, x_0\Big) = \lim_{\varepsilon \to 0^+} \underline{d}\Big(E \cap \big\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) > 0.$$

By Theorem 1.2, $f + g$ is [0]-lower continuous at x_0.

Let $f \in \mathcal{M}_a\big(\mathcal{C}_{[0]}\big)$. Suppose that there exists $x_0 \in I$ at which condition (A) is not fulfilled. Then, there exists a measurable set $E \subset I$ such that $\underline{d}(E, x_0) > 0$ and

$$\lim_{\varepsilon \to 0^+} \underline{d}\Big(E \cap \big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) = 0.$$

Hence,

$$\lim_{\varepsilon \to 0^+} \underline{d}^+\Big(E \cap \big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) = 0$$

or

$$\lim_{\varepsilon \to 0^-} \underline{d}^-\Big(E \cap \big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) = 0.$$

We may assume that

$$\lim_{\varepsilon \to 0^+} \underline{d}^+\Big(E \cap \big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) = 0.$$

By Lemma 1.1, there exists a sequence of closed intervals $\big\{[a_n, b_n]\big\}_{n \geq 1}$ such that $x_0 < b_{n+1} < a_n < b_n$ for $n \geq 1$ and

$$\overline{d}^+\left(E \setminus \bigcup_{n=1}^{\infty} [a_n, b_n], x_0\right) = \overline{d}^+\left(\bigcup_{n=1}^{\infty} [a_n, b_n] \setminus E, x_0\right) = 0.$$

Let $\big\{[c_n, d_n]\big\}_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint closed intervals such that $[a_n, b_n] \subset (c_n, d_n)$ and $\overline{d}^+\big(\bigcup_{n=1}^{\infty} ([c_n, d_n] \setminus [a_n, b_n]), x_0\big) = 0$. Let $I_n = [a_n, b_n]$ and $J_n = [c_n, d_n]$ for every $n \geq 1$. Define a function $g\colon (a, b) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in (a, x_0] \cup \bigcup_{n=1}^{\infty} I_n, \\ f(x_0) - f(x) + 1 & \text{if } x \in (x_0, b) \setminus \bigcup_{n=1}^{\infty} \operatorname{int} J_n, \\ \text{linear on every interval } [c_n, a_n], [b_n, d_n], n \in \mathbb{N}. \end{cases}$$

Clearly, g is [0]-lower continuous at every point except at x_0. Since

$$\underline{d}\left((a,x_0]\cup\bigcup_{n=1}^{\infty} I_n, x_0\right)=\underline{d}^+(E,x_0)>0$$

and g restricted to $(a,x_0]\cup\bigcup_{n=1}^{\infty} I_n$ is constant, we conclude that $g\in\mathcal{C}_{[0]}$. Take any $\varepsilon\in(0,1)$.

If $x\in(x_0,b)\setminus\bigcup_{n=1}^{\infty} J_n$, then $(f+g)(x)-(f+g)(x_0)=1$. Hence,

$$\underline{d}^+\Big(\big\{x\in I\colon |(f+g)(x)-(f+g)(x_0)|<\varepsilon\big\},x_0\Big)=$$
$$\underline{d}^+\left(\left\{x\in\bigcup_{n=1}^{\infty} I_n\colon |f(x)-f(x_0)|<\varepsilon\right\},x_0\right)=$$
$$\underline{d}^+\Big(\big\{x\in E\colon |f(x)-f(x_0)|<\varepsilon\big\},x_0\Big).$$

By assumption,

$$\lim_{\varepsilon\to 0^+}\underline{d}^+\big(E\cap\{x\colon |f(x)-f(x_0)|<\varepsilon\},x_0\big)=0.$$

Hence,

$$\lim_{\varepsilon\to 0^+}\underline{d}^+\Big(\big\{x\in I\colon |(f+g)(x)-(f+g)(x_0)|<\varepsilon\big\},x_0\Big)=0.$$

