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ON APPROXIMATION BY FUNCTIONS HAVING A STRONG ENTROPY POINT

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ABSTRACT. The paper deals with approximation of functions from the unit interval into itself by means of functions having strong entropy point. For this purpose we define a family of functions having the fixed point property: $Conn_C$ (which is a subfamily of the class $Conn$ introduced in [Korczak-Kubiak, E., Pawlak, R. J.: *Trajectories, first return limiting notions and rings of H-connected and iteratively H-connected functions*, Czechoslovak Math. J. **63** (2013), 679–700]). The main result of the paper is a theorem saying that for any function $f \in Conn_C$ and any point $x_0 \in \mathrm{Fix}(f)$ there exists a ring $R \subset Conn_C$ containing function f and in the intersection of any "graph neighbourhood of f" and "neighbourhood of f in topology of uniform convergence", one can find functions $\xi, \psi \in R$ having a strong entropy point y_0 located close to the point x_0 and being a discontinuity point of the function ξ and a continuity point of the function ψ.

1. Introduction

In the theory of discrete dynamical systems, a particular role is played by the notion of topological entropy. It is frequently thought to be a "measure of chaos". At the present time, two basic (equivalent for compact metric spaces) definitions of topological entropy are known: "covery" concept introduced by Adler, Konheim and McAndrew in [1] referring to ompact topological spaces and Bowen-Dinaburg concept ([4], [11]) connected with compact metric spaces. In the original, these two concepts were referred to continuous functions. At the beginning of this century, Čiklová showed ([5]) that the second one can also be applied to wider classes of functions.

Similarly, as in the case of entropy, also other issues connected with topological properties of dynamical systems have been examined almost exclusively with

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reference to continuous functions. However, since an important role in the analysis of this issue is played by functions from the unit interval into itself ([3], [17]), the natural consequence was to include the real functions into the scope of such research (e.g., [5], [6], [21], [23]). Simultaneously, in many papers dealing with the issue of discrete dynamical systems, considerations of particular problems were connected with some structures of functions ([2], [14], [15]). Also in the scope of real functions theory, the issues connected with dynamical systems were combined with, for example, theory of rings (e.g., [19], [20]). This paper alludes to this issue.

Analysing topological entropy from the point of view of real functions theory, it is not difficult to notice that in many cases the entropy of a function may be "focused" at some point (e.g., [21]). This observation led to distinguish so-called (strong) entropy points [22]. These considerations are directly connected with the fixed point theory, so, in a natural way, the classes of functions having a fixed point are particularly interesting in this case (Lemma 4). Simultaneously, it is important to consider the class of functions as wide as possible (in our case it will be a family $Conn_C$, which is a subfamily of the class $Conn$ introduced in [16] on the basis of "first return" theory). The obvious observation, that not all functions have a strong entropy point, makes the investigations into possibility of approximation functions from a fixed family by means of functions belonging to another family or to a fixed ring of functions important.

The direct inspiration to the research whose consequence is this paper, were the observations that, in hitherto papers, an approximation was considered independently for "graph topology" and in the space with a metric of uniform convergence, and independently in the case when strong entropy points are continuity points and when they are discontinuity points. The obtained theorem is some kind of uniform approach to this issue, which demands some specific technique of proving. Simultaneously, we aimed to ensure that the functions by means of which we make an approximation belong to a ring important from the point of view of functions theory. For that reason, we decided to use the construction described in [16] (with a slight modification).

Moreover, let us notice that using first return path systems (introduced in [18]) in our research is not accidental. Although this issue was used in analysis of problems connected, among others, with differentiability, first class of Baire and integrability (e.g., [7]–[10], [12], [13]), the genesis of it may also be found in dynamics of functions. Notice then, that according to Theorem 5 from [10]: If D is a countable and dense subset of $(0,1)$, $\{x_n\}_{n=0}^{\infty}$ is an enumeration of D and $g\colon [0,1] \longrightarrow [0,1]$ is a transitive continuous map, then there is a function $f\colon [0,1] \longrightarrow [0,1]$ topologically conjugate to g such that the first return path system determining by the ordering $\{x_0, x_1, x_2, \ldots\}$ is identical to that determined by the ordering $\{f^0(x_0), f^1(x_0), f^2(x_0), \ldots\}$.

2. Preliminaries

We will use mostly standard notations and definitions. In particular, by the letters $\mathbb{R}$ and $\mathbb{N}$ we denote the sets of all real numbers and positive integers, respectively.

