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ON SOME PROBLEM OF SIERPIŃSKI AND RUZIEWICZ CONCERNING THE SUPERPOSITION OF MEASURABLE FUNCTIONS. MICROSCOPIC HAMEL BASIS

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ABSTRACT. S. Ruziewicz and W. Sierpiński proved that each function $f\colon \mathbb{R}\to\mathbb{R}$ can be represented as a superposition of two measurable functions. Here, a strengthening of this theorem is given. The properties of Lusin set and microscopic Hamel bases are considered.

In this note, some properties concerning microscopic sets are considered.

Definition 1. We will say that a set $E \subset \mathbb{R}$ is microscopic if for each $\varepsilon > 0$ there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of intervals such that $E \subset \bigcup_{n\in\mathbb{N}} I_n$, and $m(I_n) \le \varepsilon^n$ for each $n \in \mathbb{N}$.

The notion of microscopic set on the real line was introduced by J. Appell. The properties of these sets were studied in [1]. The authors proved, among others, that the family $\mathcal{M}$ of microscopic sets is a σ-ideal, which is situated between countable sets and sets of Lebesgue measure zero, more precisely, between the σ-ideal of strong measure zero sets and sets with Hausdorff dimension zero (see [3]). Moreover, analogously as the σ-ideal of Lebesgue nullsets, $\mathcal{M}$ is orthogonal to the σ-ideal of sets of the first category, i.e., the real line can be decomposed into two complementary sets such that one is of the first category and the second is microscopic (compare [5, Lemma 2.2] or [3, Theorem 20.4]).

The paper consists of three parts. In the first one, a possibility of the representation of an arbitrary function $f\colon \mathbb{R} \to \mathbb{R}$ as a superposition of two functions measurable with respect to some σ-algebra connected with microscopic sets is studied.

The second part deals with the extended Principle of Duality between σ-ideals of microscopic sets and sets of the first category and suitable σ-algebras.

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In the third part, the Lusin set and Hamel basis in the context of microscopic sets are considered.

1. Superposition of functions

In 1933, S. Ruziewicz and W. Sierpiński proved that for each function $f\colon \mathbb{R} \to \mathbb{R}$ there exist two Lebesgue measurable functions g and h such that $f = g \circ h$ (compare [9]).

Observe that an analogous theorem also holds if we replace measurable functions with functions having the Baire property, i.e., measurable with respect to the σ-algebra of sets having the Baire property. In 1935, W. Sierpiński proved (compare [11] or [12, p. 243–247]) that for each function $f : \mathbb{R} \to \mathbb{R}$, there exist two functions g and h which are pointwise discontinued (i.e., the sets of continuity points of g and h are dense in $\mathbb{R}$) and such that $f = g \circ h$.

Let $C(g)$ and $C(h)$ denote the sets of continuity points of g and h, respectively. Clearly, the sets $C(g)$ and $C(h)$ are dense and of type G_δ, so are residual. Consequently, there exist two sets P_1 and P_2 of the first category such that $C(g) = \mathbb{R} \setminus P_1$ and $C(h) = \mathbb{R} \setminus P_2$. Clearly, the functions $g_{|(\mathbb{R}\setminus P_1)}$ and $h_{|(\mathbb{R}\setminus P_2)}$ are continuous, which is equivalent to the fact (see [8, Theorem 8.1]) that g and h have the Baire property. Finally, for each function $f : \mathbb{R} \to \mathbb{R}$ there exist two functions g and h having the Baire property such that $f = g \circ h$.

In both cases, the measurability with respect to the σ-algebra generated by Borel sets and the σ-ideal of nullsets or the σ-ideal of sets of the first category was considered, respectively.

Now, we will prove that, in our case, for the σ-ideal of microscopic sets on the real line, an analogous theorem is also valid.

