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# SOME ALGEBRAIC PROPERTIES OF FINITE BINARY SEQUENCES

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ABSTRACT. We study properties of differences of finite binary sequences with a fixed number of ones, treated as binary numbers from $\mathbb{Z}(2^m)$. We show that any binary sequence consisting of $m$ terms (except of the sequence $(1,0,\dots,0)$) can be presented as a difference of two sequences having exactly $n$ ones, whenever $\frac{1}{4}m < n < \frac{3}{4}m$.

## 1. Introduction

In the paper, we consider algebraic differences and sums of sets. It is well-known that the sets $A - A$ and $A + A$ have nonempty interiors for any set $A \subset \mathbb{R}^n$ of positive Lebesgue measure and for any second category set with Baire property. The Cantor ternary set has this property, too.

R. Ger (see [1] and later T. Banakh [6]) have stated the following problems:

Do there exist compact sets $A$'s of reals such that $A - A$ has a nonempty interior and $A + A$ or even $A + A + A$ has Lebesgue measure zero?

This question was partially answered by M. Crnjac, B. Guljaš and H. I. Miller in 1991. In the interesting paper [6], they defined a compact set

$$S := \left\{ \sum_{i=1}^{\infty} \frac{a_i}{7^i} : a_i \in \{0, 2, 6\} \right\}$$

such that $S - S = [-1, 1]$ and $S + S$ is a null set.

It is very surprising that the R. Ger and T. Banakh problem was fully resolved by British mathematicians T. H. Jackson, J. H. Williamson, D. R. Woodall, D. Connolly and J. A. Haight in the early seventies. They started from an interesting property of subsets of groups $\mathbb{Z}(p)$. In the series


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of papers [4], [7] and [8], they proved that for any positive integer $k$ there exist a number $p$ and a set $E \subset \mathbb{Z}(p)$ with $E - E = \mathbb{Z}(p)$ and a $k$-sum $E + \cdots + E \neq \mathbb{Z}(p)$. In [5], D. Connoly and J. H. Williamson proved that this property leads to the statement that for any positive integer $k$ there exists a compact subset $A$ of reals such that $A - A$ contains an interval, and a $k$-sum $A + \cdots + A$ is a null set. Their results are really impressive and almost forgotten.

In this paper, we look for sets with large differences and small 3-sums which satisfy some additional conditions. Our construction is strictly connected with the notion of statistical density of subsets of $\mathbb{N}$. We focus on finite binary sequences with small amount of ones. We also consider sets of real numbers which binary expansions have fixed density of ones. Theorem 1 leads us to a useful observation concerning the sets $A_p$ which are natural supports of Bernoulli-like measures.

For any $x \in [0, 1)$, $0.x_1x_2x_3\ldots_{(2)}$ denotes the binary expansion of $x$ with infinite many zero terms. It is well-known that the set of numbers with density of ones equal to $\frac{1}{2}$

$$A_{\frac{1}{2}} := \left\{ x \in [0, 1) : \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = \frac{1}{2} \right\}$$

has full Lebesgue measure on the interval $[0, 1)$ (see for example [3]). Therefore, for any $t \in [0, 1)$, the set $A_{\frac{1}{2}} \cap \left(A_{\frac{1}{2}} + t\right)$ is nonempty, and consequently each number $t$ from $[0, 1)$ can be represented as a difference of two numbers from $A_{\frac{1}{2}}$. Analogously, since the set $A_{\frac{1}{2}} \cap \left(t - A_{\frac{1}{2}}\right)$ is nonempty, each $t$ from $[0, 1)$ can be represented as a sum of two numbers with density of ones equal to $\frac{1}{2}$. In other words,

$$[0, 1) \subset A_{\frac{1}{2}} - A_{\frac{1}{2}} \quad \text{and} \quad [0, 1) \subset A_{\frac{1}{2}} + A_{\frac{1}{2}}.$$

In [3], P. Billingsley describes the family of probability measures $\mu_p$ which are distributions of the sums $\sum_{k=1}^{\infty} \frac{1}{2^k} X_k$, where $(X_k)$ is a sequence of independent random variables with $\Pr(X_k = 1) = p$ and $\Pr(X_k = 0) = 1 - p$. The set

$$A_p := \left\{ 0.x_1x_2x_3\ldots_{(2)} : \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = p \right\}$$

is a support of $\mu_p$. It turns out that for some $p$ the set $A_p + A_p$ is much smaller then $A_p - A_p$. It can be shown (see [2]) that for any $p \in \left(\frac{1}{4}, \frac{1}{3}\right)$, the set $A_p + A_p$ (and even $A_p + A_p + A_p$) has an empty interior, although $[0, 1) \subset A_p - A_p$.