The cardinality of a set A will be denoted by $\#(A)$.

In the paper we will consider real functions defined on the closed unit interval only. The symbol $\mathcal{C}(f)$ ($\mathcal{D}(f)$) stands for the set of all continuity (discontinuity) points of a function f. By $\Gamma(f)$ and $f \restriction A$ we denote a graph of a function f and a restriction of a function f to a set A, respectively. A set of all fixed points of a function f will be denoted by $\mathrm{Fix}(f)$.

Let f, g be real functions defined on $[0,1]$. By ρ_{uc} we denote a metric of uniform convergence defined as $\rho_{uc}(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. The symbol $B_{\rho_{uc}}(f,\varepsilon)$ will denote an open ball in the metric ρ_{uc} with a centre at f and with a radius $\varepsilon > 0$.

If $\mathcal{F}$ is a fixed class of functions and $f \in \mathcal{F}$, then the symbol $\mathcal{R}_{\mathcal{F}}(f)$ will stand for the family of all rings of functions from $\mathcal{F}$ containing f.

Let $f\colon [0,1] \to [0,1]$. Then, we define $f^0(x) = x$ and $f^i(x) = f\big(f^{i-1}(x)\big)$ for any $i \in \mathbb{N}$.

If A is a subset of the domain of $f\colon X \to Y$ and $B \subset Y$, we will say that a set A f-covers a set B (denoted by $A \underset{f}{\to} B$) if $B \subset f(A)$.

Now, we will recall a definition of entropy in the sense of Bowen-Dinaburg. It may be formulated for compact spaces. However, due to the scope of this paper, we will restrict these definitions (as well as definitions connected with strong entropy point) to the case of functions mapping the unit interval into itself.

Let $f\colon [0,1] \to [0,1]$, $\varepsilon > 0$ and $n \in \mathbb{N}$. A set $M \subset [0,1]$ is (n,ε)-separated if for each $x, y \in M$, $x \neq y$ there is $0 \le i < n$ such that $|f^i(x) - f^i(y)| > \varepsilon$.

Let

$$\mathrm{maxsep}[n,\varepsilon] = \max\big\{\#(M) : M \subset [0,1] \text{ is } (n,\varepsilon)\text{-separated set}\big\}.$$

The topological entropy of the function f is the number

$$h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \left[\frac{1}{n}\log\big(\mathrm{maxsep}[n,\varepsilon]\big)\right].$$

3. Strong entropy points

Following [22], we will introduce a notion of a strong entropy point.

Let $f\colon [0,1] \to [0,1]$. An f-bundle B_f is a pair $(\mathcal{F}, J)$ consisting of a family $\mathcal{F}$ of pairwise disjoint (nonsingletons) continuums in $[0,1]$ and a connected set $J \subset [0,1]$ (fibre of bundle) such that $A \underset{f}{\to} J$ for any $A \in \mathcal{F}$. Moreover, if we additionally assume that $A \subset J$ for all $A \in \mathcal{F}$ then such an f-bundle will be called an f-bundle with dominating fibre. By the cardinality of f-bundle $B_f = (\mathcal{F}, J)$ (denoted by $\#(B_f)$), we will mean the cardinality of the family $\mathcal{F}$.

Let $\varepsilon > 0$, $n \in \mathbb{N}$ and $B_f = (\mathcal{F}, J)$ be an f-bundle. A set $M \subset \bigcup \mathcal{F}$ is (B_f, n, ε)-separated if, for each $x, y \in M$, $x \neq y$, there is $0 \leq i < n$ such that $f^i(x), f^i(y) \in J$ and $|f^i(x) - f^i(y)| > \varepsilon$.

Let

$$\operatorname{maxsep}[B_f, n, \varepsilon] = \max\bigl\{\#(M) \colon M \subset [0,1] \text{ is } (B_f, n, \varepsilon)\text{-separated set}\bigr\}.$$

The entropy of the f-bundle B_f is the number

$$h(B_f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left[\frac{1}{n} \log\bigl(\operatorname{maxsep}[B_f, n, \varepsilon]\bigr)\right].$$

For the purpose of this paper, the following lemma will be very useful.

