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ON SOME MODIFICATION OF ŚWIĄTKOWSKI PROPERTY

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ABSTRACT. We introduce some families of functions $f\colon \mathbb{R} \to \mathbb{R}$ modifying the Darboux property analogously as it was done by [Maliszewski, A.: *On the limits of strong Świątkowski functions*, Zeszyty Nauk. Politech. Łódź. Mat. **27** (1995), 87–93], replacing continuity with $\mathcal{A}$-continuity, i.e., the continuity with respect to some family $\mathcal{A}$ of subsets in the domain. We prove that if $\mathcal{A}$ has (*)-property then the family $\mathcal{D}_{\mathcal{A}}$ of functions having $\mathcal{A}$-Darboux property is contained and dense in the family $\mathcal{DQ}$ of Darboux quasi-continuous functions.

A function $f\colon \mathbb{R} \to \mathbb{R}$ has the intermediate value property if, on each interval $(a, b) \subset \mathbb{R}$, the function f assumes every real value between $f(a)$ and $f(b)$. In 1875, J. Darboux [4] showed that this property is not equivalent to the continuity and every derivative has the intermediate value property.

The intermediate value property is usually called the Darboux property and a function having the intermediate value property is called a Darboux function.

Let $\mathcal{D}$ denote the class of Darboux functions. To simplify our notation, we will write:

$$< a, b >= \big(\min\{a, b\}, \max\{a, b\}\big).$$

In 1977, T. Mańk and T. Świątkowski [16] defined some modification of the Darboux property. They considered a family of functions with the so-called Świątkowski property.

Definition 1 ([16]). A function $f : \mathbb{R} \to \mathbb{R}$ has Świątkowski property if for each interval $(a, b) \subset \mathbb{R}$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) \in\, < f(a)\,, f(b) >$ and f is continuous at x_0.

In 1995, A. Maliszewski investigated a class of functions which possesses some stronger property.

Definition 2 ([14]). A function $f\colon \mathbb{R} \to \mathbb{R}$ has the strong Świątkowski property if for each interval $(a, b) \subset \mathbb{R}$ and for each $\lambda \in\, < f(a)\,, f(b) >$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \lambda$ and f is continuous at x_0.

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The family of all strong Świątkowski functions will be denoted by $\mathcal{D}_s$.

In 2009, Z. Grande considered some modification of strong Świątkowski property replacing the continuity with the approximate continuity.

Definition 3 ([5])**.** A function $f : \mathbb{R} \to \mathbb{R}$ has the ap-Darboux property if for each interval $(a, b) \subset \mathbb{R}$ and for each $\lambda \in < f(a), f(b) >$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \lambda$ and f is approximately continuous at x_0.

In the sequel, the family of all functions with the ap-Darboux property will be denoted by $\mathcal{D}_{ap}$.

Let $\mathcal{I}$ be a σ-ideal of the sets of the first category. In [6] and [7], we introduce a family of functions $f : \mathbb{R} \to \mathbb{R}$ modifying the Darboux property analogously as it was done by Z. Grande and replacing approximate continuity with $\mathcal{I}$-approximate continuity, i.e., the continuity with respect to $\mathcal{I}$-density topology in the domain (see [3], [19], [20], [23], [24]).

Definition 4. A function $f : \mathbb{R} \to \mathbb{R}$ has the $\mathcal{I}$-ap-Darboux property if for each interval $(a, b) \subset \mathbb{R}$ and for each $\lambda \in < f(a), f(b) >$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \lambda$ and f is $\mathcal{I}$-approximately continuous at x_0.

The family of all functions with the $\mathcal{I}$-ap-Darboux property will be denoted by $\mathcal{D}_{\mathcal{I}-ap}$.

Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ is a family of all subsets of $\mathbb{R}$. To simplify our considerations, we need the following definition.