The aim of our paper is to prove an interesting property of finite binary sequences which is the key point of the proof that the difference $A_p - A_p$ contains the interval $[0, 1)$ for $p \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Namely, we demonstrate a way to present a binary sequence of length $m$ as a difference of two sequences, each of which has exactly $n$ ones, whenever $\frac{1}{4}m < n < \frac{3}{4}m$.

Throughout the paper, unless otherwise stated, we assume that all numbers are positive integers. The set $[k,m] := \{n \in \mathbb{N} : k \le n \le m\}$ is called an *interval* (in $\mathbb{N}$). By $X_m$ we denote the set of all binary sequences with $m$ elements, and by $X_m^0$ the set of sequences starting from zero.

For a fixed sequence $x=(x_1,\dots,x_m)\in X_m$, the sets $J(x):=\{i\in[1,m]: x_i=1\}$ and $Z(x) := \{i \in [1,m] : x_i = 0\}$ are called *the set of ones* and *the set of zeros* of the sequence $x$, and we denote their cardinality by $j(x)$ and $z(x)$, respectively. If $A \subset \mathbb{N}$, then we write $J_A(x) := J(x) \cap A$, $Z_A(x) := Z(x) \cap A$, $j_A(x) := |J_A(x)|$ and $z_A(x) := |Z_A(x)|$. By $\mathbf{1}$ we denote the sequence $(1,\dots,1)$, and by $\mathbf{1}_A$ the sequence satisfying $J(x) = A$.

Sequences from $X_m$ can be treated as binary numbers from $\mathbb{Z}(2^m) := \{0,1,\dots,2^m-1\}$. We identify the sequence $(x_1,\dots,x_m)$ with the number $x_1 \times 2^{m-1} + \dots + x_{m-1} \times 2 + x_m$, and define addition $x+y$ in $X_m$ as addition modulo $2^m$ in $\mathbb{Z}(2^m)$.

Let $x,y \in X_m$. It is easy to check

- $j(x+y) \le j(x) + j(y)$.
- If $J(x) \cap J(y) = \emptyset$, then $j(x+y) = j(x) + j(y)$.
- If $J(y) = J(x) \cap [1,k]$, then

$$j(x+y) = \begin{cases} j(x) & \text{if } x_1 = 0, \\ j(x) - 1 & \text{if } x_1 = 1. \end{cases}$$

  In particular, $j(x+x) = j(x)$ or $j(x+x) = j(x) - 1$.
- If $J(y) \subset J(x)$, then $j(x+y) \le j(x)$.
- If $x := (0,1,0,1,\dots)$ (i.e., $x_i = 0$ for add $i$, and $x_i = 1$ for even $i$), then $j(x+y) \ge j(x) - j(y) + 1$.

## 2. The main result

The following theorem is the main goal of our paper.

**Theorem 1.** *Suppose that $\frac{1}{4}m<n<\frac{3}{4}m$. For any sequence $x\in X_m\setminus\{(1,0,\dots,0)\}$, there exist sequences $a,b \in X_m$ such that $x = b - a$ and $j(a) = j(b) = n$.*

**Remark 1.** We can formulate the assertion using only a sequence $a$ instead of two sequences $a$ and $b$: “there is a sequence $a$ such that $j(a) = j(x+a) = n$”.

Before starting a proof of Theorem 1, we discuss the assumptions.

**Remark 2.**

(1) Equality $j(a) = j(x+a)$ does not hold for $x := (1,0,\dots,0)$. Indeed, $j(x+a) = j(a) + 1$ if $a_1 = 0$ or $j(x+a) = j(a) - 1$ if $a_1 = 1$.

(2) It suffices to prove the assertion when $\frac{1}{4}m < n \leq \frac{1}{2}m$. Indeed, if $x = b - a$ and $j(a) = j(b) = n$, then taking $a' := \mathbf{1} - a$ and $b' := \mathbf{1} - b$, we obtain $x = a' - b'$ and $j(a') = j(b') = m - n$.

(3) It is sufficient to prove theorem for $x \in X_m^0$. Indeed, if $x_1 = 1$ than $-x \in X_m^0$ (because $x \neq (1, 0, \dots, 0)$), so there are sequences with $n$ ones such that $-x = b - a$, and consequently, $x = a - b$.

If $j(x) \leq n \leq \frac{1}{2}m$ and $x \in X_m^0$, it is easy to choose a suitable sequence $a$.

