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EQUI-CLIQUISHNESS AND THE HAHN PROPERTY

Vasyl' Nesterenko

ABSTRACT. We study the joint continuity of mappings of two variables. In particular, we show that for a Baire space X, a second countable space Y and a metric space Z, a map $f\colon X \times Y \to Z$ has the Hahn property (i.e., there is a residual subset A of X such that $A \times Y \subseteq C(f)$) if and only if f is locally equi-cliquish with respect to y and $\{x \in X\colon f^x \text{ is continuous}\}$ is a residual subset of X.

1. Introduction

Investigations of the relationships between separate and joint continuity, which began its history in classical works of Baire and Osgood, has been continued in papers of many mathematicians of the twentieth century. One of the variants of this research deals with a question whether a mapping $f\colon X \times Y \to Z$ has the Hahn property, i.e., there is a residual subset A of X such that $A \times Y \subseteq C(f)$ where $C(f)$ is the set of continuity points of f. Beginning with the condition of separate continuity, there were found a large number of sufficient conditions that a mapping $f\colon X \times Y \to Z$ has the Hahn property. These conditions relate to both properties of mapping f and properties of spaces X, Y and Z.

Calbrix–Troallic's theorem [1] is a starting point of this research: for arbitrary topological spaces X and Y, a metric space Z, a separately continuous map $f : X \times Y \to Z$ and a countable type set B in Y, the set $C_B(f) = \big\{x \in X : \{x\} \times B \subseteq C(f)\big\}$ is residual in X. As corollaries in [1], the authors obtained the following results: if X is a topological space, Y is a first (second) countable space and Z is a metric space, then $C_y(f) = C_{\{y\}}(f)$ is a residual subset of X for all $y \in Y$ ($C_Y(f)$ is a residual subset of X). These corollaries were extended in [2] for KC-functions (quasi-continuous with respect to the first variable and continuous with respect to the second variable) and in [3] for K_hC-functions (horizontally quasi-continuous and continuous with respect

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to the second variable). However, there was no analogue to Calbrix–Troallic's theorem for both KC-functions and K_hC-functions for a long time. Only in [4], Calbrix–Troallic's theorem was generalized to a wider class of functions.

Note that $F_V : X \ni x \mapsto f_x(V) \in 2^Z$ is lower quasi-continuous for each $V \in Y$ if and only if $f: X \times Y \to Z$ is weakly horizontally quasi-continuous [6].

Results of [4] and [5] were generalized in [7]: if X and Y are topological spaces, Z a metric space, $\mathcal{V}$ a countable system of subsets of Y, $\mathcal{V}_y = \{V \in \mathcal{V} : V$ is a neighbourhood of y in $Y\}$, $B(\mathcal{V}) = \{y \in Y : \mathcal{V}_y$ is a base of neighbourhood of y in $Y\}$, $f: X \times Y \to Z$ a map such that for each set $V \in \mathcal{V}$ with $V \cap B(\mathcal{V}) \neq \varnothing$ the map $F_V : X \ni x \mapsto F_V(x) = f^x(V) \subseteq Z$ is lower pseudo-quasicontinuous and lower categorical cliquish, then $R = \big\{x \in X : \{x\} \times (C(f^x) \cap B(\mathcal{V})) \subseteq C(f)\big\}$ is a residual subset of X.

All the listed results give only sufficient conditions for the existence of the set $C(f)$ of a certain type. Recently, in [8], there have been found necessary and sufficient conditions for a map $f: X \times Y \to Z$ to have the Hahn property: for a Baire space X, a second countable compact space Y, a countable base $\mathcal{V}$ of Y and a metric space Z, a map $f: X \times Y \to Z$ has the Hahn property if and only if for each $V \in \mathcal{V}$ the map $F_V : X \ni x \mapsto F_V(x) = f^x(V) \subseteq Z$ satisfies condition (A), is lower categorical cliquish, and $\big\{x \in X : C(f^x) = Y\big\}$ is a residual subset of X.

In this paper, we establish new necessary and sufficient conditions under which a mapping of two variables has the Hahn property.

2. Basic definitions and concepts

Let X and Y be topological spaces and $(Z, |\cdot - \cdot|)$ a metric space. Recall that for a non-empty set $E \subseteq Z$, the number $\operatorname{diam}(E) = \sup_{u,v \in E} |u - v|$ is called a *diameter of a set* E and, for a function $f : X \to Z$, the number $\omega_f(A) = \operatorname{diam}\big(f(A)\big)$ is called an *oscillation of* f *on the set* A.