Put
$$\mathcal{B}\triangle\mathcal{M} := \{B\triangle M : B \in \mathcal{B} \text{ and } M \in \mathcal{M}\},$$
where $\mathcal{B}$ denotes a family of Borel sets on the real line. As it was observed earlier (see [3, Theorem 20.8]), $\mathcal{B}\triangle\mathcal{M}$ is a proper subfamily of the family $\mathcal{L}$ of all Lebesgue measurable subsets of $\mathbb{R}$, i.e., $\mathcal{B}\triangle\mathcal{M} \subsetneq \mathcal{L}$.

For our purpose, we need an auxiliary lemma.

Lemma 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$. If there exists a microscopic set M such that the restriction of f to $\mathbb{R}\setminus M$ is continuous, then f is measurable with respect to $\mathcal{B}\triangle\mathcal{M}$.*

Proof. Let M be a microscopic set such that $g = f_{|(\mathbb{R}\setminus M)}$ is continuous. If U is an arbitrary open set then $g^{-1}(U)$ is open in $\mathbb{R} \setminus M$, so $g^{-1}(U) = G \setminus M$, where G is some open set. Clearly,
$$g^{-1}(U) = f^{-1}(U) \setminus M \subset f^{-1}(U) \subset g^{-1}(U) \cup M.$$
Consequently,
$$G \setminus M \subset f^{-1}(U) \subset G \cup M \quad \text{and} \quad f^{-1}(U) \in \mathcal{B}\triangle\mathcal{M}. \qquad \square$$

The converse theorem does not hold. Let A be a set of type F_σ such that A and $\mathbb{R} \setminus A$ have a positive measure on each interval (see [8, p. 37]). If f is the characteristic function of A, then f is a Borel function and there is no microscopic set M such that the restriction of f to $\mathbb{R} \setminus M$ is continuous.

Theorem 3. *For each function $f : \mathbb{R} \to \mathbb{R}$ there exist two functions g and h measurable with respect to $\mathcal{B}\triangle\mathcal{M}$ such that $f = g \circ h$.*

P r o o f. Let us consider a function φ defined as follows

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}[nx] \quad \text{for } x \in \mathbb{R}$$

(compare [11]), where $[nx]$ denotes the integer part of nx. The function φ is increasing (so pointwise discontinuous) and discontinuous for each rational number. Hence, the set $E = \varphi(\mathbb{R})$ is nowhere dense. From [4, Theorem 3], there exists an automorphism h_1 on the real line such that $h_1(E)$ is a microscopic nowhere dense set. Put $h = h_1 \circ \varphi$. Obviously, $h(\mathbb{R}) = h_1\big(\varphi(\mathbb{R})\big) = h_1(E)$. Clearly, $h^{-1}((-\infty, a)) = \varphi^{-1}\big(h_1^{-1}((-\infty, a))\big) \in \mathcal{B}\triangle\mathcal{M}$ for each $a \in \mathbb{R}$, as h_1 is a homeomorphism and φ is an increasing function, so $h^{-1}((-\infty, a))$ is a half-line (closed or open). Consequently, h is measurable with respect to $\mathcal{B}\triangle\mathcal{M}$.

Now, we define a function $g : \mathbb{R} \to \mathbb{R}$ as follows:

1. if $y \in h_1(E)$, then there exists $x \in \mathbb{R}$ such that $y = h_1\big(\varphi(x)\big) = h(x)$. In this case, we put $g(y) = f(x)$;
2. if $y \notin h_1(E)$, then we put $g(y) = 0$.

As $g_{|(\mathbb{R}\setminus h_1(E))}$ is continuous and $h_1(E) \in \mathcal{M}$, using Lemma 2, we obtain that g is measurable with respect to $\mathcal{B}\triangle\mathcal{M}$. Clearly, $f = g \circ h$. □

As $\mathcal{B}\triangle\mathcal{M} \subsetneq \mathcal{L}$, the previous Theorem 3 is a strengthening of the result of S. R u z i e w i c z and W. S i e r p i ń s k i from [9].

