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ON COMMUTATION PROPERTIES OF THE COMPOSITION RELATION OF CONVERGENT AND DIVERGENT PERMUTATIONS (PART I)

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ABSTRACT. In the paper we present the selected properties of composition relation of the convergent and divergent permutations connected with commutation. We note that a permutation on $\mathbb{N}$ is called the convergent permutation if for each convergent series $\sum a_n$ of real terms, the p-rearranged series $\sum a_{p(n)}$ is also convergent. All the other permutations on $\mathbb{N}$ are called the divergent permutations. We have proven, among others, that, for many permutations p on $\mathbb{N}$, the family of divergent permutations q on $\mathbb{N}$ commuting with p possesses cardinality of the continuum. For example, the permutations p on $\mathbb{N}$ having finite order possess this property. On the other hand, an example of a convergent permutation which commutes only with some convergent permutations is also presented.

1. Introduction

A family of all permutations of $\mathbb{N}$ will be denoted by $\mathfrak{P}$. We call $p \in \mathfrak{P}$ a convergent permutation if, for every convergent series $\sum a_n$ of real terms, the p-rearranged series $\sum a_{p(n)}$ is also convergent. A family of all convergent permutations will be denoted by $\mathfrak{C}$. Members of family $\mathfrak{D} := \mathfrak{P} \setminus \mathfrak{C}$ will be called divergent permutations.

Next, we define the following two-sided subsets of $\mathfrak{P}$ (originally defined by A. S. Kronrod [6], and independently, by R. Wituła in 1990s)

$$\mathfrak{A}\mathfrak{B} := \{p \in \mathfrak{P} \colon p \in \mathfrak{A} \wedge p^{-1} \in \mathfrak{B}\}, \quad \text{for every} \quad \mathfrak{A}, \mathfrak{B} \in \{\mathfrak{C}, \mathfrak{D}\}$$

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(see also [5], where families $\mathfrak{CD}$ and $\mathfrak{DC}$ are discussed also from the point of view of two topologies defined on $\mathfrak{P}$).

Convergent and divergent permutations, as well as the sets of these permutations distinguished here, have been intensively investigated by authors of this paper for many years (see [13]–[21]). The present paper essentially completes the results obtained so far. Let us begin with presenting the necessary terminology.

Let $p \in \mathfrak{P}$. The p-order of element $a \in \mathbb{N}$ is defined to be the smallest positive integer k satisfying relation $p^k(a) = a$. If such an integer does not exist, we set $k = \infty$. A set $\mathcal{G} \subset \mathbb{N}$ will be called the set of generators of p if for every $a \in \mathbb{N}$ there exist $b \in \mathcal{G}$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $p^k(b) = a$. Every set of generators of p, which is minimal with respect to the inclusion, will be denoted by $\mathcal{G}(p)$.

Let us also put

$$\mathcal{G}_k(p) = \{a \in \mathbb{N}\colon \text{the } p\text{-order of } a \text{ is equal to } k\},$$

for every $k \in \mathbb{N} \cup \{\infty\}$. The set of all such indices $k \in \mathbb{N}$, for which the set $\mathcal{G}_k(p)$ is nonempty, will be denoted by $\mathcal{M}(p)$.

Moreover, we set

$$\mathrm{Comm}(p) := \{q \in \mathfrak{P}\colon q \circ p = p \circ q\},$$

for every $p \in \mathfrak{P}$. For the shortness of notation, we will henceforward describe the composition $p \circ q$ (or $p \circ q(n)$) of permutations $p, q \in \mathfrak{P}$ as pq ($pq(n)$, respectively).

We say that a nonempty set $A \subset \mathbb{N}$ is a union of n **MSI** (or of at most n **MSI**, or of at least n **MSI**) if there exists a family $\mathfrak{J}$ of n (or of at most n, or of at least n, respectively) intervals of $\mathbb{N}$ which form a partition of A and $\mathrm{dist}(I, J) \geq 2$ for any two different members I, J of $\mathfrak{J}$. **MSI** is an abbreviated form of the notion of **m**utually **s**eparated **i**ntervals.