For a function $f : X \times Y \to Z$ and a point $p = (x, y) \in X \times Y$, we put $f^x(y) = f_y(x) = f(p)$. A map $f: X \times Y \to Z$ is said to be:

- *equi-cliquish with respect to* y if for each $\varepsilon > 0$ and a non-empty open subset U of X there is a non-empty open subset G of X such that $G \subseteq U$ and $\omega_{f_y}(G) < \varepsilon$ for all $y \in Y$;
- *locally equi-cliquish with respect to* y *at* $(a, b) \in X \times Y$ if for each $\varepsilon > 0$ there is a neighbourhood V of b in Y such that for each neighbourhood U of a in X there is a non-empty open subset G of X such that $G \subseteq U$ and $\omega_{f_y}(G) < \varepsilon$ for all $y \in V$.

A map f is said to be *locally equi-cliquish with respect to y* if it is such at each point of $X \times Y$. It is obvious that the equi-cliquishness with respect to y implies the locally equi-cliquishness with respect to y. Note that the notion of equi--cliquishness of sequences was defined by J. Borsík, Ľ. Holá and D. Holý in [9].

Let $F : X \to Z$ be a multi-value map which puts in accordance with each point $x \in X$ a nonempty subset $F(x) \subseteq Y$. For a set $A \subseteq X$, the set

$$F(A) = \bigcup_{x \in A} F(x)$$

is its image under the multi-value map F. A multi-value map $F\colon X \to Z$:

- *satisfies condition (A)* if, for any $\varepsilon > 0$, a non-empty open subset U of X and a dense subset E of U with $\operatorname{diam}\big(F(E)\big) < \varepsilon$, there exists a non-empty open set $G \subseteq U$ such that $\operatorname{diam}\big(F(G)\big) < \varepsilon$;
- is said to be *lower categorical cliquish* if, for each $\varepsilon > 0$ and any non--meager subset E of X, there are a somewhere dense set $A \subseteq E$ and a map $g\colon A \to Z$ such that $g(x) \in F(x)$ for all $x \in A$ and $\operatorname{diam}\big(g(A)\big) < \varepsilon$.

3. The example

For a map $f\colon X \times Y \to Z$, under certain conditions on spaces X, Y and Z, the continuity with respect to the second variable and the continuity or quasi--continuity with respect to the first variable implies the Hahn property (see Introduction). If f is cliquish with respect to the first variable and continuous with respect to the second variable, then f need not have the Hahn property. This is shown by the following example.

EXAMPLE 1. There is a function $f\colon [0,1]^2 \to [0,1]$ such that f_y is cliquish for all $y \in [0,1]$, f^x is continuous for all $x \in [0,1]$, but f does not have the Hahn property.

Construction of the example. For each positive integer n, consider the continuous function $g_n\colon [0,1] \to \mathbb{R}$ defined by the formula

$$g_n(x) = \begin{cases} 0, & \frac{1}{n} \le x \le 1, \\ 2(1 - nx), & \frac{1}{2n} \le x < \frac{1}{n}, \\ 2nx, & 0 \le x < \frac{1}{2n}. \end{cases}$$

Let $\mathbb{Q} \cap [0,1] = \{r_n : n \in \mathbb{N}\}$ be a set of all rational numbers of $[0,1]$. Define the function $f\colon [0,1]^2 \to \mathbb{R}$ by the rule

$$f(x,y) = \begin{cases} g_n(y), & x = r_n, \\ 0, & x \notin \mathbb{Q} \cap [0,1]. \end{cases}$$

It is clear that f is continuous with respect to the second variable. We show that f is cliquish with respect to the first variable. Fix any point $y_0 \in [0,1]$. Since $\frac{1}{n} \to 0$, the set of positive integers n with $\frac{1}{n} > y_0$ is finite. Then, $f_{y_0}(x) \neq 0$ for finite number of numbers x. Hence, f_{y_0} is cliquish.