Lemma 1 ([22]). *Let $f \colon [0,1] \to [0,1]$ be an arbitrary function and B_f be an f-bundle with dominating fibre. Then, $h(B_f) \geq \log\bigl(\#(B_f)\bigr)$ whenever B_f is finite, and $h(B_f) = +\infty$ whenever B_f is infinite.*

Let us now introduce a series of definitions aiming to determine a notion of a strong entropy point.

We will say that a sequence of f-bundles $B_f^k = (\mathcal{F}_k, J_k)$ converges to a point x_0 (written $B_f^k \underset{k\to\infty}{\longrightarrow} x_0$), if for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\bigcup \mathcal{F}_k \subset (x_0 - \varepsilon, x_0 + \varepsilon)$ and $\bigl(f(x_0) - \varepsilon, f(x_0) + \varepsilon\bigr) \cap J_k \neq \emptyset$ for any $k \geq k_0$.

Putting

$$E_f(x) = \left\{\limsup_{n\to\infty} h(B_f^n) \colon B_f^n \underset{n\to\infty}{\longrightarrow} x\right\},$$

we obtain a multifunction $E_f \colon [0,1] \multimap \mathbb{R} \cup \{+\infty\}$.

Let us recall an important lemma.

Lemma 2 ([22]). *If $f \colon [0,1] \to [0,1]$ and $x \in [0,1]$, then $\max E_f(x) \leq h(f)$.*

We say that a point $x_0 \in [0,1]$ is a strong entropy point of f if $x_0 \in \operatorname{Fix}(f)$ and $h(f) \in E_f(x_0)$. The family of all functions $f \colon [0,1] \to [0,1]$ having a strong entropy point will be denoted by $\mathfrak{E}_s([0,1])$.

In order to simplify further notations, let us introduce the following symbols:
$\mathfrak{E}_s^{\mathcal{C}}(f)$ – the set of all strong entropy points of a function f which are its continuity points;
$\mathfrak{E}_s^{\mathcal{D}}(f)$ – the set of all strong entropy points of a function f which are its discontinuity points.

4. H_C-connected functions and rings of functions

First, following [16], we will introduce another class of functions whose definition is based on the notions of an od-set, H-trajectory and $(H, \{d_n\}_{n\in\mathbb{N}})$-first return continuity[1]. Let $H \subset [0,1]$ be an od-set, i.e., H is open and dense in $[0,1]$. A sequence $\{d_n\}_{n\in\mathbb{N}}$ of distinct points such that $\{d_n : n \in \mathbb{N}\}$ is a dense subset of H is called an H-trajectory. A left first return path to $x \in (0,1]$ based on H-trajectory $\{d_n\}_{n\in\mathbb{N}} = \bar{d}$ (denoted by $P_l(x,\bar{d})$) is a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that t_1 is the first element of the sequence $\bar{d}$ belonging to $(0,x)$ and t_{k+1} is the first element of the sequence $\bar{d}$ belonging to (t_k, x), for $k \in \mathbb{N}$. Similarly, we define a right first return path to $x \in [0,1)$ based on H-trajectory $\{d_n\}_{n\in\mathbb{N}} = \bar{d}$ denoted by $P_r(x,\bar{d})$ (see [13], [16] for details). We say that a function $f\colon [0,1] \to \mathbb{R}$ is first return continuous from the left (right) at a point $x \in (0,1]$ ($x \in [0,1)$) with respect to the H-trajectory $\bar{d}$, if

$$\lim_{\substack{t\to x\\ t\in P_l(x,\bar{d})}} f(t) = f(x) \left(\lim_{\substack{t\to x\\ t\in P_r(x,\bar{d})}} f(t) = f(x) \right).$$

Let H be a fixed od-set and $\bar{d}$ be a fixed H-trajectory. A family of all functions $f\colon [0,1] \to \mathbb{R}$ such that f is first return continuous with respect to $\bar{d}$ from the left and right at each point $x \in H$, first return continuous from the left with respect to $\bar{d}$ at right end of any component of H, and first return continuous from the right with respect to $\bar{d}$ at left end of any component of H will be denoted by $FRC(H,\bar{d})$.