We know (see [1] [9], [10], [13]–[15])[1] that the permutation $p \in \mathfrak{P}$ is a convergent permutation if and only if there exists $c = c(p) \in \mathbb{N}$ such that for each interval I of $\mathbb{N}$ the set $p(I)$ is a union of at most c **MSI**. In the dual manner,

[1]The first mathematician who discussed permutations on $\mathbb{N}$ preserving the convergence of real series was E. Borel [2]. With respect to this fact, sometimes (but rarely, indeed) the convergent permutations are called the borelian permutations. The above given combinatoric characterization of convergent permutations was discovered by N. Bourbaki [3] (however, by whom from the Bourbaki collective specifically, we do not know). Next, the same characterization was probably found independently by R. P. Agnew [1]. The other combinatoric characterization of convergent permutations was presented by F. W. Levi in 1946 [7], which was either the first presented combinatoric characterization of convergent permutation ever or it was presented at the same time as the Bourbaki's characterization. In the same year, A. S. Kronrod published paper [6] where he characterizes the two-sided permutations from families $\mathfrak{CC}$, $\mathfrak{CD}$, $\mathfrak{DC}$, $\mathfrak{DD}$ in the same combinatoric language as Bourbaki-Agnew. This paper, important and profound, was completely unknown by English and American mathematicians until the 21st century, and indeed, it seems to be still strange (see references in [9], [10], especially in [5], [8], [11], [12]). We have not found it either in paper [4].

permutation $p \in \mathfrak{P}$ is a divergent permutation if and only if for every a, $n \in \mathbb{N}$ there exists an interval I of $\mathbb{N}$ such that the set $p(I)$ is a union of at least n **MSI** and $\min p(I) > a$. This characterization of divergent permutations will be exploited throughout the paper.

A permutation $p \in \mathfrak{P}$ will be called an almost identity permutation if there exists $n_0 \in \mathbb{N}$ such that $p(n) = n$ for every $n \in \mathbb{N}$, $n \geq n_0$.

The cardinality of the finite set A will be denoted by $|A|$.

The main aim of this paper is to investigate some commutation properties of the composition relation of the convergent and divergent permutations. We intend to prove that, under some assumptions about the given permutation p on $\mathbb{N}$ distinguishing it strictly from the almost identity permutations, we have $\operatorname{card}\bigl(\mathfrak{D}\mathfrak{D} \cap \operatorname{Comm}(p)\bigr) = \mathfrak{c}$, which is a very original property (see Theorem 2). Evidently, there also exist some convergent permutations possessing extremely strange properties, for instance, the property blocking the commutation with divergent permutations — see Example 2. Additionally, we will also prove an interesting anticommutative property described in Theorem 4.

One should also notice that in the course of discussion on the problem of commutation within the permutations from $\mathfrak{D}(1)$ (which is the family of divergent permutations p on $\mathbb{N}$ for which the set $\bigl\{n \in \mathbb{N} \colon p(\{1, 2, \ldots, n\}) = \{1, 2, \ldots, n\}\bigr\}$ is infinite), we introduce a new important set-theoretical concept of the so-called minimal and maximal chains of elements of X with respect to the given transitive binary relation on set X. It generalizes and, simultaneously, essentially completes the concept of minimal and maximal elements of X with respect to the given binary relation on set X. For the sake of extensive volume of this subject, we have separated part II completing the current paper.

2. Results

We begin by proving the theorem given below.

Theorem 1. *Let $p \in \mathfrak{P}$. The set $Comm(p)$ is countable if and only if every set $\mathcal{G}(p)$ is finite.*

Next, we present an example (Example 1) of permutation $p \in \mathfrak{C}\mathfrak{C}$ such that $\operatorname{Comm}(p) \subset \mathfrak{C}\mathfrak{C}$ and $\operatorname{card}\bigl(\operatorname{Comm}(p)\bigr) = \mathfrak{c}$. In the sequel, by Theorem 1, it follows that in this case every set $\mathcal{G}(p)$ is infinite.