2. The extended Principle of Duality

In 1934, W. S i e r p i ń s k i proved (assuming CH) that there exists a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f(E)$ is a nullset if and only if E is of the first category. In 1943, P. E r d ő s showed that a stronger version of this theorem is also valid. He proved (assuming CH) that there exists a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f = f^{-1}$ and $f(E)$ is a nullset if and only if E is a set of the first category. From Erdős's result, a theorem known as Duality Principle follows.

Duality Principle ([8, Theorem 19.4])**.** Let P be any proposition solely involving notions of measure zero, first category, and notions of pure set theory. Let P^* be the proposition obtained from P by interchanging the terms "nullset" and "set of first category" whenever they appear. Then, each of the propositions P and P^* implies the other, assuming the continuum hypothesis.

In [5], authors proved that the theorem analogous to Sierpiński–Erdős Duality Theorem for the families of microscopic sets and sets of the first category on the real line is valid. Assuming CH, it is showed ([5, Theorem 2.12]) that there exists a one-to-one mapping f of the real line onto itself such that $f = f^{-1}$, and $f(E)$ is a microscopic set if and only if E is a set of the first category.

Observe that the extended Principle of Duality, including measurability with respect to $\mathcal{B}\triangle\mathcal{M}$ and the property of Baire as dual notions, is not true. Such possibility for measurability in the sense of Lebesgue and the Baire property was considered by E. Szpilrajn in [15] (see also [8, p. 82]). The proof, in our case, is analogous with necessary changes.

For $A \subset \mathbb{R}$, let $\bar{A}$ denote the closure of A in the Euclidean topology on the real line.

Theorem 4. *There is no bijection $f : \mathbb{R} \to \mathbb{R}$ such that for an arbitrary set $E \subset \mathbb{R}$, the following conditions are fulfilled:*

1. *$f(E) \in \mathcal{M}$ if and only if E is a set of the first category,*
2. *$f(E) \in \mathcal{B}\triangle\mathcal{M}$ if and only if E is a set having the Baire property.*

Proof. Suppose that there exists a bijection $f : \mathbb{R} \to \mathbb{R}$ fulfilling the conditions 1 and 2. Let $I = [0, 1]$ and $E = f^{-1}(I)$. Obviously, $f(E) = I \in \mathcal{B}\triangle\mathcal{M}$, so E has the Baire property.

Let $x_1, x_2, \ldots$ be a countable and dense subset of E. Let I_i be an open interval containing x_i such that

$$m\big(f(I_i) \cap I\big) < \frac{1}{2^{i+1}} \qquad \text{for} \quad i \in \mathbb{N}. \tag{1}$$

Such interval exists, as

$$\{x_i\} = \bigcap_{j=1}^{\infty} \left(x_i - \frac{1}{j}, x_i + \frac{1}{j}\right), \quad \text{so} \quad f\big(\{x_i\}\big) = \bigcap_{j=1}^{\infty} f\left(\left(x_i - \frac{1}{j}, x_i + \frac{1}{j}\right)\right)$$

is a microscopic set, hence of Lebesgue measure zero. Consequently,

$$\lim_{j\to\infty} m\left(f\left(\left(x_i - \frac{1}{j}, x_i + \frac{1}{j}\right)\right)\right) = 0,$$

and for each $i \in \mathbb{N}$ there exists $j_i \in \mathbb{N}$ such that for the interval $I_i = \big(x_i - \frac{1}{j_i}, x_i + \frac{1}{j_i}\big)$, condition (1) holds.

