DE GRUYTER OPEN

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.1515/tmmp-2016-0004
Tatra Mt. Math. Publ. **65** (2016), 49–59

ON $\mathcal{J}$-CONTINUOUS FUNCTIONS

Jacek Hejduk — Anna Loranty — Renata Wiertelak

ABSTRACT. This paper presents the properties of continuous functions equipped with the $\mathcal{J}$-density topology or natural topology in the domain and the range.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ – the set of positive integers and $\mathcal{L}$ – the family of Lebesgue measurable sets in $\mathbb{R}$. By $\lambda(A)$ we shall denote the Lebesgue measure of $A \in \mathcal{L}$ and by $|I|$ – the length of an interval I. Furthermore, $\mathcal{T}_{nat}$ will denote the natural topology on $\mathbb{R}$ and $\mathcal{T}_d$ – the density topology on $\mathbb{R}$. Moreover, if A, $B \subset \mathbb{R}$ and $z \in \mathbb{R}$ then

$$A \triangle B = (A \setminus B) \cup (B \setminus A),$$
$$A + z = \{a + z : a \in A\},$$
$$zA = \{z \cdot a : a \in A\}.$$

It is well known that a point $x_0 \in \mathbb{R}$ is a **density point** of $A \in \mathcal{L}$ if

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

An equivalent form of this condition is the following one:

$$\lim_{\substack{h_1 \to 0^+,\, h_2 \to 0^+ \\ h_1 + h_2 > 0}} \frac{\lambda\left(A \cap [x_0 - h_1, x_0 + h_2]\right)}{h_1 + h_2} = 1.$$

Sometimes it is written in the form (see [6]):

$$\forall_{\{J_n\}_{n\in\mathbb{N}}} \left(0 \in \bigcap_{n\in\mathbb{N}} J_n \wedge |J_n| \underset{n\to\infty}{\longrightarrow} 0\right) \Rightarrow \lim_{n\to\infty} \frac{\lambda\left(A \cap (J_n + x_0)\right)}{|J_n|} = 1,$$

where $\{J_n\}_{n\in\mathbb{N}}$ is a sequence of non-degenerate closed intervals.

2010 Mathematics Subject Classification: 54A10, 26A15, 54A20.
Keywords: $\mathcal{J}$-density topologies, $\mathcal{J}$-approximately continuous functions.

By $\mathcal{J}$ we will denote a sequence of non-degenerate and closed intervals $\{J_n\}_{n\in\mathbb{N}}$ **tending to zero**, that means $\operatorname{diam}\big\{J_n \cup \{0\}\big\} \underset{n\to\infty}{\longrightarrow} 0$. To shorten the notation, we will write $\mathcal{J}$ instead of $\{J_n\}_{n\in\mathbb{N}}$.

From now on, the family of all sequences of intervals tending to zero will be denoted by $\Im$. Moreover, we will consider sequences which differ in only a finite number of their terms to be identical.

Definition 1. Let $\mathcal{J}\in\Im$. We shall say that a point $x_0\in\mathbb{R}$ is a **$\mathcal{J}$-density point** of a set $A \in \mathcal{L}$ if

$$\lim_{n\to\infty} \frac{\lambda(A \cap (x_0 + J_n))}{|J_n|} = 1.$$

If $A \in \mathcal{L}$ and $\mathcal{J} \in \Im$, then

$$\Phi_{\mathcal{J}}(A) := \{x \in \mathbb{R} : x \text{ is a } \mathcal{J}\text{-density point of the set } A\}.$$

Property 2 ([5]). If $A \in \mathcal{L}$ and $\mathcal{J} \in \Im$, then $\Phi_{\mathcal{J}}(A) \in \mathcal{L}$.

Moreover, this operator fulfills the following property.

Property 3 ([5]). For any sets $A, B \in \mathcal{L}$ and $\mathcal{J} \in \Im$ we have:

(1) $\Phi_{\mathcal{J}}(\emptyset) = \emptyset$, $\Phi_{\mathcal{J}}(\mathbb{R}) = \mathbb{R}$;

(2) $\lambda(A \triangle B) = 0 \Rightarrow \Phi_{\mathcal{J}}(A) = \Phi_{\mathcal{J}}(B)$;

(3) $\Phi_{\mathcal{J}}(A \cap B) = \Phi_{\mathcal{J}}(A) \cap \Phi_{\mathcal{J}}(B)$.

It turned out that the analogue of the Lebesgue Density Theorem does not hold for every sequence of intervals $\mathcal{J}$ tending to zero. Studying paper [2], we can find a construction of a sequence $\mathcal{J} \in \Im$ and a set $A \in \mathcal{L}$ of positive measure such that $\lambda(\Phi_{\mathcal{J}}(A) \cap A) = 0$. However, if we consider a subfamily $\Im_\alpha \subset \Im$ such that for any sequence $\mathcal{J} \in \Im_\alpha$ we have

$$\alpha(\mathcal{J}) := \limsup_{n\to\infty} \frac{\operatorname{diam}(J_n \cup \{0\})}{|J_n|} < \infty,$$

then the analogue of the Lebesgue Density Theorem holds.