Therefore, $f+g\notin\mathcal{C}_{[0]}$. □

We will show connections between $\mathcal{M}_a\big(\mathcal{C}_{[0]}\big)$ and the so-called T^*-continuity. To this end, we need the notion and some properties of sparse sets and definition of T^* continuous functions. Details of this notion can be found in [4], [8]. We will need only the following

Definition 3.2 ([4])**.** We say that a measurable set $E\subset\mathbb{R}$ is sparse at $x_0\in\mathbb{R}$ if for every measurable set $F\subset\mathbb{R}$, if $\overline{d}(F,x_0)<1$ then $\overline{d}(E\cup F,x_0)<1$. We say that E is sparse if E is sparse at every $x_0\in\mathbb{R}$.

Definition 3.3 ([4])**.** We say that a function $f\colon I\to\mathbb{R}$ is T^* continuous at $x_0\in I$ if for each $\varepsilon>0$ the complement of the set $\big\{x\in I\colon |f(x)-f(x_0)|<\varepsilon\big\}$ is sparse at x_0. A function $f\colon I\to\mathbb{R}$ is T^* continuous if and only if it is T^* continuous at each point of I.

(Actually, these definitions are equivalent conditions of original definitions of sparsity and T^* continuity.)

Theorem 3.2 ([4])**.** *A complement of a measurable set E is sparse at x if and only if for each measurable set $F\subset\mathbb{R}$ such that $\underline{d}(F,x)>0$ the inequality $\underline{d}(E\cap F,x)>0$ holds.*

Applying Definition 3.3 and Theorem 3.2, we have

Theorem 3.3. *A function $f\colon I \to \mathbb{R}$ is T^* continuous at $x_0 \in I$ if and only if*

$$\forall_{\substack{E\in\mathcal{L},\\ E\subset I}} \Big(\underline{d}(E,x_0) > 0 \Rightarrow \forall_{\varepsilon>0}\underline{d}\Big(E \cap \big\{x\colon |f(x)-f(x_0)| < \varepsilon\big\}, x_0\Big) > 0\Big). \qquad \text{(B)}$$

Corollary 3.1. $\mathcal{A} \subset \mathcal{M}_a\big(\mathcal{C}_{[0]}\big) \subset \mathcal{C}_{T^*}$.

Lemma 3.1. *Let $x_0 \in \mathbb{R}$ and $F = \bigcup_{n=1}^{\infty}[a_n, b_n]$, where $x_0 < b_{n+1} < a_n < b_n$ for every $n \geq 1$, $\lim_{n\to\infty} a_n = x_0$. If*

$$\lim_{n\to\infty} \frac{b_n - a_n}{a_n - x_0} = \infty \qquad (2)$$

and

$$\limsup_{n\to\infty} \frac{a_n - b_{n+1}}{b_{n+1} - x_0} < \infty, \qquad (3)$$

then

$$\forall_{\substack{E\in\mathcal{L},\\ E\subset I}} \big(\underline{d}^+(E,x_0) > 0 \Rightarrow \underline{d}^+(E\cap F, x_0) > 0\big). \qquad \text{(C)}$$

P r o o f.

According to (3), there exist $\alpha \in (1,\infty)$ and $n_1 \in \mathbb{N}$ such that $\frac{a_n - b_{n+1}}{b_{n+1}-x_0} < \alpha$ for $n \geq n_1$. Choose any measurable set $E \subset I$ satisfying $\underline{d}^+(E,x_0) > 0$. Let $\beta \in \big(0, \underline{d}^+(E,x_0)\big)$. Then we can find $\delta > 0$ such that $\frac{|E\cap[x_0,x]|}{x-x_0} > \beta$ for each $x \in (x_0, x_0+\delta)$. Choose any $n_2 \in \mathbb{N}$ for which $b_{n_2} < x_0 + \delta$. By (2), there exists $n_3 \in \mathbb{N}$ such that $b_n - a_n > \frac{2(1+\frac{\beta}{2})(1+\alpha)}{\beta}(a_n - x_0)$ for $n \geq n_3$. In particular, $b_n - a_n > \frac{2}{\beta}(a_n - x_0)$ for $n \geq \mathbb{N}$. Let $c_n = a_n + \frac{2}{\beta}(a_n - x_0)$ for $n \geq n_3$. Then, $c_n \in [a_n, b_n]$. Finally, let $n_0 = \max\{n_1, n_2, n_3\}$.