We say that the function f is H-connected with respect to H-trajectory $\bar{d} = \{d_n\}_{n\in\mathbb{N}}$ if $f \in FRC(H,\bar{d})$, and for any $x \in [0,1] \setminus H$ and any $\varepsilon > 0$, there exists $\delta \in (0,\varepsilon)$ such that the following condition

$$\text{for any component } I \text{ of the set } H \text{ if } I \cap (x-\delta, x+\delta) \neq \emptyset, \tag{1}$$

then

$$f\big(\{d_n : n = 1,2,\dots\} \cap I \cap (x-\delta,x+\delta)\big) \cap \big(f(x)-\varepsilon, f(x)+\varepsilon\big) \neq \emptyset$$

is fulfilled.
If additionally $\{d_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(f)$, then f is called $H_{\mathcal{C}}$-connected with respect to $\bar{d}$.

The symbol $Conn_{\mathcal{C}}$ ($Conn_{\mathcal{C}}([0,1])$) will denote the family of all functions $f\colon [0,1] \to \mathbb{R}$ ($f\colon [0,1] \to [0,1]$) such that there exist an od-set $H(f)$ and an $H(f)$-trajectory $\{d_n\}_{n\in\mathbb{N}}$ such that f is $H(f)_{\mathcal{C}}$-connected with respect to $\{d_n\}_{n\in\mathbb{N}}$.

Let us remind two important facts connected with the considered family $Conn_C$.

[1]Notations and definitions connected with "first return paths" agree with those given in [13], [16], [18].

Lemma 3 ([16]). *If $f \in Conn_{\mathcal{C}}$ then f has a connected graph.*

Lemma 4. *If $f \in Conn_{\mathcal{C}}([0,1])$ then* $\operatorname{Fix}(f) \neq \emptyset$.

Proof. By Lemma 3, the function f has a connected graph. Suppose that $\operatorname{Fix}(f) = \emptyset$. Then, the graph of f is a union of separated sets $A_0 = \{(x, f(x)) \in [0,1] \times [0,1] : f(x) > x\} \neq \emptyset$ and $A_1 = \{(x, f(x)) \in [0,1] \times [0,1] : f(x) < x\} \neq \emptyset$, which contradicts the fact that the graph of f is connected. Thus, $\operatorname{Fix}(f) \neq \emptyset$. □

Let $H \subset [0,1]$ be an od-set and $f : [0,1] \to \mathbb{R}$ be an $H_{\mathcal{C}}$-connected function with respect to H-trajectory $\bar{d} = \{d_n\}_{n \in \mathbb{N}}$. We will briefly present a construction of a ring R consisting of H-connected functions with respect to $\bar{d}$ such that $f \in R$ (see [16] for details).

If $H = [0,1]$ then put $R = FRC(H, \bar{d})$. Assume now that $H \neq [0,1]$. Fix a point $x \in [0,1] \setminus H$ and a positive integer n. Choose a number $\delta_x(n) \in \left(0, \frac{1}{n}\right)$ such that

$$\text{for any component } I \text{ of the set } H \text{ if } I \cap \big(x - \delta_x(n), x + \delta_x(n)\big) \neq \emptyset, \tag{2}$$

then

$$f\Big(\{d_n : n = 1, 2, \dots\} \cap I \cap \big(x - \delta_x(n), x + \delta_x(n)\big)\Big) \cap \left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\right) \neq \emptyset.$$

From each component I of the set H having nonempty intersection with the interval $\big(x - \delta_x(n), x + \delta_x(n)\big)$ choose one point $y^I_{x,n} \in \{d_n : n = 1, 2, \dots\} \cap I \cap \big(x - \delta_x(n), x + \delta_x(n)\big)$ such that $f(y^I_{x,n}) \in \big(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\big)$. Let us denote by $D(x,n)$ the set of all chosen points $y^I_{x,n}$. In the same way, we define the sets $D(x,n)$ for each pair $(x,n) \in \big([0,1] \setminus H\big) \times \mathbb{N}$.

Let R be a family of all functions $g : [0,1] \to \mathbb{R}$ fulfilling the following conditions:

1.1 $g \in FRC(H, \bar{d})$;

1.2 for any $x \in [0,1] \setminus H$ and for any $\varepsilon > 0$ there exists $n(g, x, \varepsilon) \in \mathbb{N}$ such that for any integer $n \geq n(g, x, \varepsilon)$ we have $g\big(D(x,n)\big) \subset \big(g(x) - \varepsilon, g(x) + \varepsilon\big)$.

It is easy to show that the family R is a required ring.

For fixed od-set H, H-trajectory $\bar{d}$ and H-connected function f with respect to $\bar{d}$, the symbol $\Re_f(H, \bar{d})$ will stand for the family of all rings constructed by use of the method described above.