Theorem 4 ([5]). *If $\mathcal{J} \in \Im_\alpha$ and $A \in \mathcal{L}$, then $\lambda\big(\Phi_{\mathcal{J}}(A) \triangle A\big) = 0$.*

Property 3 and Theorem 4 mean that an operator $\Phi_{\mathcal{J}} \colon \mathcal{L} \to \mathcal{L}$ is a lower density operator for every $\mathcal{J} \in \Im_\alpha$. Therefore the family

$$\mathcal{T}_{\mathcal{J}} := \big\{A \in \mathcal{L} : A \subset \Phi_{\mathcal{J}}(A)\big\}$$

is a topology on $\mathbb{R}$ such that $\mathcal{T}_d \subset \mathcal{T}_{\mathcal{J}}$ (see [5]).

Obviously, one can ask what will happen if we consider any sequence $\mathcal{J} \in \Im$. In this case, we can prove the following fact.

Theorem 5 ([5]). *For every sequence $\mathcal{J} \in \Im$ and every set $A \in \mathcal{L}$ we have*

$$\lambda\big(\Phi_{\mathcal{J}}(A) \setminus A\big) = 0.$$

By Property 3 and Theorem 5 we obtain that an operator $\Phi_{\mathcal{J}}\colon \mathcal{L} \to \mathcal{L}$ is an almost lower density operator. Theorem 10 in [5] says that the family

$$\mathcal{T}_{\mathcal{J}} = \{A \in \mathcal{L} : A \subset \Phi_{\mathcal{J}}(A)\}$$

is a topology on $\mathbb{R}$ such that $\mathcal{T}_{nat} \subset \mathcal{T}_{\mathcal{J}}$ and $\mathcal{T}_{nat} \neq \mathcal{T}_{\mathcal{J}}$.

It is easy to see that if $\mathcal{J} = \{[-\frac{1}{n}, \frac{1}{n}]\}_{n\in\mathbb{N}}$, then x_0 is a $\mathcal{J}$-density point of a set $A \in \mathcal{L}$ if and only if x_0 is a density point of A (see [6]). Moreover, if we consider an unbounded and nondecreasing sequence $\langle s\rangle = \{s_n\}_{n\in\mathbb{N}}$ of positive numbers and a sequence $\mathcal{J} = \{[-\frac{1}{s_n}, \frac{1}{s_n}]\}_{n\in\mathbb{N}}$, then the notion of a $\mathcal{J}$-density point of a set $A \in \mathcal{L}$ is equivalent to the notion of an $\langle s\rangle$-density point of A (see [3]).

From the definition of a $\mathcal{J}$-density point and a $\mathcal{J}$-density topology it is easy to conclude the following property.

Property 6. For every $\mathcal{J} \in \Im$ and every set $A \in \mathcal{L}$ the following properties hold:

(i) $\underset{x\in\mathbb{R}}{\forall}\ \underset{a\in\mathbb{R}}{\forall}\ x \in \Phi_{\mathcal{J}}(A) \Leftrightarrow (x+a) \in \Phi_{\mathcal{J}}(A+a)$,

(ii) $\underset{x\in\mathbb{R}}{\forall}\ \underset{m\in\mathbb{R}\setminus\{0\}}{\forall}\ x \in \Phi_{\mathcal{J}}(A) \Leftrightarrow mx \in \Phi_{m\mathcal{J}}(mA)$,

(iii) $\underset{a\in\mathbb{R}}{\forall}\ A \in \mathcal{T}_{\mathcal{J}} \Leftrightarrow (A+a) \in \mathcal{T}_{\mathcal{J}}$,

(iv) $\underset{m\in\mathbb{R}\setminus\{0\}}{\forall}\ A \in \mathcal{T}_{\mathcal{J}} \Leftrightarrow mA \in \mathcal{T}_{m\mathcal{J}}$.

Since for any $\mathcal{J} \in \Im$, the operator $\Phi_{\mathcal{J}}$ is an almost lower density operator, so, by [4, Theorem 25.27], we obtain immediately the following claim.