Definition 5. We will say that $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{A}$-continuous at a point $x \in \mathbb{R}$ if for each open set $V \subset \mathbb{R}$ with $f(x) \in V$ there exists a set $A \in \mathcal{A}$ such that $x \in A$ and $f(A) \subset V$. We will say that $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{A}$-continuous if f is $\mathcal{A}$-continuous at each point $x \in \mathbb{R}$.

It is not difficult to see that if $\mathcal{A}$ is the Euclidean topology τ_e, then the notion of the $\mathcal{A}$-continuity is equivalent to the notion of the continuity in the classical sense. If $\mathcal{A}$ is the density topology τ_d, then we have the approximate continuity. If $\mathcal{A}$ is the $\mathcal{I}$-density topology $\tau_{\mathcal{I}}$, then we obtain the $\mathcal{I}$-approximate continuity. If $\mathcal{A}$ is some topology τ on $\mathbb{R}$, then the $\mathcal{A}$-continuity is a continuity between $(\mathbb{R}, \tau)$ and $(\mathbb{R}, \tau_e)$.

Of course, $\mathcal{A}$ need not be a topology. Let $\overline{A}$ $(Int(A))$ denote the closure (interior) of the set A in the Euclidean topology. A set $A \subset \mathbb{R}$ is said to be semi-open if there is an open set U such that $U \subset A \subset \overline{U}$ (see [11]). It is not difficult to see that A is semi-open if and only if $A \subset \overline{Int(A)}$. The family of all semi-open sets will be denoted by $\mathcal{S}$. A function $f : \mathbb{R} \to \mathbb{R}$ is semi-continuous if for each set V open in the Euclidean topology the set $f^{-1}(V)$ is semi-open (see [11]).

Definition 6. A function $f : \mathbb{R} \to \mathbb{R}$ is quasi-continuous at a point x if for every neighbourhood U of x and for every neighbourhood V of $f(x)$ there exists a non-empty open set $G \subset U$ such that $f(G) \subset V$. A function $f : \mathbb{R} \to \mathbb{R}$ is quasi-continuous if it is quasi-continuous at each point.

A. Neubrunnová [18] proved that f is semi-continuous if and only if it is quasi-continuous.

Obviously, $\mathcal{S}$ is not a topology and if $\mathcal{A}$ is the family of semi-open sets $\mathcal{S}$, then the $\mathcal{A}$-continuity is equivalent to the quasi-continuity.

Definition 7. We will say that $f : \mathbb{R} \to \mathbb{R}$ has $\mathcal{A}$-Darboux property if for each interval $(a, b) \subset \mathbb{R}$ and each $\lambda \in < f(a), f(b) >$ there exists a point $x \in (a, b)$ such that $f(x) = \lambda$ and f is $\mathcal{A}$-continuous at x.

The family of all functions having $\mathcal{A}$-Darboux property will be denoted by $\mathcal{D}_{\mathcal{A}}$. It is easy to see that if $\mathcal{A}$ is the Euclidean topology τ_e, then $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\tau_e} = \mathcal{D}_s$, if $\mathcal{A}$ is the density topology τ_d, then $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\tau_d} = \mathcal{D}_{ap}$, and if $\mathcal{A}$ is the $\mathcal{I}$-density topology, then $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\tau_{\mathcal{I}}} = \mathcal{D}_{\mathcal{I}-ap}$.

The set A is of the first category at the point x (see [9]) if there exists an open neighbourhood G of x such that $A \cap G$ is of the first category. By $D(A)$ we will denote the set of all points x such that A is not of the first category at x.

Let $\mathcal{B}a$ be a family of all functions having the Baire property.

Definition 8. We will say that the family $\mathcal{A}$ has (*)-property if

1. $\tau_e \subset \mathcal{A} \subset \mathcal{B}a$;
2. $A \subset D(A)$ for each $A \in \mathcal{A}$.