5. Main result

Notice that for each ring $R \in \Re_f(H, \bar{d})$ we can consider a subring R' consisting of all functions from R which are continuous at each point of $\bar{d}$. Obviously, f belongs to such a subring. A family of all such subrings will be denoted by $\Re^{\mathcal{C}}_f(H, \bar{d})$.

If we have a function $\eta\colon [0,1] \longrightarrow [0,1]$ and an open set $V \subset [0,1]\times[0,1]$ containing $\Gamma(\eta)$, we can define (e.g., [21]) a set $V_\Gamma(\eta) := \{\zeta \in Conn_C : \Gamma(\zeta) \subset V\}$. The set $V_\Gamma(\eta)$ can be regarded as a "graph neighbourhood" of η.

Theorem 5. *Let $f \in Conn_{\mathcal{C}}([0,1])$ and $x_0 \in \mathrm{Fix}(f)$. For any open set V containing graph of f and any $\varepsilon > 0$ there exist a ring $R \in \mathcal{R}_{Conn_{\mathcal{C}}}(f)$, a point $y_0 \in (x_0-\varepsilon, x_0+\varepsilon)$ and functions $\xi, \psi \in R \cap V_\Gamma(f) \cap B_{\rho_{uc}}(f,\varepsilon)$ such that $y_0 \in \mathfrak{E}_s^{\mathcal{D}}(\xi) \cap \mathfrak{E}_s^{\mathcal{C}}(\psi)$.*

Proof. Let $f \in Conn_{\mathcal{C}}\big([0,1]\big)$ and $x_0 \in \mathrm{Fix}(f)$. There exist an od-set H and an H-trajectory $\bar{q}$ such that f is H-connected with respect to $\bar{q}$ and $\bar{q} \subset \mathcal{C}(f)$. Moreover, let $V \subset [0,1]\times[0,1]$ be an arbitrary open set containing $\Gamma(f)$ and $\varepsilon > 0$.

Assume that $x_0 \in (0,1)$ (the proof in the case $x_0 \in \{0,1\}$ runs analogously). Choose $n_0 \in \mathbb{N}$ such that

$$\left[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right] \times \left[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right] \subset V \text{ and } \frac{1}{n_0} < \frac{\varepsilon}{2}. \tag{3}$$

We will show that any ring from the family $\Re_f^{\mathcal{C}}(H,\bar{q})$ fulfils the requirements of the theorem. So, let $R \in \Re_f^{\mathcal{C}}(H,\bar{q})$. Consider two cases:

1. $x_0 \notin H$.

In the construction of the ring R, for the pair (x_0,n_0), a number $\delta_{x_0}(n_0) \in \big(0,\frac{1}{n_0}\big)$ with the following properties has been chosen:
for any component I of the set H such that $I \cap \big(x_0 - \delta_{x_0}(n_0), x_0 + \delta_{x_0}(n_0)\big) \neq \emptyset$ we have

$$f\big(\{q_k : k = 1,2,\dots\} \cap I \cap (x_0 - \delta_{x_0}(n_0), x_0 + \delta_{x_0}(n_0))\big) \\ \cap \big(f(x_0) - \frac{1}{n_0}, f(x_0) + \frac{1}{n_0}\big) \neq \emptyset.$$

So, let us fix a component I of the set H such that $I \cap (x_0, x_0 + \delta_{x_0}(n_0)) \neq \emptyset$. Let a, b denote the left and the right end of I, respectively. According to the construction of R, for the pair (x_0,n_0) and the component I, a point (let us call it $\hat{x}$) belonging to $\{q_k : k = 1,2,\dots\} \cap I \cap \big(x_0, x_0 + \delta_{x_0}(n_0)\big)$ and such that $f(\hat{x}) \in \big(f(x_0) - \frac{1}{n_0}, f(x_0) + \frac{1}{n_0}\big) = \big(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\big)$ has been chosen. Let

$$\sigma = \min\left\{\frac{f(\hat{x}) - x_0 + \frac{1}{n_0}}{2}, \frac{x_0 + \frac{1}{n_0} - f(\hat{x})}{2}\right\}.$$

Since $\hat{x}$ is a continuity point of f, there exists a number $\delta > 0$ such that

$$[\hat{x} - \delta, \hat{x} + \delta] \subset I \cap \big(x_0, x_0 + \delta_{x_0}(n_0)\big) \text{ and}$$
$$f\big([\hat{x} - \delta, \hat{x} + \delta]\big) \subset \big(f(\hat{x}) - \sigma, f(\hat{x}) + \sigma\big) \subset \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right). \tag{4}$$