Put

$$G = \bigcup_{i=1}^{\infty} I_i.$$

Obviously, G is open, $E \subset \bar{G}$, so $E \subset (G \cap E) \cup (\bar{G} \setminus G)$. Hence,

$$I = f(E) \subset \bigcup_{i=1}^{\infty} \big[f(I_i) \cap I\big] \cup f(\bar{G} \setminus G).$$

As $\bar{G} \setminus G$ is nowhere dense, $f(\bar{G} \setminus G)$ is microscopic, so also of Lebesgue measure zero. Consequently, using (1), we obtain

$$1 = m(I) \leq \sum_{i=1}^{\infty} m\big[f(I_i) \cap I\big] < \frac{1}{2},$$

which is a contradiction. □

3. Lusin set and Hamel basis

In this part, we will prove that the image of Lusin set under arbitrary continuous function is microscopic. Recall that an uncountable set $A \subset \mathbb{R}$ is called a Lusin set if it has a countable intersection with every set of the first category. The construction of such a set using continuum hypothesis was given independently by N. Lusin (1914) and P. Mahlo (1913).

M. Kuczma proved (see [6, Lemma 3.4.1]) that for an arbitrary set $E \subset \mathbb{R}^n$ and an arbitrary continuous function $f : E \to \mathbb{R}^m$ $(n, m \in \mathbb{N})$, there exists a decomposition

$$E = A \cup B,$$

where $A \cap B = \emptyset$, A is of the first category and $f(B)$ is a nullset. By a slight modification of the above mentioned construction, we obtain the following

Lemma 5. *Let $E \subset \mathbb{R}$ be an arbitrary set and let $f : E \to \mathbb{R}$ be an arbitrary continuous function. The set E can be represented as the union of two disjoint sets A and B such that A is of the first category and $f(B)$ is a microscopic set.*

Proof. If E is finite, the lemma is obvious (we can put $A = E$ and $B = \emptyset$). Assume that E is infinite. Let $\{p_k\}_{k \in \mathbb{N}}$ be a sequence of points of E, dense in E. Let $n \in \mathbb{N}$. By the continuity of f, for each $k \in \mathbb{N}$, there exists a positive number δ_{n_k} such that

$$f((p_k - \delta_{n_k}, p_k + \delta_{n_k})) \subset \left(f(p_k) - \frac{1}{2(n+1)^k}, f(p_k) + \frac{1}{2(n+1)^k}\right).$$

Put

$$E_n = E \setminus \bigcup_{k=1}^{\infty} (p_k - \delta_{n_k}, p_k + \delta_{n_k}), \quad A = \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad B = E \setminus A.$$

Obviously,

$$E = A \cup B \quad \text{and} \quad A \cap B = \emptyset.$$

Assume $y \in \operatorname{int} \bar{E_n}$. Then there exists an open interval $(y-\eta, y+\eta) \subset \bar{E_n}$. Hence, $(y-\eta, y+\eta) \cap E_n \neq \emptyset$. So, there exists $p_k \in (y-\eta, y+\eta)$. Consequently,

$$(p_k - \delta_{n_k}, p_k + \delta_{n_k}) \cap (y-\eta, y+\eta) \neq \emptyset.$$

On the other hand,

$$(p_k - \delta_{n_k}, p_k + \delta_{n_k}) \cap E_n = \emptyset,$$

so the interval $(p_k - \delta_{n_k}, p_k + \delta_{n_k}) \cap (y-\eta, y+\eta)$ does not contain any point of E_n. Simultaneously, $(y-\eta, y+\eta) \subset \bar{E_n}$, so, in each subinterval of the interval $(y-\eta, y+\eta)$, a point of E_n can be found–a contradiction.

Finally, $\operatorname{int} \bar{E_n} = \emptyset$, so E_n is nowhere dense for $n \in \mathbb{N}$ and, consequently, A is of the first category.