Fix any $x \in (x_0, a_{n_0})$. There exists $k > n_0$ such that $x \in [b_{k+1}, b_k]$.

If $x \in [c_k, b_k]$, then

$$\begin{aligned} \big|E\cap F\cap[x_0,x]\big| &\geq \big|E\cap[a_k,x]\big| \\ &\geq \big|E\cap[x_0,x]\big| - (a_k - x_0) \\ &\geq \beta(x-x_0) - (a_k - x_0). \end{aligned} \qquad (4)$$

Moreover, $x - x_0 \geq c_k - x_0 = \big(\frac{2}{\beta}+1\big)(a_k - x_0)$. Hence, $a_k - x_0 \leq \frac{\beta}{2}(x-x_0)$. Therefore,

$$\big|E\cap F\cap[x_0,x]\big| \geq \beta(x-x_0) - \frac{\beta}{2}(x-x_0) = \frac{\beta}{2}(x-x_0). \qquad (5)$$

If $x \in [b_{k+1}, c_k]$, then

$$\big|E\cap F\cap[x_0,x]\big| \geq |E\cap[a_{k+1},b_{k+1}]| \geq \big|E\cap[x_0,b_{k+1}]\big| - (a_{k+1}-x_0) \qquad (6)$$

Moreover, $x - x_0 \ge b_{k+1} - x_0$,

$$\begin{aligned} x - x_0 \le c_k - x_0 &= \left(1 + \tfrac{2}{\beta}\right)(a_k - x_0) \\ &\le \left(1 + \tfrac{2}{\beta}\right)(b_{k+1} - x_0 + a_k - b_{k+1}) \\ &\le \left(1 + \tfrac{2}{\beta}\right)\big(b_{k+1} - x_0 + \alpha(b_{k+1} - x_0)\big) \\ &= \left(1 + \tfrac{2}{\beta}\right)(1+\alpha)(b_{k+1} - x_0) \end{aligned}$$

and

$$a_{k+1} - x_0 \le \tfrac{2}{\beta}(b_{k+1} - a_{k+1}).$$

Thus,

$$\begin{aligned} &\frac{|E \cap F \cap [x_0, x]|}{x - x_0} \\ &\ge \frac{1}{(1+\frac{2}{\beta})(1+\alpha)} \cdot \frac{|E \cap [x_0, b_{k+1}]|}{b_{k+1} - x_0} - \frac{\frac{\beta}{2(1+\frac{\beta}{2})(1+\alpha)}(b_{k+1} - a_{k+1})}{b_{k+1} - x_0} \\ &\ge \frac{1}{(1+\frac{2}{\beta})(1+\alpha)} \left(\frac{|E \cap [x_0, b_{k+1}]|}{b_{k+1} - x_0} - \frac{\beta(b_{k+1} - x_0)}{2(b_{k+1} - x_0)} \right) \\ &\ge \frac{1}{(1+\frac{2}{\beta})(1+\alpha)} \left(\beta - \frac{\beta}{2}\right) \\ &= \frac{\beta}{2(1+\frac{2}{\beta})(1+\alpha)}. \end{aligned} \tag{7}$$

By (5) and (7), we have

$$\frac{|E \cap F \cap [x_0, x]|}{x - x_0} \ge \frac{\beta}{2(1+\frac{2}{\beta})(1+\alpha)} \quad \text{for every } \ x \in (x_0, a_{n_0}).$$