Theorem 7. *Let $\mathcal{J} \in \Im$.*

(i) $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ *is neither a first countable, nor a second countable, nor a separable, nor a Lindelöf space,*

(ii) $\lambda(A) = 0$ *if and only if A is a closed and discrete set with respect to a topology $\mathcal{T}_{\mathcal{J}}$,*

(iii) *a set $A \subset \mathbb{R}$ is compact with respect to a topology $\mathcal{T}_{\mathcal{J}}$ if and only if A is finite.*

We say that a sequence of intervals $\mathcal{J} = \{[a_n, b_n]\}_{n\in\mathbb{N}} \in \Im$ is **right-side (left-side) tending to zero** if there exists $n_0 \in \mathbb{N}$ such that $b_n > 0$ ($a_n < 0$) for $n \geq n_0$ and

$$\lim_{n\to\infty} \frac{\min\{0, a_n\}}{b_n} = 0 \quad \left(\lim_{n\to\infty} \frac{\max\{0, b_n\}}{a_n} = 0\right).$$

Obviously, if for a sequence $\mathcal{J} \in \Im$ there exists $n_0 \in \mathbb{N}$ such that $J_n \subset [0, \infty)$ ($J_n \subset (-\infty, 0]$) for $n > n_0$, then $\mathcal{J}$ is right-side (left-side) tending to zero.

A sequence of intervals $\mathcal{J} \in \Im$ is **one-side tending to zero** if it is right-side or left-side tending to zero.

Theorem 8. *If $\mathcal{J}$ is a sequence of intervals tending to zero, then $[0,b) \in \mathcal{T}_{\mathcal{J}}$ for $b > 0$ ($(a,0] \in \mathcal{T}_{\mathcal{J}}$ for $a < 0$) if and only if the sequence $\mathcal{J}$ is right-side (left-side) tending to zero.*

Proof. We give the proof only for the case when the sequence $\mathcal{J}$ is right-side tending to zero; the second case is left to the reader. Sufficiency. Let $\mathcal{J} = \{[a_n, b_n]\}_{n\in\mathbb{N}}$ be a sequence of intervals right-side tending to zero and $b > 0$. Without the loss of generality we may assume that $0 < b_n < b$ for $n \in \mathbb{N}$. It is sufficient to show that $0 \in \Phi_{\mathcal{J}}([0,b))$. We prove that for every increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ there exists subsequence $\{n_{k_m}\}_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \frac{\lambda([0,b) \cap J_{n_{k_m}})}{|J_{n_{k_m}}|} = 1. \tag{1}$$

Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers. If $a_{n_k} \geq 0$ for infinitely many $k \in \mathbb{N}$, then we choose subsequence $\{n_{k_m}\}_{m\in\mathbb{N}}$ such that $a_{n_{k_m}} \geq 0$ for $m \in \mathbb{N}$. Hence, $J_{n_{k_m}} \subset [0,b)$ for $m \in \mathbb{N}$. Therefore, condition (1) is fulfilled.

Now, we assume that $a_{n_k} \geq 0$ only for finite numbers $k \in \mathbb{N}$. Then, there is a $k_1 \in \mathbb{N}$ such that for $k > k_1$ we have $a_{n_k} < 0$ and

$$\frac{\lambda([0,b) \cap J_{n_k})}{|J_{n_k}|} = \frac{|J_{n_k}| - \lambda([a_{n_k}, 0))}{|J_{n_k}|} = 1 - \frac{-a_{n_k}}{|J_{n_k}|} \geq 1 + \frac{a_{n_k}}{b_{n_k}}.$$

The above and the assumption that the sequence $\mathcal{J}$ is right-side tending to zero implies condition (1).

We conclude that, in both cases, $0 \in \Phi_{\mathcal{J}}([0,b))$.

Necessity. Let $\mathcal{J} = \{[a_n, b_n]\}_{n\in\mathbb{N}} \in \Im$ and $[0,b) \in \mathcal{T}_{\mathcal{J}}$, where $b > 0$. Obviously, $0 \in \Phi_{\mathcal{J}}([0,b))$ and $b_n \leq 0$ only for finitely many $n \in \mathbb{N}$. Without loss of generality we may assume that $b_n > 0$ for $n \in \mathbb{N}$. Suppose that $\mathcal{J}$ is not right-side tending to zero. Then,

$$\limsup_{n\in\mathbb{N}} \frac{|\min\{0, a_n\}|}{b_n} = \beta > 0.$$

So, there exists subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{|\min\{0, a_{n_k}\}|}{b_{n_k}} = \beta$$

and also that $a_{n_k} < 0$ for $k \in \mathbb{N}$. Moreover, there exists $k_0 \in \mathbb{N}$ such that

$$\frac{1}{2}\beta b_{n_k} < |a_{n_k}| < \frac{3}{2}\beta b_{n_k} \qquad \text{for} \quad k > k_0.$$

Then,

$$\frac{\lambda([0,b)\cap J_{n_k})}{|J_{n_k}|} = \frac{\lambda(J_{n_k}) - \lambda([a_{n_k},0))}{|J_{n_k}|} = 1 - \frac{|a_{n_k}|}{b_{n_k} + |a_{n_k}|}$$
$$\leq 1 - \frac{1/2\beta b_{n_k}}{b_{n_k} + 3/2\beta b_{n_k}} = 1 - \frac{\beta}{2+3\beta}.$$