Now, we will prove that $f(B)$ is microscopic. We have

$$B = E \setminus A = E \setminus \bigcup_{n=1}^{\infty} E_n =$$
$$\bigcap_{n=1}^{\infty} \left[E \setminus \left(E \setminus \bigcup_{k=1}^{\infty} (p_k - \delta_{n_k}, p_k + \delta_{n_k}) \right) \right] \subset \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (p_k - \delta_{n_k}, p_k + \delta_{n_k}).$$

Hence,

$$f(B) \subset f\left(\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (p_k - \delta_{n_k}, p_k + \delta_{n_k}) \right) \subset \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} f((p_k - \delta_{n_k}, p_k + \delta_{n_k})) \subset$$
$$\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \left(f(p_k) - \frac{1}{2(n+1)^k}, f(p_k) + \frac{1}{2(n+1)^k} \right).$$

Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0+1} < \varepsilon$. Clearly,

$$f(B) \subset \bigcup_{k=1}^{\infty} \left(f(p_k) - \frac{1}{2(n_0+1)^k}, f(p_k) + \frac{1}{2(n_0+1)^k} \right)$$

and

$$m\left(\left(f(p_k) - \frac{1}{2(n_0+1)^k}, f(p_k) + \frac{1}{2(n_0+1)^k} \right) \right) = \frac{1}{(n_0+1)^k} < \varepsilon^k$$

for each $k \in \mathbb{N}$. □

The next theorem is a strengthening of the result obtained by W. Sierpiński ([10] or [6, Theorem 3.4.1]) and gives an interesting property of Lusin sets.

Theorem 6. *If $E \subset \mathbb{R}$ is a Lusin set and $f : E \to \mathbb{R}$ a continuous function, then $f(E)$ is microscopic.*

Proof. By the previous lemma, the set E can be represented as a union of two disjoint sets A and B where A is of the first category and $f(B)$ is microscopic. As E is a Lusin set, each uncountable subset of E is of the second category. Hence A and, consequently, $f(A)$ is countable. Finally, $f(E)$ is microscopic. □

Remark 7. In particular, any Lusin set is microscopic as the image of itself by the identity function. So, Lusin sets are "very small" from the point of view of measure theory (as well Lebesgue as Hausdorff measure). From the point of view of topology, the situation is quite different: each Lusin set is of the second category and without the Baire property (compare [6, Theorem 3.4.2]).

Now, let us consider a Hamel basis of $\mathbb{R}$. Recall that a Hamel basis is any base of the linear space $(\mathbb{R}^n; \mathbb{Q}; +; \cdot)$. The properties of Hamel bases were investigated by M. Kuczma [6]. It is observed there, among others, that every Hamel basis has the power of continuum ([6, Theorem 4.2.3]). W. Sierpiński in [13] (see Théoréme I) proved that every measurable Hamel basis is a nullset (compare also [6, Corollary 11.2.1]), and under the assumption of continuum hypothesis, there exists a Hamel basis $H \subset \mathbb{R}$ which is a Lusin set (compare [14] or [6, Corollary 11.6.1]). So, from the Remark 7, it follows

Corollary 8. *There exists a Hamel basis which is a microscopic set.*

We can prove even more: there exists a microscopic Hamel basis which has the $(*)$ property.

Definition 9. ([6], Chapter 3.3) The set $A \subset \mathbb{R}$ has $(*)$ property if any set B with Baire property such that $B \subset A$ or $B \subset \mathbb{R} \setminus A$ is of the first category.

The $(*)$ property is a topological analogue of the saturated non-measurability ([7] or [6, p. 56]).

For our purpose, we will use the following class of sets introduced by R. Ger and M. Kuczma in [2] (see also [6, Chapter 9.1]):

$$\mathcal{U} = \{T \subset \mathbb{R} : \text{every convex function } f\colon D \to \mathbb{R}, \text{ where } T \subset D \subset \mathbb{R}$$
and
$$D \text{ is convex and open, bounded above on } T \text{ is continuous in } D\}.$$

Theorem of M. Mehdi says (compare [6, Theorem 9.3.2]) that if $T \subset \mathbb{R}$ contains a second category set with the Baire property, then $T \in \mathcal{U}$. From Corollary 9.3.2 in [6], it follows that each set from the family $\mathcal{U}$ contains a Hamel basis.