We also prove the following results.

Theorem 2. *Let $p \in \mathfrak{P}$. If there exists $k \in \mathbb{N} \cup \{\infty\}$ such that for any set $\mathcal{G}(p)$ the set $\mathcal{G}(p) \cap \mathcal{G}_k(p)$ is infinite, then*

$$card\bigl(\mathfrak{D}\mathfrak{D} \cap Comm(p)\bigr) = \mathfrak{c}. \tag{1}$$

Corollary 3. *If the order of p is finite then property* (1) *holds true.*

Theorem 4. *Suppose that $p \in \mathfrak{P}$ is not an almost identity permutation of $\mathbb{N}$, which means that $p(n) \neq n$ for infinitely many $n \in \mathbb{N}$. Then, for every $\mathfrak{U} = \mathfrak{CC}$, $\mathfrak{CD}$, $\mathfrak{DC}$ or $\mathfrak{DD}$, there exists a set of power continuum of all permutations $q \in \mathfrak{U}$ such that $qp(n) \neq pq(n)$ for infinitely many $n \in \mathbb{N}$.*

At the end of this paper, an example (Example 2) of permutations $p, q \in \mathfrak{DD}$ such that $qp \in \mathfrak{CC}$ and $pq \in \mathfrak{DD}$ will be given.

3. Proofs

Proof of Theorem 1. First, we present an auxiliary and a rather obvious result without giving proof.

Lemma 5. *We have*

$$Comm(p) \subseteq Comm(p^n) \tag{2}$$

for every $p \in \mathfrak{P}$ and $n \in \mathbb{N}$. Moreover, if $q \in Comm(p)$ then the p-orders of a and $q(a)$ are the same for every $a \in \mathbb{N}$, and the set $q\big(\mathcal{G}(p)\big)$ gives also a set of generators of p, which is minimal with respect to the inclusion for every $\mathcal{G}(p)$.

Hence, if $q \in \mathrm{Comm}(p)$ and the set $\mathcal{G} = \mathcal{G}(p)$ is fixed, we can identify permutation q with the choice function $\Psi = \Psi(q)$ on family

$$\Big\{\{p^n(a)\colon n \in \mathbb{Z}\}\colon a \in \mathcal{G}\Big\}$$

defined by

$$\Psi\Big\{\{p^n(a)\colon n \in \mathbb{Z}\}\Big\} = q(\mathcal{G}) \cap \{p^n(a)\colon n \in \mathbb{Z}\},$$

for each $a \in \mathcal{G}$, and simultaneously, with the family of permutations q_i of sets $\mathcal{G} \cap \mathcal{G}_i$, $i \in \mathcal{M}$, defined by relation

$$\{q_i(a)\} = \mathcal{G} \cap \Big\{p^n\big(q(a)\big)\colon n \in \mathbb{Z}\Big\},$$

for each $a \in \mathcal{G} \cap \mathcal{G}_i$ and $i \in \mathcal{M}$. Let us notice that if

$$\mathrm{card}\Big(\{p^n(a)\colon n \in \mathbb{Z}\}\Big) \geq 2$$

for infinitely many generators $a \in \mathcal{M}$ then the set of all choice functions Ψ, defined as above, has the power of continuum. On the other hand, if

$$\mathrm{card}\Big(\{p^n(a)\colon n \in \mathbb{Z}\}\Big) = 1$$

for infinitely many generators $a \in \mathcal{M}$, then there exists the continuum infinite set of all permutations q_1 of the set $\mathcal{G} \cap \mathcal{G}_1$. This means that $\operatorname{card}\bigl(\operatorname{Comm}(p)\bigr) = \mathfrak{c}$ whenever $\mathcal{G}$ is infinite. If $\mathcal{G}$ is finite, then

$$\operatorname{card}\bigl(\operatorname{Comm}(p)\bigr) = \prod_{i \in \mathcal{M}} |\mathcal{G} \cap \mathcal{G}_i|! \left|\bigl\{p^n(\min \mathcal{G}_i) \colon n \in \mathbb{Z}\bigr\}\right|^{|\mathcal{G} \cap \mathcal{G}_i|}.$$