It is not difficult to see that the wide class of topologies has (*)-property, for example, Euclidean topology, $\mathcal{I}$-density topology, topologies constructed in [10] by E. Łazarow, R. A. Johnson and W. Wilczyński or topology constructed by R. Wiertelak in [22]. Some families of sets which are not the topologies also have (*)-property, for example, the family of semi-open sets, however, it does not have the density topology.

Definition 9. We will say that $f : \mathbb{R} \to \mathbb{R}$ has q-property if for each $(a, b) \subset \mathbb{R}$ and for each non-empty open interval $(C, D) \subset f((a, b))$ there exists a non-empty open interval $(c, d) \subset (a, b)$ such that $f((c, d)) \subset (C, D)$.

Lemma 1. *Darboux function $f : \mathbb{R} \to \mathbb{R}$ has q-property if and only if f is quasi-continuous.*

Proof. $\Rightarrow$: Let $x \in \mathbb{R}$. There are two cases:

1. f is constant on some neighbourhood of x. Then, f is continuous at x.

2. f is constant on no neighbourhood of x. Let (a, b) and (p, q) be arbitrary intervals such that $x \in (a, b)$ and $f(x) \in (p, q)$. From the Darboux property, $f((a, b))$ is a non-degenerate interval and

$$f(x) \in (p, q) \cap f((a, b)).$$

Hence, $(p, q) \cap f((a, b))$ is a non-degenerate interval and it contains some non-empty interval (C, D). From q-property, there exists a non-empty open interval $(c, d) \subset (a, b)$ such that $f((c, d)) \subset (C, D) \subset (p, q)$.

$\Leftarrow$: Suppose $f : \mathbb{R} \to \mathbb{R}$ is quasi-continuous and $(a, b) \subset \mathbb{R}$. Let (C, D) be an arbitrary non-empty open interval contained in $f((a, b))$. Then, there exists a point $x \in (a, b)$ such that $f(x) \in (C, D)$. Hence, from Definition 6, there exists a non-empty open interval $(c, d) \subset (a, b)$ such that $f((c, d)) \subset (C, D)$. □

Lemma 2. *If $\mathcal{A}$ has (*)-property and $f \in \mathcal{D}_{\mathcal{A}}$, then f has q-property.*

Proof. Let (a, b) be such that $f((a, b))$ is a non-degenerate interval, and let $(C, D) \subset f((a, b))$.

Fix $y \in (C, D)$ and a number $\epsilon > 0$ with $[y - \epsilon, y + \epsilon] \subset (C, D)$. Clearly,

$$[y - \epsilon, y + \epsilon] \subset f((a, b)).$$

Since $f \in \mathcal{D}_{\mathcal{A}}$, there exists a point $x \in (a, b)$ such that $f(x) = y$ and f is $\mathcal{A}$-continuous at x. Hence, we can find a set $A_x \in \mathcal{A}$ satisfying $x \in A_x$ and $f(A_x) \subset (y - \epsilon, y + \epsilon)$.

As $\mathcal{A}$ has (*)-property, A_x has the Baire property and is not of the first category at x. Hence, $A_x \cap (a, b)$ has the Baire property and is not of the first category at x, too. Consequently,

$$A_x \cap (a, b) = G \Delta P = (G \backslash P) \cup (P \backslash G),$$

where G is open and P is of the first category in the Euclidean topology. Obviously, $G \neq \emptyset$, hence, there exists an interval $(c, d) \subset G$. Observe that $(c, d) \backslash P \subset (a, b)$, so, $(c, d) \subset (a, b)$ and $(c, d) \backslash P \subset A_x$, i.e., $f\big((c, d) \backslash P\big) \subset f(A_x) \subset (y - \epsilon, y + \epsilon)$.