Therefore,

$$\underline{d}^{+}(E \cap F, x_0) \ge \frac{\beta}{2(1+\frac{2}{\beta})(1+\alpha)} > 0.$$

□

Theorem 3.4. *$\mathcal{A} \subsetneq \mathcal{M}_a\big(\mathcal{C}_{[0]}\big) \subsetneq \mathcal{C}_{T^*}$.*

P r o o f. We only have to prove that $\mathcal{M}_a\big(\mathcal{C}_{[0]}\big) \setminus \mathcal{A} \ne \emptyset$ and $\mathcal{C}_{T^*} \setminus \mathcal{M}_a\big(\mathcal{C}_{[0]}\big) \ne \emptyset$.

Let $x_n = \frac{1}{n!}, y_n = \frac{x_n + x_{n+1}}{2}, u_n = \frac{x_n + y_n}{2}$ (we may assume that $[0,1] \subset I$). Obviously, $\lim_{n\to\infty} \frac{x_n - x_{n+1}}{x_{n+1}} = \lim_{n\to\infty} \frac{y_n - x_{n+1}}{x_{n+1}} = \infty$. Define $f\colon I \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{for } x \in (a,0] \cup \{x_1, x_2, \dots\} \cup [x_1, b), \\ 1 & \text{for } x \in \bigcup_{n=1}^{\infty} [y_n, u_n], \\ \text{linear on every interval } [x_{n+1}, y_n], [u_n, x_n], n = 1, 2, \dots \end{cases}$$

The function f is continuous at every point except at 0. Take $\varepsilon \in (0,1)$. Then

$$\{x \in I\colon |f(x)-f(0)|<\varepsilon\} =$$
$$(a,0] \cup \bigcup_{n=1}^{\infty} \left[x_{n+1}, x_{n+1}+\varepsilon(y_n-x_{n+1})\right] \cup \bigcup_{n=1}^{\infty} \left[u_n+(1-\varepsilon)(x_n-u_n), x_n\right] \cup [x_1,b)$$

Notice that the set

$$\bigcup_{n=1}^{\infty} \left[x_{n+1}, x_{n+1}+\varepsilon(y_n-x_{n+1})\right]$$

fulfills conditions (2) and (3) from Lemma 3.1. Indeed, if

$$a_n = x_{n+1} \quad \text{and} \quad b_n = x_{n+1}+\varepsilon(y_n-x_{n+1}),$$

then

$$\lim_{n\to\infty} \frac{b_n-a_n}{a_n} = \lim_{n\to\infty} \frac{x_{n+1}+\varepsilon(y_n-x_{n+1})-x_{n+1}}{x_{n+1}} = \lim_{n\to\infty} \frac{\varepsilon(y_n-x_{n+1})}{x_{n+1}} = \infty$$

and

$$\begin{aligned} \frac{a_n-b_{n+1}}{b_{n+1}} &= \frac{x_{n+1}-x_{n+2}-\varepsilon(y_{n+1}-x_{n+2})}{x_{n+2}+\varepsilon(y_{n+1}-x_{n+2})} \\ &\le \frac{(2-\varepsilon)(y_{n+1}-x_{n+2})}{\varepsilon(y_{n+1}-x_{n+2})} \\ &= \frac{2-\varepsilon}{\varepsilon}. \end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \frac{a_n-b_{n+1}}{b_{n+1}} < \infty.$$

By Lemma 3.1 and Theorem 3.3, f is T^*-continuous at 0 and $f \in \mathcal{C}_{T^*}$. Notice that

$$\{x \in \mathbb{R}\colon |f(x)-f(0)|<\varepsilon\} \cap \bigcup_{n=1}^{\infty} \left[x_{n+1}+\varepsilon\frac{x_n-x_{n+1}}{2}, u_n\right] = \emptyset.$$