It is easy to verify that the above cardinal number is equal to $\aleph_0$. This fact completes the proof. □

Example 1. Let us put

$$p\left(i + 2^{-1}n(n-1)\right) = i (\operatorname{mod} n) + 1 + 2^{-1}n(n-1),$$

for every $i = 1, \dots, n$ and $n \in \mathbb{N}$. Certainly, $p \in \mathfrak{CC}$. It is not hard to show (see the previous considerations) that if a permutation $q \in \operatorname{Comm}(p)$, then q acts in the following way

$$q\left(i + 2^{-1}n(n-1)\right) = (a_n + i)(\operatorname{mod} n) + 1 + 2^{-1}n(n-1),$$

for $i = 1, \dots, n$ and $n \in \mathbb{N}$, where $\{a_n\}$ is a sequence of positive integers chosen such that

$$a_n \in \bigl\{1 + 2^{-1}n(n-1), 2 + 2^{-1}n(n-1), \dots, 2^{-1}n(n+1)\bigr\}, \qquad n \in \mathbb{N}. \quad (3)$$

It is clear that $q \in \mathfrak{CC}$ and that the set of all sequences $\{a_n\}$, $n \in \mathbb{N}$, satisfying relation (3) has the power of continuum. Hence,

$$\operatorname{card}\bigl(\operatorname{Comm}(p)\bigr) = \mathfrak{c}.$$

Proof of Theorem 2. Let us fix $k \in \mathbb{N}$ as in the hypothesis of theorem. Next, we select four sequences A_n, B_n, C_n and D_n, $n \in \mathbb{N}$, of the sets of positive integers in such a way that

1. $|A_n| = |B_n| = |C_n| = n$,
   for every $n \in \mathbb{N}$ and each of sets D_n, $n \in \mathbb{N}$, is infinite.
2. The p-order of any element $a \in \bigcup_{n \in \mathbb{N}} (A_n \cup B_n \cup C_n \cup D_n)$ is equal to k.
3. $\bigl\{p^n(a) \colon n \in \mathbb{Z}\bigr\} \cap \bigl\{p^n(b) \colon n \in \mathbb{Z}\bigr\} = \emptyset$,
   for any two different $a, b \in \bigcup_{n \in \mathbb{N}} (A_n \cup B_n \cup C_n \cup D_n)$.
4. $|a - c| \geq \max A_n$ and $|b - d| \geq \max B_n$,
   for any two different $a, c \in C_n$ and $b, d \in D_n$ and for each $n \in \mathbb{N}$.

Suppose that $\alpha_i^{(n)}$, $\beta_i^{(n)}$ and $\gamma_i^{(n)}$, $i = 1, \dots, n$, are the sequences of all members of sets A_n, B_n and C_n, respectively, for every $n \in \mathbb{N}$.

Next, we associate some permutation $q = q(\Delta)$ of $\mathbb{N}$ which will be constructed below with each countable family Δ of one-to-one sequences

$$\bigl\{\delta_i^{(n)} \in D_n \colon i = 1, \dots, n\bigr\}, \quad n \in \mathbb{N}.$$

To define this permutation, first we introduce some notation.