Obviously, there exists a point $y_0 \in (\hat{x} - \delta, \hat{x} + \delta) \setminus \{q_n : n \in \mathbb{N}\}$. Moreover, there exists a number $\delta_1 > 0$ such that

$$[y_0 - \delta_1, y_0 + \delta_1] \subset (\hat{x} - \delta, \hat{x} + \delta)$$

and consequently

$$f\big([y_0 - \delta_1, y_0 + \delta_1]\big) \subset \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right). \tag{5}$$

Fix points $y \in (y_0 - \delta_1, y_0) \setminus \{q_n : n \in \mathbb{N}\}$ and $z \in \big(y_0 + \frac{\delta_1}{2}, y_0 + \delta_1\big) \setminus \{q_n : n \in \mathbb{N}\}$. By (5), we have $f(y), f(z) \in \big(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\big)$.

Now, we will construct a function ξ.

Let $\{s_k\}_{k\in\mathbb{N}}$ be a right first return path to y_0 based on H-trajectory $\bar{q}$. Without loss of generality, we can assume that $\{s_k\} \subset \big(y_0, y_0 + \frac{\delta_1}{2}\big)$. Define the function $\xi\colon [0,1] \to [0,1]$ as follows: $\xi(y_0) = y_0$, $\xi(x) = f(x)$ for $x \in [0, y] \cup [z, 1]$; $\xi(x) = y_0$ for $x = s_k$ $(k \geq 1)$; ξ continuous on $(y_0, s_1]$ and such that $\xi\big([s_{k+1}, s_k]\big) = [y_0 - \delta_1, y_0 + \delta_1]$ $(k \geq 1)$, ξ linear on $[y, y_0]$ and on $[s_1, z]$.

We will show that $\xi \in R$, i.e., it fulfils conditions 1.1 and 1.2 and $\bar{q} \subset \mathcal{C}(\xi)$. Condition 1.1 and the fact that $\bar{q} \subset \mathcal{C}(\xi)$ are easily seen.

Condition 1.2: Let $\varepsilon_1 > 0$. For $x \in \big([0,1] \setminus H\big) \setminus \{a, b\}$ we have $\xi(x) = f(x)$ and we can choose $n(\xi, x, \varepsilon_1)$, such that

$$\Big(x - \delta_x\big(n(\xi, x, \varepsilon_1)\big), x + \delta_x\big(n(\xi, x, \varepsilon_1)\big)\Big) \cap I = \emptyset.$$

Then ξ and f coincide at the points of the sets $D(x, n)$, for $n \geq n(\xi, x, \varepsilon_1)$. Thus, $\xi\big(D(x, n)\big) \subset \big(\xi(x) - \varepsilon_1, \xi(x) + \varepsilon_1\big)$. For $x \in \{a, b\}$ we also have $\xi(x) = f(x)$ and one can choose $n(\xi, x, \varepsilon_1)$, such that $\big(x - \delta_x(n(\xi, x, \varepsilon_1)), x + \delta_x(n(\xi, x, \varepsilon_1))\big) \cap (y, z) = \emptyset$. Then for $n \geq n(\xi, x, \varepsilon_1)$ functions ξ and f coincide at the points of the sets $D(x, n)$. Thus, $\xi\big(D(x, n)\big) \subset \big(\xi(x) - \varepsilon_1, \xi(x) + \varepsilon_1\big)$, which completes the proof of Condition 1.2.

So, we have shown that $\xi \in R$.

What is more, it is easy to notice that

$$y_0 \in \mathrm{Fix}(\xi) \cap \mathcal{D}(\xi).$$

Now, we will show that

$$\xi \in \mathfrak{E}_s\big([0,1]\big).$$

Put $F_k = \big\{[s_{2i}, s_{2i-1}] : i = k, k+1, \ldots\big\}$ and $J_k = [y_0 - \delta_1, y_0 + \delta_1]$ for $k = 1, 2, \ldots$ Let $B_k = (F_k, J_k)$, $k \in \mathbb{N}$.

