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CORRIGENDUM TO A CATEGORY ANALOGUE OF THE GENERALIZATION OF LEBESGUE DENSITY TOPOLOGY

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ABSTRACT. The notion of $\mathcal{A}_I$-density point introduced in Wojdowski, W. *A topology stronger than the Lebesgue density topology*, in: Real Functions, Density Topology and Related Topics, Łódź Univ. Press, 2011, pp. 73–80 [WO1]. leads to the operator $\Phi_{\mathcal{A}_I}(A)$ which is not a lower density operator. We present a counterexample giving a corrected definition which should be used in [WO1] to keep all results valid.

In [WO] we introduced a notion of an $\mathcal{A}_I$-density density point of a set with the Baire property in the following way. Let S be the σ-algebra of sets having the Baire property on the real line $\mathbb{R}$ and $I \subset S$ the σ-ideal of sets of first category.

Let $\mathcal{A}_I$ be the family of subsets of interval $[-1,1]$ that are from S and have 0 as its I-density point.

Definition 1. We shall say that $x \in \mathbb{R}$ is an $\mathcal{A}_I$-density point of $A \in S$, if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_I$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere on $[-1,1]$ to χ_B.

In contrast to what was incorrectly claimed in [WO], the density operator $\Phi_{\mathcal{A}_I}(A)$ defined as a set of all $\mathcal{A}_I$-density points of A is not monotonic and thus, it is not a lower density. We shall present a counterexample and show how to modify the definition of an $\mathcal{A}_I$-density point so that the operator $\Phi_{\mathcal{A}_I}(A)$ is a lower density.

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A counterexample

Let $D = \left(0, \frac{1}{2}\right)$. Then, D is an open set such that $(0,1) \setminus D \in S \setminus I$. Let $\{c_n\}_{n \in N}$ be an arbitrary sequence of real numbers decreasing to 0, such that $c_1 < 1$, $\frac{c_{n+1}}{c_n} < \frac{1}{2}$ and $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$. We define a measurable set U as

$$U = \bigcup_{n=1}^{\infty} \left[(c_n \cdot D) \cap (c_{n+1}, c_n)\right].$$

Let $A = -U \cup U$.

By Proposition 3 of [WO], 0 is an $\mathcal{A}_I$-density point of A according to Definition 1. In [WO] it is also shown that 0 fails to be an I-density point of A.

Now, let

$$D_1 = \left[0, \frac{1}{2}\right) \cup \left(\frac{3}{4}, \frac{4}{4}\right),$$

$$D_2 = \left[0, \frac{1}{2}\right) \cup \left(\frac{5}{8}, \frac{6}{8}\right) \cup \left(\frac{7}{8}, \frac{8}{8}\right),$$

and consecutively,

$$D_n = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2} + \bigcup_{k=1}^{2^{n-1}} \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right)\right).$$

Let $\{c_n\}_{n\in\mathbb{N}}$ be defined as above. We define a set $E \in S$ as

$$E = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} \left[(c_n \cdot D_n) \cap (c_{n+1}, c_n)\right].$$

Clearly, E is a superset of A.

We shall show now that 0 is not an $\mathcal{A}_I$-density point of E: Consider the sequence $\{c_n\}_{n\in\mathbb{N}}$. Suppose that for some $B \in \mathcal{A}_I$, and for some subsequence $\{c_{n_m}\}$,

$$\chi_{\left(\left(\frac{1}{c_{n_m}} E\right) \cap \left[\frac{1}{2}, 1\right]\right)} \xrightarrow{a.e.} \chi_B \qquad \text{on} \quad \left[\frac{1}{2}, 1\right].$$

We shall consider two cases:

a) The set $B \cap \left[\frac{1}{2}, 1\right] \in S \setminus I$.
By definition, for every $n \in \mathbb{N}$, the set D_n is periodic within interval $\left(\frac{1}{2}, 1\right)$ with period $\frac{1}{2^n}$. We have

$$\frac{1}{c_{n_m}} E \cap \left(\frac{1}{2}, 1\right) = D_{n_m} \cap \left(\frac{1}{2}, 1\right) \qquad \text{for every} \quad m \in \mathbb{N}.$$