Let G and G^* denote the subsets of the following sets

$$\mathbb{N} \setminus \Big\{a:\ a = p^m(b),\ \text{where } b \in \bigcup_{n\in\mathbb{N}} (A_n \cup D_n) \text{ and } m \in \mathbb{Z}\Big\}$$

and

$$\mathbb{N} \setminus \Big\{c:\ c = p^m(d),\ \text{where } d \in \bigcup_{n\in\mathbb{N}} (B_n \cup C_n) \text{ and } m \in \mathbb{Z}\Big\},$$

respectively, such that

$$G \cup \bigcup_{n\in\mathbb{N}} (A_n \cup D_n) \quad \text{and} \quad G^* \cup \bigcup_{n\in\mathbb{N}} (B_n \cup C_n)$$

form two sets of generators of p which are minimal with respect to the inclusion. Further, the sets of all members of G and G^*, having the p-order equal to k, will be denoted by G_1 and G_1^*, respectively. Let us observe that by assumptions (1)–(3) both G_1 and G_1^* are infinite. Without loss of generality we can assume that $G \setminus G_1 = G^* \setminus G_1^*$. At last, let us choose any bijection ϕ of set $G \cup D$ onto G^* satisfying conditions

$$\phi(G_1 \cup D) = G_1^* \quad \text{and} \quad \phi(n) = n \qquad \text{for} \quad n \in G \setminus (G_1 \cup D),$$

where $D := \big(\bigcup_{n\in\mathbb{N}} D_n\big) \setminus \bigcup \Delta$.

We are now in a position to define the desired permutation q. Set

$$q\big(p^m(a)\big) = p^m(b), \qquad m \in \mathbb{Z},$$

for $a = \alpha_i^{(n)}, \delta_i^{(n)}, c$ and $b = \gamma_i^{(n)}, \beta_i^{(n)}, \phi(c)$, respectively, and for each $i = 1, \dots, n$, $n \in \mathbb{N}$ and $c \in G \cup D$. We remark that the correspondence (taken over all Δ's) between the family Δ and the above defined permutation q is an injection, which is clear from assumption (2) and from the following equation

$$q\left(p^m\big(\delta_i^{(n)}\big)\right) = p^m\big(\beta_i^{(n)}\big)$$

for each $i = 1, \dots, n$, $n \in \mathbb{N}$, $m \in \mathbb{Z}$, i.e.,

$$q\left(\{p^m(d) \colon d \in \bigcup \Delta \text{ and } m \in \mathbb{Z}\}\right) = \left\{p^m(b) \colon b \in \bigcup_{n\in\mathbb{N}} B_n \text{ and } m \in \mathbb{Z}\right\}.$$

Since the set of continuum power of different families Δ does exist, then the set of continuum power of different permutations q exists as well. Direct consequence of the definition of q is that p and q are commutative. It remains to prove that $q \in \mathfrak{D}\mathfrak{D}$. For this purpose, let us observe that the definition of q implies the following inclusions

$$C_n \subset q\big([\min A_n, \max A_n]\big)$$

and

$$\left\{\delta_1^{(n)}, \dots, \delta_n^{(n)}\right\} \subset q^{-1}\big([\min B_n, \max B_n]\big) \quad \text{for every} \quad n \in \mathbb{N}.$$

This, by the assumption (4), gives that any of the sets

$$q\big([\min A_n, \max A_n]\big) \quad \text{and} \quad q^{-1}\big([\min B_n, \max B_n]\big)$$

is a union of at least n **MSI** for each $n \in \mathbb{N}$. This means that $q \in \mathfrak{D}\mathfrak{D}$, as claimed. The proof is now completed. $\square$

P r o o f o f C o r o l l a r y 3. Let us fix $\mathcal{G}(p)$. By hypothesis, there is $k \in \mathbb{N}$ such that the p-order of any element $a \in G$ is not bigger than k. Hence, there exists $l \leq k$ such that the set $\mathcal{G}(p) \cap \mathcal{G}_l(p)$ is infinite. Then, the assertion of Corollary 3 follows immediately from Theorem 2. $\square$

P r o o f o f T h e o r e m 4. We begin this proof by fixing two sequences I_n and J_n, $n \in \mathbb{N}$, of intervals of $\mathbb{N}$ and two sequences a_n and b_n, $n \in \mathbb{N}$, of positive integers satisfying the following conditions:

1. $|I_n| = |J_n| = 2n$,
2. $a_n \neq p(a_n)$ and $a_n \neq b_n \neq p(a_n)$,
3. $I_n < J_n < \big\{a_n,\, p(a_n),\, b_n\big\} < I_{n+1}$, for each $n \in \mathbb{N}$.