Fix k. Obviously, F_k is a family of pairwise disjoint (nonsingletons) continuums and J_k is a connected set. For every $i \in \{k, k+1, \ldots\}$ we have $J_k = \xi\big([s_{2i}, s_{2i-1}]\big)$, so $[s_{2i}, s_{2i-1}] \underset{\xi}{\to} J_k$. Moreover, it is easy to see that B_k is a ξ-bundle with a dominating fibre.

We will prove that

$$\text{a sequence of } \xi\text{-bundles } \{B_k\}_{k\in\mathbb{N}} \text{ is convergent to } y_0. \tag{6}$$

So, let $\eta > 0$ and $k_0 \in \mathbb{N}$ be such that $s_{2k_0-1} \in (y_0, y_0+\eta)$. Then, for $k \geq k_0$, we have $\bigcup F_k \subset \bigcup F_{k_0} \subset (y_0-\eta, y_0+\eta)$ and $(\xi(y_0)-\eta, \xi(y_0)+\eta) \cap J_k = (y_0-\eta, y_0+\eta) \cap [y_0-\delta_1, y_0+\delta_1] \neq \emptyset$. The proof of (6) is completed.

By Lemma 1, we conclude that $h(B_k) = +\infty$ for any $k \in \mathbb{N}$. So, we have $\max E_\xi(y_0) = +\infty$ and Lemma 2 implies that $h(\xi) = +\infty$. Thus, $h(\xi) \in E_\xi(y_0)$. Hence, and from the fact that $y_0 \in \mathrm{Fix}(\xi)$, we obtain that y_0 is a strong entropy point of ξ, so $\xi \in \mathfrak{E}_s\big([0,1]\big)$.

We claim that

$$\Gamma(\xi) \subset V. \tag{7}$$

Indeed, V contains the graph of f, and the graph of ξ differs from the graph of f only on the interval (y,z). According to our assumptions, we have

$$(y,z) \subset \left[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right].$$

Notice that

$$\xi((y_0,s_1)) = [y_0-\delta_1, y_0+\delta_1] \subset \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right).$$

Moreover, according to (5),

$$\xi(y), \xi(z) \in \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right), \quad \xi(y_0) = \xi(s_1) = y_0 \in \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right)$$

and ξ is linear on $[y, y_0]$ and on $[s_1, z]$. Thus,

$$\Gamma\big(\xi \restriction (y,z)\big) \subset \left[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right] \times \left[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right] \subset V.$$

Hence, we obtain inclusion (7).

Consequently,

$$\xi \in V_\Gamma(f).$$

Now, we will show that

$$\xi \in B_{\rho_{uc}}(f, \varepsilon). \tag{8}$$

Indeed, if $x \in [0,y] \cup [z,1]$, then $f(x) = \xi(x)$. For $x \in (y,z)$ we have $\xi(x) \in \big[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\big]$, and by (5), $f(x) \in \big(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\big)$. Consequently, the inequality $|f(x) - \xi(x)| < \frac{2}{n_0}$ is true for $x \in (y,z)$, and the proof of (8) is completed.

Summarizing,

$$\xi \in R \cap V_\Gamma(f) \cap B_{\rho_{uc}}(f,\varepsilon) \quad \text{and} \quad y_0 \in \mathfrak{E}_s^{\mathcal{D}}(\xi).$$

Now, we will start with the construction of the function $\psi\colon [0,1] \to [0,1]$.

For any $n \in \mathbb{N}$ choose points

$$y_0 + \frac{\delta_1}{2^{n+1}} = c_0^n < c_1^n < \cdots < c_{2^{n+1}}^n < c_{2^{n+1}+1}^n = y_0 + \frac{\delta_1}{2^n}.$$

Define ψ as follows: $\psi(x) = f(x)$ for $x \in [0, y] \cup [z, 1]$; $\psi(y_0) = y_0$; ψ linear on $[y, y_0]$; for any $n \in \mathbb{N}$:

- $\psi\big(\big[y_0 + \frac{\delta_1}{2^{n+1}}, y_0 + \frac{\delta_1}{2^n}\big]\big) = \big[y_0 + \frac{\delta_1}{2^{n+1}}, y_0 + \frac{\delta_1}{2^n}\big]$,
- $\psi(c_{2i}^n) = y_0 + \frac{\delta_1}{2^{n+1}}$ and $\psi(c_{2i+1}^n) = y_0 + \frac{\delta_1}{2^n}$ for $i \in \{0, 1, \ldots, 2^n\}$,
- $\psi \restriction [c_{2i}^n, c_{2i+1}^n]$ is a linear and increasing function, for $i \in \{0, 1, \ldots, 2^n\}$,
- $\psi \restriction [c_{2i-1}^n, c_{2i}^n]$ is a linear and decreasing function, for $i \in \{1, \ldots, 2^n\}$;
- ψ is linear on $\big[y_0 + \frac{\delta_1}{2}, z\big]$.