Simultaneously, if a Borel set X spans the real line then, from Theorem 11.4.3 in [6], it follows that there exists a Burstin basis H relative to X (A Hamel basis $H \subset X$ is called a Burstin basis relative to X if and only if H intersects every uncountable Borel subset of X). Theorem 11.4.2 in [6] says that if the set $X \subset \mathbb{R}$ is residual, then every Burstin basis relative to X has $(*)$ property.

These facts will be useful in the argument of the next theorem.

Theorem 10. *There exists a Hamel basis which is a microscopic set and has* $(*)$ *property.*

Proof. In [5, Lemma 2.2], it is proved that the real line can be represented as a union of two complementary sets A and B such that A is a microscopic set of type G_δ and B is of the first category. Thus, A is of the second category and has the Baire property. By the Theorem of Mehdi, $A \in \mathcal{U}$. Moreover, from [6, Corollary 9.3.2], we can conclude that A contains a Hamel basis. Thus, A is a Borel set which spans the real line, so A satisfies assumptions of [6, Theorem 11.4.3]. Therefore, A contains a Burstin basis H relative to A. Since A is residual, from [6, Theorem 11.4.2], basis H satisfies condition $(*)$. Obviously, H is microscopic as a subset of the microscopic set A. □

REFERENCES

[1] APPELL, J.—D'ANIELLO, E.—VÄTH, M.: *Some remarks on small sets,* Ric. Mat. **50** (2001), 255–274.

[2] GER, R.—KUCZMA, M.: *On the boundedness and continuity of convex functions and additive functions,* Aequationes Math. **4** (1970), 157–162.

[3] HORBACZEWSKA, G.—KARASIŃSKA, A.—WAGNER-BOJAKOWSKA, E.: *Properties of the σ-ideal of microscopic sets,* Chapter 20 in Traditional and Present-Day Topics in Real Analysis, Łódź University Press, Łódź, 2013, pp. 325–344.

[4] KARASIŃSKA, A.—WAGNER-BOJAKOWSKA, E.: *Homeomorphisms of linear and planar sets of the first category into microscopic sets,* Topology Appl. **159** (2012), 1894–1898.

[5] KARASIŃSKA, A.—POREDA, W.—WAGNER-BOJAKOWSKA, E.: *Duality principle for microscopic sets*, in: Real Functions, Density Topology and Related Topics (M. Filipczak et al., eds.), Łódź University Press, Łódź, 2011, pp. 83–87.

[6] KUCZMA, M.: *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality* (2nd ed.), Birkhäuser, Basel, 2009.

[7] KUCZMA, M.: *On some analogies between measure and category and their applications in the theory of additive functions,* Ann. Math. Sil. **1** (1985), 155–162.

[8] OXTOBY, J. C.: *Measure and Category. A survey of the analogies between topological and measure spaces* (2nd ed.), in: Grad. Texts in Math., Vol. 2, Springer-Verlag, New York, 1980.

[9] RUZIEWICZ, S.—SIERPIŃSKI, W.: *Un théoréme sur les familles de fonctions,* Mathematica (Cluj) **7** (1933), 89–91.

[10] SIERPIŃSKI, W.: *Sur un ensemble non dénombrable, dont toute image continue est de mesure nulle,* Fund. Math. **11** (1928), 302–304.

[11] SIERPIŃSKI, W.: *Sur un probléme de M. Ruziewicz concernant les superpositions de functions jouissant de la propriété de Baire,* Fund. Math. **24** (1935), 12–16.
[12] SIERPIŃSKI, W.: *Oeuvres Choisies,* T. III, PWN, Warszawa, 1975.
[13] SIERPIŃSKI, W.: *Sur la question de la mesurabilité de la base de Hamel,* Fund. Math. **1** (1920), 105–111.
[14] SIERPIŃSKI, W.: *La base de M. Hamel et la propriété de Baire,* Publ. Math. Univ. Beograd **4** (1935), 221–225.
[15] SZPILRAJN, E.: *Remarques sur les fonctions complétement additives d'ensemble et sur les ensembles joissant de la propriété de Baire,* Fund. Math. **22** (1934), 303–311.

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