It implies that $0 \notin \Phi_{\mathcal{J}}([0,b))$. This contradiction finishes the proof. □

A direct consequence of the above theorem is

Theorem 9. *If a sequence of intervals $\mathcal{J}$ is tending to zero then $[a,b) \in \mathcal{T}_{\mathcal{J}}$ ($(a,b] \in \mathcal{T}_{\mathcal{J}}$) for $a < b$ if and only if the sequence $\mathcal{J}$ is right-side (left-side) tending to zero.*

2. Continuous functions

For $\mathcal{J} \in \Im$ we consider four families of continuous functions defined as follows:

$$\mathcal{C}_{nat,\,nat} = \{f\colon (\mathbb{R}, \mathcal{T}_{nat}) \to (\mathbb{R}, \mathcal{T}_{nat}) \text{ and } f \text{ is continuous}\},$$
$$\mathcal{C}_{nat,\,\mathcal{J}} = \{f\colon (\mathbb{R}, \mathcal{T}_{nat}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{J}}) \text{ and } f \text{ is continuous}\},$$
$$\mathcal{C}_{\mathcal{J},\,nat} = \{f\colon (\mathbb{R}, \mathcal{T}_{\mathcal{J}}) \to (\mathbb{R}, \mathcal{T}_{nat}) \text{ and } f \text{ is continuous}\},$$
$$\mathcal{C}_{\mathcal{J},\,\mathcal{J}} = \{f\colon (\mathbb{R}, \mathcal{T}_{\mathcal{J}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{J}}) \text{ and } f \text{ is continuous}\}.$$

Property 10. For $\mathcal{J} \in \Im$ the family $\mathcal{C}_{nat,\,\mathcal{J}}$ consists of constant functions.

P r o o f. Let $f \in \mathcal{C}_{nat,\,\mathcal{J}}$ and $a, b \in \mathbb{R}$ be such that $a < b$. Then $f([a,b])$ is a nonempty and compact set with respect to the topology $\mathcal{T}_{\mathcal{J}}$. By Theorem 7, $f([a,b])$ is finite. Moreover, it is a connected set in $\mathcal{T}_{nat}$, so $f([a,b])$ is a singleton. Hence, $f(a) = f(b)$, and the function f is constant. □

Property 11. For $\mathcal{J} \in \Im$, the following inclusions hold:

(i) $\mathcal{C}_{nat,\,\mathcal{J}} \subsetneq \mathcal{C}_{nat,\,nat} \subset \mathcal{C}_{\mathcal{J},\,nat}$,

(ii) $\mathcal{C}_{nat,\,\mathcal{J}} \subsetneq \mathcal{C}_{\mathcal{J},\,\mathcal{J}} \subset \mathcal{C}_{\mathcal{J},\,nat}$.

P r o o f. All the inclusions are the consequence of the fact that $\mathcal{T}_{nat} \subset \mathcal{T}_{\mathcal{J}}$. The inclusions $\mathcal{C}_{nat,\,\mathcal{J}} \subset \mathcal{C}_{nat,\,nat}$ and $\mathcal{C}_{nat,\,\mathcal{J}} \subset \mathcal{C}_{\mathcal{J},\,\mathcal{J}}$ are proper because the identical function is a member of $\mathcal{C}_{nat,\,nat}$ and $\mathcal{C}_{\mathcal{J},\,\mathcal{J}}$ but not $\mathcal{C}_{nat,\,\mathcal{J}}$. □

Property 12. If $\mathcal{J}$ is a sequence of intervals one-side tending to zero then:

(i) $\mathcal{C}_{nat,\,nat} \setminus \mathcal{C}_{\mathcal{J},\,\mathcal{J}} \neq \emptyset$,

(ii) $\mathcal{C}_{\mathcal{J},\,\mathcal{J}} \setminus \mathcal{C}_{nat,\,nat} \neq \emptyset$.

P r o o f. We give the proof only for the case when the sequence $\mathcal{J}$ is right side--tending to zero; the second case is left to the reader. To show the first condition, we consider the function $f(x) = -x^2$. Obviously, $f \in \mathcal{C}_{nat,\,nat} \subset \mathcal{C}_{\mathcal{J},\,nat}$. By Theorem 9, $[-1,1) \in \mathcal{T}_{\mathcal{J}}$ but $f^{-1}([-1,1)) = [-1,1] \notin \mathcal{T}_{\mathcal{J}}$. Thus, $f \notin \mathcal{C}_{\mathcal{J},\,\mathcal{J}}$, and condition (i) is proved.