Now, let us prove that $f((c, d)) \subset [y - \epsilon, y + \epsilon]$. Put

$$V = \mathbb{R} \backslash [y - \epsilon, y + \epsilon].$$

Clearly,

$$f^{-1}(V) \cap (c, d) = (c, d) \backslash f^{-1}\big([y - \epsilon, y + \epsilon]\big) \tag{1}$$

and

$$(c, d) \backslash P \subset f^{-1}\Big(f\big((c, d) \backslash P\big)\Big) \subset f^{-1}\big([y - \epsilon, y + \epsilon]\big). \tag{2}$$

From (1) and (2), we obtain

$$f^{-1}(V) \cap (c, d) \subset P. \tag{3}$$

Assume that $f((c,d)) \cap V \neq \emptyset$, i.e., there exists $x_1 \in f^{-1}(V) \cap (c,d)$. From (3), we can find a point $x_2 \in (x_1, d) \setminus f^{-1}(V)$. The set V is open, hence there exists a point $Z \in V \cap < f(x_1), f(x_2) >$. As $f \in \mathcal{D}_{\mathcal{A}}$, we can find a point $z \in (x_1, x_2)$ such that $f(z) = Z$ and f is $\mathcal{A}$-continuous at z, i.e., there exists a set $A_z \in \mathcal{A}$ such that $z \in A_z$ and

$$f(A_z) \subset V. \tag{4}$$

By (3) and (4), we have

$$A_z \cap (x_1, x_2) \subset f^{-1}(V) \cap (c,d) \subset P,$$

i.e., $z \in A_z \cap (x_1, x_2)$, and A_z is of the first category at z, which is a contradiction, as $\mathcal{A}$ has (*)-property and $A_z \in \mathcal{A}$. Hence, $f((c,d)) \cap V = \emptyset$, i.e., $f((c,d)) \subset [y-\epsilon, y+\epsilon] \subset (C,D)$. □

From now on, by $\mathcal{Q}$ we will denote the family of quasi-continuous functions and by $\mathcal{DQ}$ the family of quasi-continuous functions with the Darboux property.

Theorem 1. *If $\mathcal{A}$ has (*)-property, then $\mathcal{D}_s \subset \mathcal{D}_{\mathcal{A}} \subset \mathcal{DQ}$.*

P r o o f. It follows easily from Lemma 1 and Lemma 2. □

Let us show that these inclusions may be proper for different families $\mathcal{A}$.

Lemma 3. *There exists a family $\mathcal{A} \neq \tau_e$ with (*)-property such that $\mathcal{D}_s = \mathcal{D}_{\mathcal{A}}$.*

P r o o f. Let $\mathcal{A} = \tau_h$, where $\tau_h = \{G \setminus P : G \in \tau_e \text{ and } P \in \mathcal{I}\}$ is a H. H a s h i m o t o topology (see [8]). Obviously, $\tau_e \subset \tau_h$, so $\mathcal{D}_s \subset \mathcal{D}_{\tau_h}$.

Let us show that $\mathcal{D}_{\tau_h} \subset \mathcal{D}_s$. Fix a function $f \in \mathcal{D}_{\tau_h}$, an interval (a,b) such that $f(a) \neq f(b)$ and $Z \in < f(a), f(b) >$. Hence, as f has the τ_h-Darboux property, there exists a point $z \in (a,b)$ such that $f(z) = Z$ and f is τ_h-continuous at z.

Let us observe that f is τ_e-continuous at z. Indeed, fix $\epsilon > 0$. Hence, there exists a set $A \in \tau_h$ such that $z \in A$ and

$$f(A) \subset \left(f(z) - \frac{\epsilon}{2}, f(z) + \frac{\epsilon}{2}\right).$$

The set A is open in τ_h, therefore $A = G \setminus P$, where G is open and P is of the first category. Obviously $A \subset G$, so $z \in G$, and there exists $\delta > 0$ such that $(z-\delta, z+\delta) \subset G$. Hence $(z-\delta, z+\delta) \setminus P \subset G \setminus P = A$, so

$$f\big((z-\delta, z+\delta) \setminus P\big) \subset \left(f(z) - \frac{\epsilon}{2}, f(z) + \frac{\epsilon}{2}\right).$$