The function f not only does not satisfy the Hahn property but also, $(x,0) \notin C(f)$ for all $x \in [0,1]$ because $\omega_f(x,0) = 1$ for all $x \in [0,1]$. □

4. Points of continuity of equi-cliquish mappings

Let $\mathcal{V}$ be a system of subset of Y. For a point $y \in Y$, set

$$\mathcal{V}(y) = \{V \in \mathcal{V} \colon V \text{ is a neighborhood of } y \text{ in } Y\}$$

and

$$B(\mathcal{V}) = \big\{y \in Y \colon \mathcal{V}(y) \text{ is a base of neighbourhoods of } y \text{ in } Y\big\}.$$

For a map $f \colon X \times Y \to Z$ and a non-empty subset V of Y, we consider the multi--value map $F_V \colon X \ni x \mapsto F_V(x) = f^x(V) \subseteq Z$.

We will use the following theorem.

Theorem 4.1 ([8, Theorem 1]). *Let X and Y be topological spaces, Z a metric space, $\mathcal{V}$ a countable system of subsets of Y, and let $f \colon X \times Y \to Z$ be a map such that for each set $V \in \mathcal{V}$ with $V \cap B(\mathcal{V}) \neq \varnothing$, the map F_V satisfies condition (A) and is lower categorical cliquish. Then,*

$$R = \big\{x \in X : \{x\} \times \big(C(f^x) \cap B(\mathcal{V})\big) \subseteq C(f)\big\}$$

is a residual subset of X.

Theorem 4.2. *Let X and Y be topological spaces, Z a metric space, and let $f \colon X \times Y \to Z$ be an equi-cliquish map with respect to y. Then, for each non--empty set $V \in Y$, the map F_V satisfies condition (A) and is lower categorical cliquish.*

P r o o f. Let V be a non-empty subset of Y. Fix any $\varepsilon > 0$, an arbitrary non--empty open subset U of X and any subset E of X such that $U \subseteq \overline{E}$ and $\operatorname{diam}\big(F_V(E)\big) < \varepsilon$. Put $\varepsilon_1 = \frac{\varepsilon - \operatorname{diam}(F_V(E))}{4} > 0$. Since f is equi-cliquish with respect to y, there is a non-empty open subset G of X such that $G \subseteq U$ and $\omega_{f_y}(G) < \varepsilon_1$ for all $y \in Y$. We show that $\omega_f(G \times V) < \varepsilon$. Fix any points $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in G \times V$ and a point $u \in G \cap E$. Then,

$$\begin{aligned} |f(p_1) - f(p_2)| &\leq |f(p_1) - f(u, y_1)| + |f(u, y_1) - f(u, y_2)| + |f(u, y_2) - f(p_2)| \\ &< \varepsilon_1 + \operatorname{diam}(F_V(E)) + \varepsilon_1 \\ &= \frac{\varepsilon + \operatorname{diam}(F_V(E))}{2}. \end{aligned}$$

So, $\mathrm{diam} F_V(G) = \omega_f(G \times V) \le \frac{\varepsilon + \mathrm{diam}(F_V(E))}{2} < \frac{\varepsilon+\varepsilon}{2} < \varepsilon$. Hence, F_V satisfies condition (A).

Now, we show that F_V is lower categorical cliquish. Fix $\varepsilon > 0$ and an arbitrary non-meager subset E of X. Since E is non-meager, it is somewhere dense. Let E be a dense set in a non-empty open subset U of X, i.e., $U \subseteq \overline{E}$. From the equi--cliquishness with respect to y of f, it follows that there is a non-empty open subset G of X such that $G \subseteq U$ and $\omega_{f_y}(G) < \varepsilon$ for all $y \in Y$. Fix any point $b \in V$ and consider $A = G \cap E$. It is clear that A is dense in G and $A \subseteq E$. Consider the map $g \colon A \to Z$ for which $g(x) = f_b(x)$, $x \in A$. Then, $g(x) \in F_V(x)$ for all $x \in A$, and $\mathrm{diam}\big(g(A)\big) = \mathrm{diam}\big(f_b(A)\big) = \omega_{f_b}(A) \le \omega_{f_b}(G) < \varepsilon$. Hence, F_V is lower categorical cliquish. □

A subset B of a space Y is said to be a *set of countable type* [1] if there is a system $\mathcal{V}_B = \{V_n : n \in \mathbb{N}\}$ of open subsets V_n of Y such that $\mathcal{V}_B(y) = \{V_n \colon y \in V_n\}$ is a base of neighbourhoods of y in Y for each $y \in B$. A space Y is a set of countable type if and only if Y is the second countable space. For the first countable space Y, for all $y \in Y$, the set $\{y\}$ is of countable type.