To prove the second condition, we define the function

$$h(x) = x - k \qquad \text{for} \quad x \in [k, k+1),\ k \in \mathbb{Z}.$$

It is easy to see that for every set $A \subset \mathbb{R}$

$$h^{-1}(A) = \bigcup_{k\in\mathbb{Z}} \Big(\big(A \cap [0,1)\big) + k \Big)$$

holds.

Thus for every set $A \in \mathcal{T}_{\mathcal{J}}$, by Theorem 9 and Property 6, we have that $h^{-1}(A) \in \mathcal{T}_{\mathcal{J}}$. Hence, $h \in \mathcal{C}_{\mathcal{J},\,\mathcal{J}} \subset \mathcal{C}_{\mathcal{J},\,nat}$. Since

$$h^{-1}\left(\left(-1, \frac{1}{2}\right)\right) = \bigcup_{k\in\mathbb{Z}} \left[k, k + \frac{1}{2}\right) \notin \mathcal{T}_{nat},$$

we get that $h \notin \mathcal{C}_{nat,\,nat}$ and condition (ii) is satisfied. □

A direct consequence of this proof is the following property.

Property 13. Let $\mathcal{J}$ be a sequence of intervals one-side tending to zero. Then the inclusions

(i) $\mathcal{C}_{nat,\,nat} \subset \mathcal{C}_{\mathcal{J},\,nat}$,

(ii) $\mathcal{C}_{\mathcal{J},\,\mathcal{J}} \subset \mathcal{C}_{\mathcal{J},\,nat}$

are proper.

The subsequent terminology is needed in the reminder of this section.

Either of the sets

$$\bigcup_{n\in\mathbb{N}} (a_n, b_n), \qquad \bigcup_{n\in\mathbb{N}} [a_n, b_n]$$

is a **right interval set at a point** x_0 if $x_0 < b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = x_0$.

In the case when $a_n < b_n < a_{n+1} < x_0$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = x_0$ it is called a **left interval set at a point** x_0.

We will call the union of a right interval set and a left interval set at the same point x_0 a **both interval set at a point** x_0. A set A is an **interval set at a point** x_0 if it is a right interval or a left interval or a both interval set at a point x_0. An interval set at a point 0 is simply called an **interval set**.

Theorem 14. *For every sequence of intervals $\mathcal{J} \in \Im$ there exists an interval set B consisting of open intervals such that 0 is an $\mathcal{J}$-density point of B.*

Proof. Let $J_n = [a_n, b_n]$ for $n \in \mathbb{N}$ and define

$$M_r = \{n \in \mathbb{N}\colon b_n > 0\}, \quad M_l = \{n \in \mathbb{N}\colon a_n < 0\}.$$

Clearly, at least one of these sets is infinite. There are three possibilities:

1^0 The set M_r is infinite and the set M_l is finite. We can assume that the set M_l is empty, hence $a_n \geq 0$ for $n \in \mathbb{N}$. Let $c_1 = \frac{1}{2}$ and

$$\begin{aligned} I(1) &= \left\{k \in \mathbb{N}\colon J_k \cap \left(\frac{1}{2}, 1\right) \neq \emptyset\right\}, \\ j(1) &= \max\Big(\{1\} \cup \big\{k \in I(1)\big\}\Big), \\ z_1 &= \min\left(\left\{\frac{1}{2}\right\} \cup \left\{\left|J_k \cap \left(\frac{1}{2}, 1\right)\right| : k \in I(1)\right\}\right), \\ d_1 &= c_1 + 2^{-j(1)} z_1. \end{aligned}$$

If we define c_k, $I(k)$, $j(k)$, z_k and d_k for $k = 1, 2, \ldots, n-1$, then there exists a natural number $i(n)$ such that $2^{1-i(n)} \leq |[c_{n-1}, d_{n-1}]|$. We put $c_n = 2^{-i(n)}$ and

$$\begin{aligned} I(n) &= \big\{k \in \mathbb{N}\colon J_k \cap (c_n, 2c_n) \neq \emptyset\big\}, \\ j(n) &= \max\Big(\{n\} \cup \big\{k \in I(n)\big\}\Big), \\ z_n &= \min\Big(\{c_n\} \cup \big\{|J_k \cap (c_n, 2c_n)|\colon k \in I(n)\big\}\Big), \\ d_n &= c_n + 2^{-j(n)} z_n. \end{aligned}$$

Notice that

$$d_n \leq 2c_n \leq |[c_{n-1}, d_{n-1}]| < c_{n-1} \tag{2}$$

and

$$|[c_n, d_n]| \leq 2^{-j(n)} |J_k| \leq 2^{-k} |J_k| \quad \text{for } k \in \mathbb{N}. \tag{3}$$