In the next step we associate each of the infinite set $\mathcal{M} \subseteq \mathbb{N}$ with some permutations q_i of $\mathbb{N}$, for $i = 1, 2, 3, 4$, definitions of which depend strictly on $\mathcal{M}$. Let us assume that an infinite subset $\mathcal{M}$ of $\mathbb{N}$ is given. We define the permutations $q_i = q_i(\mathcal{M})$ of $\mathbb{N}$, for $i = 1, 2, 3, 4$, in the following way

4. any of these four permutations $q_1, \ldots, q_4$ is a transposition of elements a_n and b_n for each $n \in \mathcal{M}$,
5. $q_i(j + \min I_n) = \begin{cases} 2j + \min I_n & \text{for} \quad j = 0, 1, \ldots, n-1, \\ 2(j-n) + 1 + \min I_n & \text{for} \quad j = n, n+1, \ldots, 2n-1, \end{cases}$
   for each $i = 1$ or 2 and $n \in \mathbb{N}$,
6. $q_i(2j + \min J_n) = j + \min J_n$ for $j = 0, 1, \ldots, n-1$ and
   $q_i(2(j-n) + 1 + \min J_n) = j + \min J_n$ for $j = n, n+1, \ldots, 2n-1$ and
   for each $i = 1$ or 3 and $n \in \mathbb{N}$,
7. $q_i(n) = n$ for each $i = 1, 2, 3, 4$ and for all others $n \in \mathbb{N}$.

By applying assumptions the (3)–(7), we can assert that $q_1 \in \mathfrak{D}\mathfrak{D}$, $q_2 \in \mathfrak{D}\mathfrak{C}$, $q_3 \in \mathfrak{C}\mathfrak{D}$ and $q_4 \in \mathfrak{C}\mathfrak{C}$. Next, combining the assumptions (2), (3) and (4) with the following facts:

$$q_i(I) = I \text{ for } I = I_n \text{ or } J_n,\ n \in \mathbb{N}$$

and

$$q_i(k) = k \text{ whenever } k \notin \bigcup_{n \in \mathbb{N}} \big(I_n \cup J_n \cup \{a_n, b_n\}\big),$$

we get

$$q_i p(\alpha) \neq p q_i(\alpha) \text{ for every } i = 1, 2, 3, 4, \text{ and for all } \alpha = a_n,\, b_n, \text{ and } n \in \mathcal{M}. \tag{4}$$

In other words, relation (4) is fulfilled for infinitely many $\alpha \in \mathbb{N}$ and for each $i = 1, 2, 3, 4$. Moreover, in view of the assumptions (2)–(7), any of the mappings $\mathcal{M} \to q_i(\mathcal{M})$, for every $i = 1, 2, 3, 4$, (taken over all infinite subsets $\mathcal{M}$ of $\mathbb{N}$) is one-to-one. Since the family of all countable subsets of $\mathbb{N}$ has the power of continuum, therefore we obtain a set of continuum power of all permutations q_i, for each $i = 1, 2, 3, 4$, as desired. $\square$

Below, we present an example of permutations $p, q \in \mathfrak{D}\mathfrak{D}$ such that $qp \in \mathfrak{C}\mathfrak{C}$ and $pq \in \mathfrak{D}\mathfrak{D}$.