Theorem 4.3. *Let X and Y be topological spaces, Z a metric space, B a subset countable type of Y, $f : X \times Y \to Z$ an equi-cliquish map with respect to y, and let $M = \big\{x \in X : B \subseteq C(f^x)\big\}$ be a residual subset of X. Then, there is a residual subset A of X such that $A \times B \subseteq C(f)$.*

P r o o f. By Theorem 4.2, it follows that for each non-empty subset V of Y, the map F_V satisfies condition (A) and is lower categorical cliquish. For $\mathcal{V}_B$, we have that $B(\mathcal{V}_B) = B$. Then, by Theorem 4.1, it follows that

$$R = \big\{x \in X \colon \{x\} \times \big(C(f^x) \cap B(\mathcal{V}_B)\big) \subseteq C(f)\big\}$$

is a residual subset of X. Note that $C(f^x) \cap B(\mathcal{V}_B) = B$ for all $x \in M$. The set $A = R \cap M$ is residual in X and $A \times B \subseteq C(f)$. □

The following result is a consequence of Theorem 4.3.

Theorem 4.4. *Let X be a topological space, Y a second countable space, Z a metric space, $f \colon X \times Y \to Z$ an equi-cliquish map with respect to y, and $M = \big\{x \in X : C(f^x) = Y\big\}$ be a residual subset of X. Then, f has the Hahn property.*

5. Points of continuity of locally equi-cliquish mappings

Theorem 5.1. *Let X and Y be topological spaces, Z a metric space, $\mathcal{V} = \{V_n \colon n \in \mathbb{N}\}$ a countable system of subsets of Y, and let $f \colon X \times Y \to Z$ be a locally equi-cliquish map with respect to y. Then,*

$$R = \big\{x \in X : \{x\} \times \big(C(f^x) \cap B(\mathcal{V})\big) \subseteq C(f)\big\}$$

is a residual subset of X.

P r o o f. Suppose that the complement

$$E_0 = X \setminus R = \big\{x \in X : \exists\, y_x \in C(f^x) \cap B(\mathcal{V}) | p_x = (x, y_x) \notin C(f)\big\}$$

is a non-meager subset of X. Since f is locally equi-cliquish with respect to y, for each $x \in E_0$ and for each $\varepsilon > 0$, there is a neighbourhood $V_\varepsilon(x)$ of y_x in Y such that for each neighbourhood U of x in X, there is a non-empty open subset G of X such that $G \subseteq U$ and $\omega_{f_y}(G) < \varepsilon$ for all $y \in V_\varepsilon(x)$.

For positive integers n and m, put

$$A_{m,n} = \Big\{x \in E_0 : \omega_f(p_x) > \frac{1}{m}, V_n \subseteq V_{\frac{1}{3m}}(x), \\ V_n \text{ is a neighbourhood of } y_x, \omega_{f^x}(V_n) < \frac{1}{3m}\Big\}.$$

If $x \in E_0$ then $p_x = (x, y_x) \notin C(f)$, $y_x \in C(f^x)$, and $\mathcal{V}(y_x)$ is a base of neighbourhoods of y in Y. Therefore, $\omega_f(p_x) > 0$ and $\omega_{f^x}(y_x) = 0$. There are positive integers n and m such that $\omega_f(p_x) > \frac{1}{m}$, $V_n \subseteq V_{\frac{1}{3m}}(x)$ and $\omega_{f^x}(V_n) < \frac{1}{3m}$. It is clear that $\bigcup_{m,n=1}^{\infty} A_{m,n} = E_0$. Since E_0 is non-meager, there are positive integers n_0 and m_0 such that $E = A_{m_0,n_0}$ is a somewhere dense set. Put $U = int\overline{E} \neq \varnothing$.

Fix any point $a \in E \cap U$. By locally equi-cliquishness with respect to y of f, it follows that there is a non-empty open subset G of X such that $G \subseteq U$ and $\omega_{f_y}(G) < \frac{1}{3m_0}$ for all $y \in V_{\frac{1}{3m_0}}(a)$. Since $V_{n_0} \subseteq V_{\frac{1}{3m_0}}(a)$, $\omega_{f_y}(G) < \frac{1}{3m_0}$ for all $y \in V_{n_0}$. We show that $\omega_f(G \times V_{n_0}) \leq \frac{1}{m_0}$. Fix any points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2) \in G \times V_{n_0}$ and $u \in G \cap E$. Then,

$$|f(p_1) - f(p_2)| \leq |f(p_1) - f(u, y_1)| + |f(u, y_1) - f(u, y_2)| + |f(u, y_2) - f(p_2)| \\ < \frac{1}{3m_0} + \frac{1}{3m_0} + \frac{1}{3m_0} = \frac{1}{m_0}.$$