Putting

$$B = \bigcup_{n \in \mathbb{N}} (d_{n+1}, c_n),$$

we obtain that B is a right interval set. Moreover,

$$(J_n \setminus B) \subset J_n \cap \big([0, d_{k(n)+1}] \cup [c_{k(n)}, d_{k(n)}]\big),$$

where $k(n) = \min\big\{k \in \mathbb{N}\colon J_n \cap [c_k, d_k] \neq \emptyset\big\}$ for $n \in \mathbb{N}$. From (3) we obtain

$$|J_n \cap [c_{k(n)}, d_{k(n)}]| \leq |[c_{k(n)}, d_{k(n)}]| \leq 2^{-n} |J_n|.$$

In addition, from (2) and (3) we have

$$|J_n \cap [0, d_{k(n)+1}]| \le d_{k(n)+1} \le |[c_{k(n)}, d_{k(n)}]| \le 2^{-n}|J_n|.$$

Hence, for every $n \in \mathbb{N}$ we have

$$\lambda(J_n \setminus B) \le 2^{1-n}|J_n|. \tag{4}$$

Therefore,

$$\lim_{n\to\infty} \frac{\lambda(B \cap J_n)}{|J_n|} = \lim_{n\to\infty} \frac{|J_n| - \lambda(J_n \setminus B)}{|J_n|}$$
$$\ge \lim_{n\to\infty} \frac{|J_n| - 2^{1-n}|J_n|}{|J_n|} \ge \lim_{n\to\infty} 1 - 2^{1-n} = 1.$$

For that reason, 0 is an $\mathcal{J}$-density point of B.

2^0 The set M_l is infinite and the set M_r is finite. In this case we consider sequence of intervals $-\mathcal{J} = \{-J_n\}_{n\in\mathbb{N}}$. From the first part of the proof, there exists a right interval set C consisting of open intervals such that 0 is an $(-\mathcal{J})$-density point of C. Putting $B = -C$, we obtain a left interval set B composed of open intervals such that $0 \in \Phi_{\mathcal{J}}(B)$.

3^0 Both sets, M_l and M_r, are infinite. Let $M_l = \{l_n : n \in \mathbb{N}\}$ and $M_r = \{r_n : n \in \mathbb{N}\}$. We then consider sequences of intervals $\mathcal{J}_L = \{L_n\}_{n\in\mathbb{N}}$ and $\mathcal{J}_R = \{R_n\}_{n\in\mathbb{N}}$, where

$$\begin{aligned} L_n &= J_n \cap (-\infty, 0] && \text{for } n \in M_l, \\ L_n &= (-J_n) \cap (-\infty, 0] && \text{for } n \notin M_l, \\ R_n &= J_n \cap [0, \infty) && \text{for } n \in M_r, \\ R_n &= (-J_n) \cap [0, \infty) && \text{for } n \notin M_r. \end{aligned}$$

As in the first part, we define B_r. It is a right interval set consisting of open intervals such that it is an $\mathcal{J}_R$-density point of B_r. Similarly we define a left interval set B_l consisting of open intervals such that 0 is an $\mathcal{J}_L$-density point of B_l. Then, the set $B = B_l \cup B_r$ is the interval set composed of open intervals. We must show that 0 is an $\mathcal{J}$-density point of B.

It follows from (4) that

$$\lambda(L_n \setminus B) \le 2^{1-n}|L_n| \quad \text{and} \quad \lambda(R_n \setminus B) \le 2^{1-n}|R_n| \quad \text{for} \quad n \in \mathbb{N}.$$

Therefore,

$$\lambda(J_n \setminus B) \le \lambda(L_n \setminus B) + \lambda(R_n \setminus B) \le 2^{1-n}|L_n| + 2^{1-n}|R_n| \le 2^{2-n}|J_n| \quad \text{for} \quad n \in \mathbb{N}.$$

It implies that

$$\lim_{n\to\infty} \frac{\lambda(B \cap J_n)}{|J_n|} \ge \lim_{n\to\infty} 1 - 2^{2-n} = 1.$$

As a result, $0 \in \Phi_{\mathcal{J}}(B)$. □

As a simple consequence of the proof of the previous theorem, we obtain the following:

Theorem 15. *For every sequence of intervals $\mathcal{J}$ tending to zero there exists a sequence of intervals $\mathcal{K}$ tending to zero such that topologies generated by $\mathcal{J}$ and $\mathcal{K}$, respectively, are incomparable.*

Proof. The set $C = [-1,1] \setminus \big(B \cup \{0\}\big)$, where B is the set from the previous proof, is an interval set composed of closed intervals. We order them in the sequence $\mathcal{K}$. Then, $0 \in \Phi_{\mathcal{J}}(B)$, $0 \notin \Phi_{\mathcal{K}}(B)$, $0 \notin \Phi_{\mathcal{J}}(C)$, $0 \in \Phi_{\mathcal{K}}(C)$. It is easy to conclude that $\mathcal{T}_{\mathcal{J}} \setminus \mathcal{T}_{\mathcal{K}} \neq \emptyset$ and $\mathcal{T}_{\mathcal{K}} \setminus \mathcal{T}_{\mathcal{J}} \neq \emptyset$. □

Property 16. Let $\mathcal{J} \in \Im$ and A be an open interval set such that $0 \in \Phi_{\mathcal{J}}(A)$. Then, there exists an interval set $B \subset A$ composed of closed intervals such that $0 \in \Phi_{\mathcal{J}}(B)$.