Analogously as in the proof of Lemma 2 (replacing (c,d) with $(z-\delta, z+\delta)$ and $(y-\epsilon, y+\epsilon)$ with $[f(z) - \epsilon/2, f(z) + \epsilon/2]$) we obtain that

$$f((z-\delta, z+\delta)) \subset \left[f(z) - \frac{\epsilon}{2}, f(z) + \frac{\epsilon}{2}\right].$$

Consequently,

$$f\left((z-\delta, z+\delta)\right) \subset \big(f(z)-\epsilon, f(z)+\epsilon\big),$$

so f is τ_e-continuous at z and $f \in \mathcal{D}_s$. □

We will say that the sets of the form $\bigcup_{n=1}^{\infty}(a_n, b_n)$ or $\bigcup_{n=1}^{\infty}[a_n, b_n]$ are right-hand interval sets (left-hand interval sets) at $x \in \mathbb{R}$ if $b_{n+1} < a_n < b_n$ ($b_n < a_{n+1} < b_{n+1}$) for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = x$.

Lemma 4. *There exists a family $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ with (*)-property such that $\mathcal{D}_s \subsetneq \mathcal{D}_{\mathcal{A}} \subsetneq \mathcal{DQ}$.*

Proof. Let $\mathcal{A} = \tau_{\mathcal{I}}$, where $\tau_{\mathcal{I}}$ denotes the $\mathcal{I}$-density topology. Then, $\mathcal{A}$ has (*)-property, so $\mathcal{D}_s \subset \mathcal{D}_{\mathcal{A}} \subset \mathcal{DQ}$.

Let us prove $\mathcal{D}_s \subsetneq \mathcal{D}_{\tau_{\mathcal{I}}} \subsetneq \mathcal{DQ}$.

Assume that $A = \bigcup_{n=1}^{\infty}(a_n, b_n)$ is a right interval set at zero and put

$$f_A(x) = \begin{cases} 1-x & \text{for } x \le 0, \\ 1-\frac{1}{n} & \text{for } x \in [a_n, b_n],\ n \in \mathbb{N}, \\ 0 & \text{for } x = \frac{a_n+b_{n+1}}{2},\ n \in \mathbb{N} \text{ and for } x \in [b_1, \infty), \\ \text{linear} & \text{on the intervals } \left[b_{n+1}, \frac{a_n+b_{n+1}}{2}\right],\ \left[\frac{a_n+b_{n+1}}{2}, a_n\right],\ n \in \mathbb{N}. \end{cases}$$

Obviously, f_A is continuous at each point $x \in \mathbb{R}$, $x \neq 0$, and assumes value 1 only at 0. Hence, $f_A \notin \mathcal{D}_s$. On the other hand, it is easy to see, that $f_A \in \mathcal{DQ}$ for each right-hand interval set A.

Let B be a right-hand interval set at 0 such that 0 is a right-hand $\mathcal{I}$-density point of B (see [6]). Obviously, $f_B \in \mathcal{D}_{\tau_{\mathcal{I}}} \setminus \mathcal{D}_s$.

Let C be a right-hand interval set at 0 such that 0 is a right-hand $\mathcal{I}$-dispersion point of C. Then, f_C is not $\mathcal{I}$-approximately continuous at 0 (see [6] for more details) and assumes value 1 only at point 0, so $f_C \in \mathcal{DQ} \setminus \mathcal{D}_{\tau_{\mathcal{I}}}$. □

Lemma 5. *There exists a family $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ with (*)-property such that $\mathcal{D}_{\mathcal{A}} = \mathcal{DQ}$.*

Proof. Let us observe that the family $\mathcal{S}$ of semi-open sets has (*)-property. Fix $A \in \mathcal{S}$. As $A \subset \overline{Int(A)}$, the set A is a union of two sets, where the first is open and the second is nowhere dense, so A has the Baire property. Fix a point $x \in A$ and an interval (a,b) such that $x \in (a,b)$. Hence $x \in \overline{Int(A)}$, so $(a,b) \cap Int(A) \neq \emptyset$, i.e., $(a,b) \cap A$ is of the second category and $x \in D(A)$. Consequently, $A \subset D(A)$.