On the other hand,

$$\begin{aligned} \left|\left[x_{n+1}+\varepsilon\frac{x_n-x_{n+1}}{2}, u_n\right]\right| &= \left|\left[x_{n+1}+\varepsilon\frac{x_n-x_{n+1}}{2}, x_{n+1}+\frac{3}{4}(x_n-x_{n+1}\right]\right| \\ &= \frac{3}{4}(x_n-x_{n+1}) - \frac{\varepsilon}{2}(x_n-x_{n+1}) \\ &= \frac{3-2\varepsilon}{4}(x_n-x_{n+1}) \end{aligned}$$

and

$$\overline{d}^{+}\Big(I \setminus \{x \in I\colon |f(x)-f(0)|<\varepsilon\}, 0\Big) \ge \limsup_{n\to\infty} \frac{\frac{3-2\varepsilon}{4}(x_n-x_{n+1})}{u_n}.$$

Hence,

$$\overline{d}^+\Big(I \setminus \big\{x \in I\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) \geq \limsup_{n\to\infty} \frac{|[0,u_n] \cap (I \setminus \{x \in I\colon |f(x)-f(0)| < \varepsilon\})|}{u_n} \geq \limsup_{n\to\infty} \frac{\frac{3-2\varepsilon}{4}(x_n - x_{n+1})}{x_{n+1} + \frac{3}{4}(x_n - x_{n+1})} = \limsup_{n\to\infty} \frac{1}{\frac{4}{3-2\varepsilon}\frac{x_{n+1}}{x_n - x_{n+1}} + \frac{3}{3-2\varepsilon}}.$$

Therefore,

$$\overline{d}^+\Big(I \setminus \big\{x \in I\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) \geq \frac{3-2\varepsilon}{3} = 1 - \frac{2}{3}\varepsilon$$

and

$$\underline{d}^+\Big(\big\{x \in \mathbb{R}\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) \leq 1 - \left(1 - \frac{2}{3}\varepsilon\right) = \frac{2}{3}\varepsilon.$$

Hence,

$$\lim_{\varepsilon\to 0^+} \underline{d}^+\Big(\big\{x \in I\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) = 0.$$

Finally,

$$\lim_{\varepsilon\to 0^+} \underline{d}^+\Big(E \cap \big\{x \in I\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) = 0$$

and $f \notin \mathcal{M}_a\big(\mathcal{C}_{[0]}\big)$. Thus $f \in \mathcal{C}_{T^*} \setminus \mathcal{M}_a\big(\mathcal{C}_{[0]}\big)$.

Next, define $g\colon I \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{for } x \in (a,0] \cup \bigcup_{n=1}^{\infty}[x_{n+1}, y_n] \cup [x_1, b), \\ 1 & \text{for } x \in \{u_1, u_2, \ldots\}, \\ \text{linear on every interval } [y_n, u_n], [u_n, x_n], n = 1, 2, \ldots \end{cases}$$

Obviously, g is continuous at every point except at 0 and g is not approximately continuous at 0. It is easy to see that $\{[x_{n+1}, y_n]\}_{n\in\mathbb{N}}$ satisfy conditions (2) and (3) from Lemma 3.1. Since g restricted to $(a,0] \cup \bigcup_{n=1}^{\infty}[x_{n+1}, y_n]$ is constant, g satisfies condition (A). Hence, $g \in \mathcal{M}_a\big(\mathcal{C}_{[0]}\big) \setminus \mathcal{A}$. □

4. Maximal multiplicative class

Definition 4.1. Let $\mathcal{F}$ be any family of real valued functions defined on I. The set

$$\mathcal{M}_m(\mathcal{F}) = \{g\colon \forall_{f\in\mathcal{F}}\ f \cdot g \in \mathcal{F}\}$$

is called a maximal multiplicative family for $\mathcal{F}$.

Remark 4.1. Let f be a constant function, $f(x) = 1$ for each x. If $f \in \mathcal{F}$, then $\mathcal{M}_a(\mathcal{F}) \subset \mathcal{F}$.