Proof. Let $A = \bigcup_{k\in\mathbb{N}} A_k$, where A_k are disjoint open intervals. Putting

$$N_k := \{i \in \mathbb{N} \colon J_i \cap A_k \neq \emptyset\} \quad \text{for every } \ k \in \mathbb{N},$$

we obtain a finite set. Then we define

$$\begin{aligned} j_k &:= \max\big(\{k\} \cup \{i \colon i \in N_k\}\big),\\ l_k &:= \min\big(\{|A_k|\} \cup \{|J_i| \colon i \in N_k\}\big). \end{aligned}$$

Let B_k be a closed subinterval of A_k such that $\lambda(A_k \setminus B_k) \leq 2^{-(k+j_k)} l_k$ for $k \in \mathbb{N}$. Observe that for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$ we have

$$\begin{aligned} \lambda(B_k \cap J_n) &= \lambda(A_k \cap J_n) - \lambda((A_k \setminus B_k) \cap J_n)\\ &\geq \lambda(A_k \cap J_n) - 2^{-(k+j_k)} l_k\\ &\geq \lambda(A_k \cap J_n) - 2^{-(k+n)} |J_n|. \end{aligned}$$

Hence, the following holds for $B = \bigcup_{k\in\mathbb{N}} B_k$ and any $n \in \mathbb{N}$:

$$\begin{aligned} \lambda(B \cap J_n) &= \lambda\Bigg(\bigcup_{k\in\mathbb{N}} B_k \cap J_n\Bigg) = \sum_{k\in\mathbb{N}} \lambda(B_k \cap J_n)\\ &\geq \sum_{k\in\mathbb{N}} \Big(\lambda(A_k \cap J_n) - 2^{-(k+n)} |J_n|\Big)\\ &= \sum_{k\in\mathbb{N}} \lambda(A_k \cap J_n) - \sum_{k\in\mathbb{N}} 2^{-(k+n)} |J_n| = \lambda(A \cap J_n) - 2^{-n} |J_n|. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{\lambda(B \cap J_n)}{|J_n|} \geq \lim_{n\to\infty} \frac{\lambda(A \cap J_n) - 2^{-n}|J_n|}{|J_n|} = \lim_{n\to\infty} \frac{\lambda(A \cap J_n)}{|J_n|} - 2^{-n}.$$

Since $0 \in \Phi_{\mathcal{J}}(A)$, we have $0 \in \Phi_{\mathcal{J}}(B)$. □

Property 17. Let $\mathcal{J} \in \Im$. Then, there exists $\mathcal{K} \in \Im$ such that $\mathcal{T}_{\mathcal{J}} \neq \mathcal{T}_{\mathcal{K}}$, $\mathcal{C}_{\mathcal{J},\mathcal{J}} \neq \mathcal{C}_{\mathcal{K},\mathcal{K}}$ and $\mathcal{C}_{\mathcal{J},nat} \neq \mathcal{C}_{\mathcal{K},nat}$.

Proof. 1^o Suppose that the sequence $\mathcal{J}$ is left-side tending to zero. Then, as a sequence $\mathcal{K}$, we put any sequence right-side tending to zero and we define the function $f\colon \mathbb{R} \to \mathbb{R}$ in the following way: $f(x) = \chi_{[0,\infty)}$. Obviously, Theorem 9 implies that $\mathcal{T}_{\mathcal{J}} \neq \mathcal{T}_{\mathcal{K}}$. Moreover, it is easy to see that $f \notin \mathcal{C}_{\mathcal{J},nat}$. So, we obtain that

$$f \in \mathcal{C}_{\mathcal{K},\mathcal{K}} \subset \mathcal{C}_{\mathcal{K},nat} \quad \text{and} \quad f \notin \mathcal{C}_{\mathcal{J},nat} \supset \mathcal{C}_{\mathcal{J},\mathcal{J}}.$$

2^o Suppose that the sequence $\mathcal{J}$ is not left-side tending to zero. Then, as a sequence $\mathcal{K}$, we put any sequence left-side tending to zero and we define function $f\colon \mathbb{R} \to \mathbb{R}$ in the following way: $f(x) = \chi_{(0,\infty)}$. Arguments similar to those above show that

$$f \in \mathcal{C}_{\mathcal{K},\mathcal{K}} \subset \mathcal{C}_{\mathcal{K},nat} \quad \text{and} \quad f \notin \mathcal{C}_{\mathcal{J},nat} \supset \mathcal{C}_{\mathcal{J},\mathcal{J}}. \qquad \square$$

Let $\mathcal{J} = \{J_n\}_{n\in\mathbb{N}}$ and $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ be sequences of intervals. Then, the sequence ordered in an arbitrary fashion containing all intervals of the sequences $\mathcal{J}$ and $\mathcal{K}$, and denoted by $\mathcal{J} \cup \mathcal{K}$, is called **the union of sequences** $\mathcal{J}$ and $\mathcal{K}$.