By Theorem 1, we have $\mathcal{D}_{\mathcal{S}} \subset \mathcal{DQ}$.

Let $f \in \mathcal{DQ}$. Fix $(a,b) \subset \mathbb{R}$ such that $f(a) \neq f(b)$, and let $Z \in \langle f(a), f(b) \rangle$. From the Darboux property, there exists $z \in (a,b)$ such that $f(z) = Z$. As f is quasi-continuous at z, $f \in \mathcal{D}_{\mathcal{S}}$.

Consequently, we can put $\mathcal{A} = \mathcal{S}$. □

Let $\mathcal{U}$ be a family of all functions such that for each $a < b$ and for each set $A \subset [a,b]$ with $card\,(A) < card\,(\mathbb{R})$, the set $f\big([a,b] \setminus A\big)$ is dense in $< f\,(a)\,, f\,(b) >$. Let $\mathcal{QU} = \mathcal{Q} \cap \mathcal{U}$. As $\mathcal{D} \subset \mathcal{U}$ (for more details, see [2]), we have $\mathcal{DQ} \subset \mathcal{QU}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is cliquish at the point x if for every neighbourhood U of x and for each $\epsilon > 0$ there exists a non-empty open set $G \subset U$ such that $|f\,(y) - f\,(z)\,| < \epsilon$ for each $y, z \in G$. A function $f : \mathbb{R} \to \mathbb{R}$ is cliquish if it is cliquish at each point.

It is noted (see [15]) that each quasi-continuous function is cliquish and a function f is cliquish if and only if the set of discontinuity points of f is of the first category. Hence the sum of two cliquish functions is cliquish.

Let $\mathcal{C}liq$ denote the family of all cliquish functions.

From the Lindenbaum's Theorem ([12]), it is well-known that an arbitrary function can be represented as a sum of two Darboux functions (see also [1, Chapter I, Theorem 4.1]). In [5], Z. Grande proved that every real function f defined on some interval is the sum of two functions from $\mathcal{D}_{ap}$.

W. Sierpiński in [21] proved that an arbitrary function can be represented as a limit of a pointwise convergent sequence of Darboux functions. Z. Grande showed the analogous result for his family, i.e., that every real function f defined on some interval is the limit of a pointwise convergent sequence of functions from $\mathcal{D}_{ap}$. As for each family $\mathcal{A}$ having (*)-property $\mathcal{D}_{\mathcal{A}} \subset \mathcal{DQ}$it is easily seen that analogous results for our family do not hold, moreover:

Theorem 2. *If $\mathcal{A}$ has (*)-property, then:*

1. *$f \in \mathcal{D}_{\mathcal{A}} + \mathcal{D}_{\mathcal{A}}$ if and only if $f \in \mathcal{D}_s + \mathcal{D}_s$ if and only if $f \in \mathcal{DQ} + \mathcal{DQ}$ if and only if $f \in \mathcal{C}liq$;*
2. *f is cliquish if and only if f is a limit of pointwise convergent sequence of functions having $\mathcal{A}$-Darboux property;*
3. *$f \in \mathcal{QU}$ if and only if f is a limit of uniformly convergent sequence of functions having $\mathcal{A}$-Darboux property.*

Proof. Let $\mathcal{A}$ be an arbitrary family having (*)-property.