Example 2. To construct the desired permutations p and q, we will use the partition of $\mathbb{N}$ given by the increasing sequence I_n, $n \in \mathbb{N}$, of intervals such that $|I_n| = 6n$, $n \in \mathbb{N}$. Additionally, let us suppose that $I_n^{(i)}$, for $i = 1, \ldots, 6$, form the increasing sequence of subintervals of I_n having the same cardinality, i.e.,

$$|I_n^{(i)}| = n \quad \text{for each} \quad i = 1, \ldots, 6 \quad \text{and} \quad n \in \mathbb{N}.$$

The permutation q maps any interval I_n, $n \in \mathbb{N}$, onto itself, but in such a way that

$$\begin{aligned} q\big(i + \min I_n^{(1)}\big) &= 2i + \min I_n^{(3)}, \\ q\big(i + \min I_n^{(2)}\big) &= 2i + 1 + \min I_n^{(3)}, \\ q\big(2i + \min I_n^{(5)}\big) &= i + \min I_n^{(5)}, \\ q\big(2i + 1 + \min I_n^{(5)}\big) &= i + \min I_n^{(6)}, \end{aligned}$$

for $i = 0, 1, \ldots, n-1$, and

$$q\big(i + \min I_n^{(3)}\big) = i + \min I_n^{(1)},$$

for $i = 0, 1, \ldots, 2n-1$. Then, in particular, we obtain

$$q\big(I_n^{(1)}\big) = \Big\{2i + \min I_n^{(3)} : i = 0, 1, \ldots, n-1\Big\}$$

and

$$q^{-1}\big(I_n^{(5)}\big) = \Big\{2i + \min I_n^{(5)} : i = 0, 1, \ldots, n-1\Big\},$$

for every $n \in \mathbb{N}$.
Thus, any of the sets $q\big(I_n^{(1)}\big)$ and $q^{-1}\big(I_n^{(5)}\big)$ is a union of precisely n **MSI** for each $n \in \mathbb{N}$. In consequence, $q \in \mathfrak{D}\mathfrak{D}$.

Now, we construct the permutation p. As above, the permutation p maps any interval I_n, $n \in \mathbb{N}$, onto itself, but in a different way than it is done by permutation q.

Namely, we put

$$\begin{aligned} p\big(2i+\min I_n^{(1)}\big) &= i+\min I_n^{(1)},\\ p\big(2i+1+\min I_n^{(1)}\big) &= i+\min I_n^{(2)},\\ p\big(i+\min I_n^{(5)}\big) &= 2i+\min I_n^{(5)},\\ p\big(i+\min I_n^{(6)}\big) &= 2i+1+\min I_n^{(5)}, \end{aligned}$$

for $i = 0,1,\dots,n-1$, and $p(i) = i$ for $i \in I_n^{(3)} \cup I_n^{(4)}$. This gives

$$p\big(I_n^{(5)}\big) = \Big\{2i+\min I_n^{(5)} : i = 0,1,\dots,n-1\Big\}$$

and

$$p^{-1}\big(I_n^{(1)}\big) = \Big\{2i+\min I_n^{(1)} : i = 0,1,\dots,n-1\Big\},$$

for every $n \in \mathbb{N}$, which yields $p \in \mathfrak{D}\mathfrak{D}$.

It only remains to verify that $qp \in \mathfrak{C}\mathfrak{C}$ and $pq \in \mathfrak{D}\mathfrak{D}$. The fact that $pq \in \mathfrak{D}\mathfrak{D}$ follows from the relation

$$\rho\big(I_n^{(1)}\big) = \Big\{2i+\min I_n^{(3)} : i = 0,1,\dots,n-1\Big\}$$

which is fulfilled for each $\rho = pq$ or $(pq)^{-1}$. It is also easy to show that $qp \in \mathfrak{C}\mathfrak{C}$. This can be seen by combining the following facts:

$$\begin{aligned} qp(I_n) &= I_n,\\ qp\big(J+\min I_n^{(1)}\big) &= J+\min I_n^{(3)},\\ qp\big(J+\min I_n^{(3)}\big) &= q\big(J+\min I_n^{(3)}\big) = J+\min I_n^{(1)}, \end{aligned}$$

for any interval $J \subset \{0,1,\dots,2n-1\}$ and, finally,

$$qp(i) = i,$$

for any $i \in I_n^{(5)} \cup I_n^{(6)}$ and $n \in \mathbb{N}$.

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