Thus, for every $\delta > 0$, there is an $m_0 \in \mathbb{N}$ such that $\frac{1}{c_{n_m}} E \cap \left(\frac{1}{2}, 1\right)$ is periodic with period $\frac{1}{2^{n_m}} < \delta$, for every $m > m_0$. Thus, B has to be periodic within interval $\left(\frac{1}{2}, 1\right)$ with period $\frac{1}{2^n}$, $n \in \mathbb{N}$. This implies that the set of $\mathcal{A}_I$-density points of B is dense in $\left(\frac{1}{2},1\right)$ and thus B is residual in $\left(\frac{1}{2},1\right)$. Since

$$-\left(D_{n_m}\cap\left(\frac{1}{2},1\right)\right)=\left(D^c_{n_m}\cap\left(\frac{1}{2},1\right)\right)-\frac{3}{2},$$

we may repeat the above argument replacing sets D_n on $\left(\frac{1}{2},1\right)$ with their complements and we similarly obtain that the complement of B is residual in $\left(\frac{1}{2},1\right)$, a contradiction.

b) The set $B\cap[0,1]\in I$.
Similarly as in a), we may show that also $B^c\cap[0,1]\in I$ a contradiction. Apparently, 0 cannot be an $\mathcal{A}_I$-density point of E in the sense of $\mathcal{A}_I$-density point as defined in [WO].

Finally, we have $A\subset E$ but $0\in\Phi_{\mathcal{A}_I}(A)\setminus\Phi_{\mathcal{A}_I}(E)$, i.e., $\Phi_{\mathcal{A}_I}(E)$ is not monotonic. In particular, part (4) of Theorem 1 in [WO] is false.

A new definition

Following the ideas from [WO1], we replace the Definition 1 in [WO] with

Definition 2. We shall say that $x\in\mathbb{R}$ is an $\mathcal{A}_I$-density point of $A\in S$, if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B\in\mathcal{A}_I$ such that the sequence $\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\}_{m\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere on B to 1.

The part (4) of Theorem 1 in [WO] can be now proved as follows

Theorem 1. *The mapping $\Phi_{\mathcal{A}_I}: S\to 2^{\mathbb{R}}$ has the following properties:*

(0) *For each $A\in S$, $\Phi_{\mathcal{A}_I}(A)\in S$.*
(1) *For each $A\in S$, $A\sim\Phi_{\mathcal{A}_I}(A)$.*
(2) *For each $A,B\in S$, if $A\sim B$ then $\Phi_{\mathcal{A}_I}(A)=\Phi_{\mathcal{A}_I}(B)$.*
(3) *$\Phi_{\mathcal{A}_I}(\emptyset)=\emptyset$, $\Phi_{\mathcal{A}_I}(\mathbb{R})=\mathbb{R}$.*
(4) *For each $A,B\in S$, $\Phi_{\mathcal{A}_I}(A\cap B)=\Phi_{\mathcal{A}_I}(A)\cap\Phi_{\mathcal{A}_I}(B)$.*

Proof. (4) Observe first that if $A\subset B$, A, $B\in S$, then $\Phi_{\mathcal{A}_I}(A)\subset\Phi_{\mathcal{A}_I}(B)$, so

$$\Phi_{\mathcal{A}_I}(A\cap B)\subset\Phi_{\mathcal{A}_I}(A)\cap\Phi_{\mathcal{A}_I}(B)\,.$$

To prove the opposite inclusion, assume $x\in\Phi_{\mathcal{A}_I}(A)\cap\Phi_{\mathcal{A}_I}(B)$. Let $\{t_n\}_{n\in N}$ be an arbitrary sequence of real numbers decreasing to zero. From $x\in\Phi_{\mathcal{A}_I}(A)$, by definition, there is its subsequence $\{t_{n_m}\}_{m\in N}$ and a set $A_1\in\mathcal{A}_I$ such that the sequence

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in N}$$

of characteristic functions converges I-almost everywhere on A_1 to 1. Similarly, for $\{t_{n_m}\}_{m\in N}$ from $x \in \Phi_{\mathcal{A}_I}(B)$, by definition, there is a subsequence $\left\{t_{n_{m_k}}\right\}_{k\in N}$ and a set $B_1 \in \mathcal{A}_I$ such that the sequence

$$\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot(A-x)\cap[-1,1]}\right\}_{k\in N}$$

of characteristic functions converges I-almost everywhere on B_1 to 1. It is clear that the sequence

$$\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot\big((A\cap B)-x\big)\cap[-1,1]}\right\}_{k\in N}$$

converges I-almost everywhere on $A_1 \cap B_1$ to 1, i.e., x is a $\Phi_{\mathcal{A}_I}$-density point of $A \cap B$. □

With the Definition 2, all results of [WO] stay valid. Since we do not require any convergence of the sequence

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in N}$$

on the set $[-1,1] \setminus B$, some proofs may be even shorter, for example, we may omit case $a < 1$ in part a) in proof of Proposition 3 in [WO].

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