Hence, $\omega_f(G \times V_{n_0}) \leq \frac{1}{m_0}$. It is clear that $\omega_f(u, y_u) \leq \omega_f(G \times V_{n_0}) \leq \frac{1}{m_0}$. However, $\omega_f(p_u) > \frac{1}{m_0}$ because $u \in E$. This contradiction proves that R is a non-meager set. □

By virtue of corollaries, we obtain the following results.

Theorem 5.2. *Let X and Y be topological spaces, Z a metric space, B a subset countable type of Y, $f : X \times Y \to Z$ a locally equi-cliquish map with respect to y, and let $M = \big\{x \in X : B \subseteq C(f^x)\big\}$ be a residual subset of X. Then, there is a residual subset A of X such that $A \times B \subseteq C(f)$.*

Theorem 5.3. *Let X be a topological space, Y a second countable space, Z a metric space, $f : X \times Y \to Z$ a locally equi-cliquish map with respect to y, and let $M = \big\{x \in X : C(f^x) = Y\big\}$ be a residual subset of X. Then, f has the Hahn property.*

6. A characterization of the Hahn property

Theorem 6.1. *Let X be a Baire space, Y a compact space, Z a metric space and $f\colon X \times Y \to Z$ a map with the Hahn property. Then, f is equi-cliquish with respect to y, and $M = \big\{x \in X\colon C(f^x) = Y\big\}$ is a residual subset of X.*

P r o o f. Since f has the Hahn property, $C_Y(f) = \big\{x \in X\colon \{x\} \times Y \subseteq C(f)\big\}$ is a residual subset of X. Then, M is a residual subset of X because $C_Y(f) \subseteq M$.

Fix any $\varepsilon > 0$. Consider a non-empty open subset U of X. Since X is a Baire space, $C_Y(f)$ is dense in X. Therefore, there is a point $a \in U$ such that $\{a\}\times Y \subseteq C(f)$. Then, for each $y \in Y$, there are open neighbourhoods U_y of a in X and V_y of y in Y such that $\omega_f(U_y \times V_y) < \varepsilon$. It is clear that $\{V_y : y \in V\}$ is an open covering of Y. Since Y is a compact space, there are points $y_1, y_2, \dots, y_n$ such that $Y = \bigcup_{k=1}^{n} V_{y_k}$. Put $G = U \cap \big(\bigcap_{k=1}^{n} U_{y_k}\big)$. Then, $\omega_{f_y}(G) < \varepsilon$ for all $y \in Y$. Hence, f is equi-cliquish with respect to y. □

Theorem 6.2. *Let X be a Baire space, Y a locally compact space, Z a metric space, and let $f\colon X \times Y \to Z$ be a map with the Hahn property. Then, f is locally equi-cliquish with respect to y and $M = \big\{x \in X\colon C(f^x) = Y\big\}$ is a residual subset of X.*

P r o o f. The set M is residual in X (see Theorem 6.1).

Fix any point $p_0 = (x_0, y_0) \in X\times Y$ and $\varepsilon > 0$. Let V be a compact neighbourhood of y_0 in Y. The restriction $f|_{X\times V}$ has the Hahn property. Therefore, $f|_{X\times V}$ is equi-cliquish with respect to y (Theorem 6.1). Hence, f is locally equi-cliquish with respect to y at p_0. □

Theorem 6.3. *Let X be a Baire space, Y a second countable locally compact space, and let Z be a metric space. A map $f\colon X \times Y \to Z$ has the Hahn property if and only if f is locally equi-cliquish with respect to y and $M = \{x \in X : f^x$ is continuous$\}$ is a residual subset of X.*

Theorem 6.4. *Let X be a Baire space, Y a second countable compact space, and let Z be a metric space. A map $f\colon X \times Y \to Z$ has the Hahn property if and only if f is equi-cliquish with respect to y and $M = \{x \in X : f^x$ is continuous$\}$ is a residual subset of X.*

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Chernivtsi National University
Department of Mathematical
Kotsubinsky Str. 2
58012 Chernivtsi
UKRAINE
E-mail: math.analysis.chnu@gmail.com