1. From Theorem 1 we obtain $\mathcal{D}_s \subset \mathcal{D}_{\mathcal{A}} \subset \mathcal{DQ} \subset \mathcal{C}liq$. Consequently, $\mathcal{D}_s + \mathcal{D}_s \subset \mathcal{D}_{\mathcal{A}} + \mathcal{D}_{\mathcal{A}} \subset \mathcal{DQ} + \mathcal{DQ} \subset \mathcal{C}liq + \mathcal{C}liq = \mathcal{C}liq$. On the other hand, by [13, Corollary 3.4, Chapter II], we have $\mathcal{C}liq \subset \mathcal{D}_s + \mathcal{D}_s$, so $\mathcal{C}liq = \mathcal{D}_s + \mathcal{D}_s = \mathcal{D}_{\mathcal{A}} + \mathcal{D}_{\mathcal{A}} = \mathcal{DQ} + \mathcal{DQ}$.
2. By [14, Corollary 6], the function f is cliquish if and only if there exists a pointwise convergent sequence of Darboux quasi-continuous functions $\{f_n\}_{n\in\mathbb{N}}$ such that $f = \lim_{n\to\infty} f_n$, and if and only if there exists a pointwise convergent sequence of strong Świątkowski functions $\{g_n\}_{n\in\mathbb{N}}$ such

that $f = \lim_{n\to\infty} g_n$. So, by Theorem 1, we have that f is a limit of pointwise convergent sequence of functions having $\mathcal{A}$-Darboux property if and only if f is cliquish.

3. In [17, Theorem 4], the author proved that $f \in \mathcal{QU}$ if and only if f is a limit of uniformly convergent sequence of Darboux quasi-continuous functions. On the other hand, by [14, Corollary 5], $f \in \mathcal{QU}$ if and only if f is a limit of uniformly convergent sequence of strong Świątkowski functions. So, by Theorem 1, $f \in \mathcal{DQ}$ if and only if f is a limit of uniformly convergent sequence of functions having $\mathcal{A}$-Darboux property.

$\square$

Let us introduce a metric ρ in the space $\mathcal{UQ}$ in the following way:

$$\rho(f,g) = \min\Big\{1, \sup\big\{|f(t) - g(t)| : t \in \mathbb{R}\big\}\Big\}.$$

Corollary 1. *If $\mathcal{A}$ has (*)-property, then $\mathcal{D}_{\mathcal{A}}$ is dense in $(\mathcal{DQ}, \rho)$, and the closure of $\mathcal{D}_{\mathcal{A}}$ equals $\mathcal{QU}$.*

Proof. Let $\mathcal{A}$ be a family having (*)-property. By the previous theorem, f is a limit of uniformly convergent sequence of functions having $\mathcal{A}$-Darboux property if and only if $f \in \mathcal{QU}$, so $\mathcal{D}_{\mathcal{A}}$ is dense in $(\mathcal{QU}, \rho)$, and the closure of $\mathcal{D}_{\mathcal{A}}$ equals $\mathcal{QU}$. As $\mathcal{DQ} \subset \mathcal{QU}$, $\mathcal{D}_{\mathcal{A}}$ is dense in $(\mathcal{DQ}, \rho)$. $\square$

Clearly, if $\mathcal{A}_1 \subset \mathcal{A}_2$ then $\mathcal{D}_{\mathcal{A}_1} \subset \mathcal{D}_{\mathcal{A}_2}$. The opposite implication does not hold: there exist uncomparable families $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{D}_{\mathcal{A}_1} \subset \mathcal{D}_{\mathcal{A}_2}$. For example, if $\mathcal{A}_1$ is Hashimoto topology and $\mathcal{A}_2$ is a family of! semi-open sets, then, by Lemmas 3 and 5, we have $\mathcal{D}_{\mathcal{A}_1} = \mathcal{D}_s \subsetneq \mathcal{DQ} = \mathcal{D}_{\mathcal{A}_2}$. Simultaneously, if P is a set of the first category dense in $\mathbb{R}$, then $\mathbb{R}\backslash P$ is open in the Hashimoto topology but not semi-open. On the other hand, $[0,1]$ is semi-open but not open in the Hashimoto topology.

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