Lemma 4.1. *Let $f\colon I \to \mathbb{R}$ be a function from $\mathcal{C}_{[0]}$. If there exists $x_0 \in I$ such that f does not fulfill condition* (A) at x_0 and $f(x_0) \neq 0$, then there exists $g\colon I \to \mathbb{R}$ such that $g \in \mathcal{C}_{[0]}$ and $f \cdot g \notin \mathcal{C}_{[0]}$.

P r o o f.

By assumptions, there exists a measurable set $E \subset I$ such that $\underline{d}(E, x_0) > 0$ and

$$\lim_{\varepsilon\to 0^+} \underline{d}\Big(E \cap \big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) = 0.$$

Again, we may assume that

$$\lim_{\varepsilon\to 0^+} \underline{d}^+\Big(E \cap \big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) = 0.$$

There exists a sequence of pairwise disjoint closed intervals $\{[a_n, b_n]\}_{n\geq 1}$ such that $x_0 < b_{n+1} < a_n < b_n$ for every $n \geq 1$ and

$$\overline{d}^+\left(E \setminus \bigcup_{n=1}^{\infty} [a_n, b_n], x_0\right) = \overline{d}^+\left(\bigcup_{n=1}^{\infty} [a_n, b_n] \setminus E, x_0\right) = 0.$$

Let $\big\{[c_n, d_n]\big\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint closed intervals such that $[a_n, b_n] \subset (c_n, d_n)$ and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty} \big([c_n, d_n] \setminus [a_n, b_n]\big), x_0\right) = 0.$$

Let $I_n = [a_n, b_n]$ and $J_n = [c_n, d_n]$ for every $n \geq 1$. Define $g\colon (a, b) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in (a, x_0] \cup \bigcup_{n=1}^{\infty} I_n, \\ 0 & \text{if } x \in (x_0, b) \setminus \bigcup_{n=1}^{\infty} \operatorname{int} J_n, \\ \text{linear on every interval } [c_n, a_n], [b_n, d_n], n \in \mathbb{N}. \end{cases}$$

Obviously, g is continuous at every point except of x_0. Since

$$\underline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, x_0\right) = \underline{d}^+(E, x_0) > 0$$

and g restricted to $\{x_0\} \cup \bigcup_{n=1}^{\infty} I_n$ is constant, we have $g \in \mathcal{C}_{[0]}$. If

$$x \in (x_0, b) \setminus \bigcup_{n=1}^{\infty} J_n\,,$$

then $(fg)(x) - (fg)(x_0) = -f(x_0)$.

Since

$$\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}(J_n \setminus I_n), x_0\right) = 0,$$

we have

$$\underline{d}^{+}\Big(\big\{x \in I\colon |(fg)(x) - (fg)(x_0)| < \varepsilon\big\}, x_0\Big) = \underline{d}^{+}\left(\left\{x \in \bigcup_{n=1}^{\infty} I_n\colon |f(x) - f(x_0)| < \varepsilon\right\}, x_0\right) = \underline{d}^{+}\Big(\big\{x \in E\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) = \underline{d}^{+}\Big(E \cap \big\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) \quad \text{for all } \varepsilon \in \big(0, |f(x_0)|\big).$$

By assumption,

$$\lim_{\varepsilon\to 0^+} \underline{d}^{+}\Big(E \cap \big\{x\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) = 0.$$

Hence,

$$\lim_{\varepsilon\to 0^+} \underline{d}^{+}\Big(\big\{x \in I\colon |(fg)(x) - (fg)(x_0)| < \varepsilon\big\}, x_0\Big) = 0.$$

Therefore, $fg \notin \mathcal{C}_{[0]}$. □

Definition 4.2. Let $\mathcal{W}_{[0]}$ be a set of all measurable functions $f\colon I \to \mathbb{R}$ such that at every $x_0 \in I$ at which f does not fulfill condition (A), the following two conditions hold

(W1) $f(x_0) = 0$,

(W2) for each measurable $E \subset I$ such that

$$\underline{d}(E, x_0) > 0 \quad \text{and} \quad E \supset \big\{x \in I\colon f(x) = 0\big\},$$

we have

$$\lim_{\varepsilon\to 0^+} \underline{d}\Big(E \cap \big\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\big\}, x_0\Big) > 0.$$

Theorem 4.1. $\mathcal{M}_m\big(\mathcal{C}_{[0]}\big) = \mathcal{W}_{[0]}$.