Property 18. The sequences of intervals $\mathcal{J}$ and $\mathcal{K}$ are tending to zero if and only if the sequence $\mathcal{J} \cup \mathcal{K}$ is tending to zero.

Property 19. If $\mathcal{J} \in \Im$ and $\mathcal{K} \in \Im$, then

$$\mathcal{T}_{\mathcal{J}\cup\mathcal{K}} = \mathcal{T}_{\mathcal{J}} \cap \mathcal{T}_{\mathcal{K}}.$$

Proof. It is a direct consequence of the following fact:

$$\Phi_{\mathcal{J}\cup\mathcal{K}}(A) = \Phi_{\mathcal{J}}(A) \cap \Phi_{\mathcal{K}}(A). \qquad \square$$

Property 20. Let $\mathcal{J} \in \Im$ and $\mathcal{K} \in \Im$. Then,

(i) $\mathcal{C}_{\mathcal{J},nat} \cap \mathcal{C}_{\mathcal{K},nat} = \mathcal{C}_{\mathcal{J}\cup\mathcal{K},nat}$,

(ii) $\mathcal{C}_{nat,\mathcal{J}} \cap \mathcal{C}_{nat,\mathcal{K}} = \mathcal{C}_{nat,\mathcal{J}\cup\mathcal{K}}$,

(iii) $\mathcal{C}_{\mathcal{J},\mathcal{J}} \cap \mathcal{C}_{\mathcal{K},\mathcal{K}} \subsetneq \mathcal{C}_{\mathcal{J}\cup\mathcal{K},\mathcal{J}\cup\mathcal{K}}$.

Proof. The condition (i) and the inclusion (iii) are evident by Property 19. The condition (ii) follows from Property 10. Let

$$\mathcal{J} = \left\{\left[-\frac{1}{n}, 0\right]\right\}_{n\in\mathbb{N}}, \qquad \mathcal{K} = \left\{\left[0, \frac{1}{n}\right]\right\}_{n\in\mathbb{N}}.$$

Then, $\mathcal{J}$ is left-side tending to zero and $\mathcal{K}$ is right-side tending to zero. It is easy to observe that $\mathcal{T}_{\mathcal{J}\cup\mathcal{K}}$ is the density topology. Thus, the function $f(x) = -x$

belongs to the family $\mathcal{C}_{\mathcal{J}\cup\mathcal{K},\mathcal{J}\cup\mathcal{K}}$. By Theorem 9, we have that $[0,1) \in \mathcal{T}_{\mathcal{K}}$, whereas $f^{-1}([0,1)) = (-1,0] \notin \mathcal{T}_{\mathcal{K}}$. It implies that

$$f \notin \mathcal{C}_{\mathcal{K},\mathcal{K}} \quad \text{so} \quad f \notin (\mathcal{C}_{\mathcal{J},\mathcal{J}} \cap \mathcal{C}_{\mathcal{K},\mathcal{K}}). \qquad \square$$

REFERENCES

[1] BRUCKNER, A. M.: *Differentiation of Real Functions*. Lecture Notes in Math., Vol. 659, Springer Verlag, Berlin, 1978.

[2] CSÖRNYEI, M.: *Density theorems revisited*, Acta Sci. Math. **64** (1998), 59–65.

[3] FILIPCZAK, M.—FILIPCZAK, T.—HEJDUK, J.: *On the comparison of the density type topologies*, Atti Semin. Mat. Fis. Univ. Modena **LII** (2004), 1–11.

[4] HEJDUK, J.—WIERTELAK, R.: *On the Abstract Density Topologies Generated by Lower and Almost Lower Density Operators*. Traditional and Present-day Topics in Real Analysis, Łodź University Press, 2013.

[5] HEJDUK, J.—WIERTELAK, R.: *On the generalization of density topologies on the real line*, Math. Slovaca **64** (2014), 1267–1276.

[6] WILCZYŃSKI, W.: *Density topologies*, in: Handbook of Measure Theory (E. Pap., ed.), North Holland, Amsterdam, 2002, pp. 675–702.

Received December 3, 2014

Łódź University
Faculty of Mathematics
and Computer Science
Banacha 22
PL–90-238 Łódź
POLAND

E-mail: hejduk@math.uni.lodz.pl
loranta@math.uni.lodz.pl
wiertelak@math.uni.lodz.pl