It is easy to notice that

$$y_0 \in \mathrm{Fix}(\psi) \cap \mathcal{C}(\psi). \tag{9}$$

Clearly, $\psi \in R$ (the proof of this fact is analogous to that showing $\xi \in R$).

Now, we claim that

$$y_0 \in \mathfrak{E}_s^{\mathcal{C}}(\psi). \tag{10}$$

Indeed, for $m \in \mathbb{N}$ put $P_m = (K_m, T_m)$, where

$$K_m = \big\{[c_{2i}^m, c_{2i+1}^m] : i = 0, 1, \ldots, 2^m\big\}$$

and

$$T_m = \left[y_0 + \frac{\delta_1}{2^{m+1}}, y_0 + \frac{\delta_1}{2^m}\right].$$

It is easy to show that P_m is a ψ-bundle with a domainating fibre for $m = 1, 2, \ldots$ and a sequence $\{P_m\}_{m \in \mathbb{N}}$ is convergent to y_0.

By Lemma 1 we conclude that $\limsup_{m \to \infty} h(P_m) = +\infty$. So, we have $h(\psi) = +\infty$. Thus, $h(\psi) \in E_\psi(y_0)$. Hence, and by (9), we obtain (10).

Now, we will show that

$$\Gamma(\psi) \subset V. \tag{11}$$

Indeed, the graph of ψ differs from the graph of f only on the interval (y, z). It is easy to notice that

$$(y, z) \subset \left[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right]$$

and

$$\psi\left(\left(y_0, y_0 + \frac{\delta_1}{2}\right)\right) \subset \left[y_0, y_0 + \frac{\delta_1}{2}\right] \subset \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right).$$

Moreover,

$$\psi(y_0) = y_0 \in \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right),$$

according to (5),

$$\psi(y) \in \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right),$$

and ψ is linear on $[y, y_0]$.

Furthermore,

$$\psi(z) \in \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right)$$

(by (5)),

$$\psi\left(y_0 + \frac{\delta_1}{2}\right) = y_0 + \frac{\delta_1}{2} \in \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right),$$

and ψ is linear on $\left[y_0 + \frac{\delta_1}{2}, z\right]$. Thus,

$$\Gamma\big(\psi \restriction (y, z)\big) \subset \left[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right] \times \left[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right] \subset V,$$

which completes the proof of inclusion (11).

Consequently,

$$\psi \in V_\Gamma(f).$$

It remains to show that

$$\psi \in B_{\rho_{uc}}(f, \varepsilon). \tag{12}$$

Evidently, $\psi(x) = f(x)$, for $x \in [0, y] \cup [z, 1]$. For $x \in (y, z)$ we have

$$\psi(x) \in \left[x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right]$$

and by (5)

$$f(x) \in \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right).$$

Hence, for $x \in (y, z)$, the following inequality is true $|f(x) - \psi(x)| < \frac{2}{n_0}$ and we finally obtain (12).

Summarizing,

$$\psi \in R \cap V_\Gamma(f) \cap B_{\rho_{uc}}(f, \varepsilon) \quad \text{and} \quad y_0 \in \mathfrak{E}_s^{\mathcal{C}}(\psi).$$

We now turn to the second case:

2. $x_0 \in H$.

Let I' be such a component of the set H that $x_0 \in I'$. Let a', b' denote the left and the right end of I', respectively. Since f is first return continuous from the right with respect to $\bar{q}$ at x_0, there exists a point $\hat{x}' \in \left(x_0, x_0 + \frac{1}{n_0}\right) \cap I'$ belonging to right first return path at x_0 based on $\bar{q}$, such that

$$f(\hat{x}') \in \left(f(x_0) - \frac{1}{n_0}, f(x_0) + \frac{1}{n_0}\right) = \left(x_0 - \frac{1}{n_0}, x_0 + \frac{1}{n_0}\right).$$

Obviously, $\hat{x}'$ is an element of H-trajectory $\bar{q}$, so it is a continuity point of f. The rest of the proof runs as in the case $x_0 \notin H$. □

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