Proof. Assume that $f\colon I \to \mathbb{R}$ satisfies conditions (W1) and (W2). If f fulfills condition (A) at $x_0 \in I$, then, repeating arguments from the proof of Theorem 3.1, we can easily prove that $f \cdot g$ is [0]-lower continuous at x_0 for every $g \in C_{[0]}$.

Assume that f does not satisfy condition (A) at x_0. By (W1), we have $f(x_0) = 0$. Let $N_f = \big\{x \in I\colon f(x) = 0\big\}$. Take any $g \in C_{[0]}$. There exists a measurable set $E \subset I$ such that $x_0 \in E$, $g_{|E}$ is continuous at x_0 and $\underline{d}(E, x_0) = \lambda > 0$.

For every $\varepsilon > 0$ there exists $\delta > 0$ for which $E \cap (x_0 - \delta, x_0 + \delta) \subset \{x \in I : |g(x)| < \varepsilon\}$. Therefore,

$$\lim_{\varepsilon \to 0^+} \underline{d}\Big(\{x \in I : |(f \cdot g)(x)| < \varepsilon\}, x_0\Big) \geq \lim_{\varepsilon \to 0^+} \underline{d}\Big((E \cup N_f) \cap \{x \in I : |f(x)| < \varepsilon\}, x_0\Big) > 0,$$

by condition (W2). Hence $f \cdot g$ is [0]-continuous at x_0. Since x_0 was arbitrary, $f \cdot g \in C_{[0]}$.

Let $f \in \mathcal{M}_m(C_{[0]})$ and assume that f does not fulfill condition (A) at x_0. By Lemma 4.1, $f(x_0) = 0$. Choose any measurable set $E \subset I$ such that $N_f \subset E$ and $\underline{d}(E, x_0) > 0$. We can find four sequences $(I_n = [a_n, b_n])_{n \in \mathbb{N}}$, $(J_n = [c_n, d_n])_{n \in \mathbb{N}}$, $(I'_n = [a'_n, b'_n])_{n \in \mathbb{N}}$ and $(J'_n = [c'_n, d'_n])_{n \in \mathbb{N}}$ of pairwise disjoint closed intervals such that

$$c'_n < a'_n < b'_n < d'_n < c_{n+1}', \quad d_{n+1} < c_n < a_n < b_n < d_n,$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} a'_n = x_0,$$

$$\overline{d}^- \left(E \setminus \bigcup_{n=1}^{\infty} I_n', x_0\right) = \overline{d}^- \left(\bigcup_{n=1}^{\infty} I'_n \setminus E, x_0\right) = 0,$$

$$\overline{d}^+ \left(E \setminus \bigcup_{n=1}^{\infty} I_n, x_0\right) = \overline{d}^+ \left(\bigcup_{n=1}^{\infty} I_n \setminus E, x_0\right) = 0,$$

and

$$\overline{d}^- \left(\bigcup_{n=1}^{\infty} (J'_n \setminus I'_n), x_0\right) = \overline{d}^+ \left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\right) = 0.$$

Fix $n \in \mathbb{N}$. Since

$$\lim_{\alpha \to \infty} \Big|\{x \in [d_{n+1}, c_n] \setminus N_f : |\alpha \cdot f(x)| < 1\}\Big| = 0,$$

there exists $\alpha_n \in \mathbb{R}$ such that

$$\Big|\{x \in [d_{n+1}, c_n] \setminus N_f : |\alpha_n \cdot f(x)| < 1\}\Big| < \frac{|[a_{n+1}, b_{n+1}]|}{n}.$$

It follows that

$$\overline{d}^+ \left(\bigcup_{n=1}^{\infty} \{x \in [d_{n+1}, c_n] \setminus N_f : |\alpha_n \cdot f(x)| < 1\}, x_0\right) = 0.$$

Similarly, for each $n \in \mathbb{N}$ there exists $\beta_n \in \mathbb{R}$ such that

$$\overline{d}^- \left(\bigcup_{n=1}^{\infty} \{x \in [d'_n, c'_{n+1}] \setminus N_f : |\beta_n \cdot f(x)| < 1\}, x_0\right) = 0.$$

Define $g\colon I \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{for} \quad x \in \{x_0\} \cup \bigcup_{n=1}^{\infty} (I_n \cup I'_n) \cup (a, c'_1] \cup [d_1, b), \\ \alpha_n & \text{for} \quad x \in [d_{n+1}, c_n], \quad n = 1, 2, \dots, \\ \beta_n & \text{for} \quad x \in [d'_n, c'_{n+1}], \quad n = 1, 2, \dots, \\ & \text{linear on every interval} \quad [c_n, a_n], [b_n, d_n], [c'_n, a'_n], [b'_n, d'_n], n \geq 1. \end{cases}$$

It is clear that g is continuous at every point except at x_0. Moreover, $\underline{d}\big(\{x \in I\colon g(x) = g(x_0)\}, x_0\big) > 0$. Thus, $g \in C_{[0]}$. By assumptions about f, we have $f \cdot g \in C_{[0]}$. In particular, $f \cdot g$ is [0]-lower continuous at x_0. Since $(f \cdot g)(x_0) = 0$, we have

$$\lim_{\varepsilon \to 0^+} \underline{d}\Big(\big\{x \in I\colon |(f \cdot g)(x)| < \varepsilon\big\}, x_0\Big) > 0.$$

On the other hand,

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \underline{d}\Big(\big\{x \in I\colon |(f \cdot g)(x)| < \varepsilon\big\}, x_0\Big) \\ &\leq \lim_{\varepsilon \to 0^+} \underline{d}\left(\bigcup_{n=1}^{\infty} (I_n \cup I'_n) \cap \big\{x\colon |f(x)| < \varepsilon\big\}, x_0\right) \\ &\quad + \lim_{\varepsilon \to 0^+} \overline{d}\left(\bigcup_{n=1}^{\infty} \Big([d_{n+1}, c_n] \cap \big\{x\colon |\alpha_n \cdot f(x)| < \varepsilon\big\}\Big)\right. \\ &\quad \left. \cup \Big([d'_n, c'_{n+1}] \cap \big\{x\colon |\beta_n \cdot f(x)| < \varepsilon\big\}\Big) \setminus N_f, x_0\right) \\ &\quad + \overline{d}\left(N_f \setminus \bigcup_{n=1}^{\infty} I_n, x_0\right) + \overline{d}\left(\bigcup_{n=1}^{\infty} \big((J_n \setminus I_n) \cup (J'_n \setminus I'_n)\big), x_0\right) \\ &= \lim_{\varepsilon \to 0^+} \underline{d}\Big(E \cap \big\{x\colon |f(x)| < \varepsilon\big\}, x_0\Big). \end{aligned}$$

Hence, condition (W2) holds. □

Corollary 4.1. *If $f\colon I \to \mathbb{R}$ is such that at every $x_0 \in I$ at which f does not fulfill condition* (A), the following two conditions hold

(W1′) $f(x_0) = 0$,

(W2′) $\underline{d}\big(\{x \in I\colon f(x) = 0\}, x_0\big) > 0$,

then

$$f \in \mathcal{M}_m\big(\mathcal{C}_{[0]}\big).$$

Corollary 4.2.

$$\mathcal{M}_m\big(\mathcal{C}_{[0]}\big) \subsetneq \mathcal{M}_a\big(\mathcal{C}_{[0]}\big).$$

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