**Lemma 1.** *Suppose that $n \leq \frac{1}{2}m$. For any sequence $x \in X_m^0$ satisfying $j(x) \leq n$, there exists a sequence $a \in X_m^0$ such that $j(a) = j(x + a) = n$.*

P r o o f. If $j(x) = n$, then we set $a := x$. Suppose that $j(x) < n$ and write $k := j(x)$. Thus $j(x + x) = j(x) = k < n$, and since

$$m - |J(x) \cup J(x + x)| \geq 2n - 2k > n - k,$$

there exists a set $A \subset [2, m] \setminus \big(J(x) \cup J(x + x)\big)$ with $n - k$ elements. Taking $a := x + \mathbf{1}_A$, we obtain the desired sequence. □

In the proof of Theorem 1, we will use several lemmas. We first prove that a fixed sequence is a difference of two sequences each of which has exactly $n$ ones if it can be presented as a difference of two sequences with at most $n$ ones.

**Lemma 2.** *Let $n \leq \frac{1}{2}m$ and $x \in X_m^0$. If there exists a sequence $b \in X_m^0$ such that $j(b) \leq n$ and $j(x + b) \leq n$, then there is a sequence $a \in X_m^0$ such that $j(a) = j(x + a) = n$.*

P r o o f. We can require that $j(x) > n$ because otherwise, the assertion follows from Lemma 1. We consider four cases.

(I) $k := j(b) = j(x + b) \leq n$.

Of course, we can assume that $k < n$. Since

$$|[2, m] \setminus J(b) \setminus J(x + b)| \geq m - 1 - 2k \geq 2(n - k) - 1 \geq n - k,$$

there exists a set $B \subset [2, m] \setminus J(b) \setminus J(x + b)$ consisting of $n - k$ elements. Taking $a := b + \mathbf{1}_B$, we obtain $j(a) = j(b) + |B| = n$ and

$$j(x + a) = j\big((x + b) + \mathbf{1}_B\big) = j(x + b) + |B| = n\,.$$

(II) $j(b) < n$ and $j(x + b) = n$.

Let $r := n - j(b)$. We will define disjoint sets $B, C \subset J(x)$ such that $j(\mathbf{1}_{B\cup C}) = j(x + \mathbf{1}_{B\cup C}) \leq n$. If $J(b) \cap J(x) = \emptyset$, then we put $B := \emptyset$. If $J(b) \cap J(x) \neq \emptyset$, we define $B := \{p_1, \dots, p_{k_0}\} \subset J(b) \cap J(x)$ by recursion. As $p_1$, we set the last common one of $b$ and $x$, i.e., $p_1 := \max\big(J(b) \cap J(x)\big)$. Suppose that numbers

$p_1 > \cdots > p_i$ are chosen, and that $J(b) \cap J(x + \mathbf{1}_{p_1} + \cdots + \mathbf{1}_{p_i}) \neq \emptyset$. As $p_{i+1}$, we take the last common one of $b$ and $x + \mathbf{1}_{p_1} + \cdots + \mathbf{1}_{p_i}$, i.e.,

$$p_{i+1} := \max\bigl(J(b) \cap J(x + \mathbf{1}_{p_1} + \cdots + \mathbf{1}_{p_i})\bigr).$$

Let $x' := x + \mathbf{1}_B$. Of course, $x'$ and $b$ have no common one. Thus, from $x + b = x' + \mathbf{1}_{J(b)\setminus B}$, it follows that

$$j(x') = j(x+b) - |J(b) \setminus B| = n - (j(b) - k_0) = r + k_0\,.$$

We look for a set $C$ disjoint with $B$, and such that each of the sequences $\mathbf{1}_{B\cup C}$ and $x + \mathbf{1}_{B\cup C}$ has $r + k_0$ ones. Write $C := J(x') \cap J(x)$, i.e., $C$ is a set of "old ones" in $x'$. Clearly, $B \cap C = \emptyset$. Since the set $J(x') \setminus J(x)$ of "new ones" satisfies $|J(x') \setminus J(x)| = |B| = k_0$, we have

$$|C| = j(x') - |J(x') \setminus J(x)| = (r + k_0) - k_0 = r\,.$$

Note that zero is a successor of any "new one" in $x'$. Thus, zero is a predecessor of any series of "old ones". Consequently,

$$j(x + \mathbf{1}_{B\cup C}) = j(x' + \mathbf{1}_C) = j(x') = r + k_0 = |B \cup C|\,.$$

Since $r + k_0 \leq n$, the assertion follows from (I).

(III) $j(b) = n$ and $j(x+b) < n$.

Let us set up ones from $b$ in increasing order $J(b) = \{p_1, \ldots, p_n\}$. We consider sequences $x + \mathbf{1}_{A_i}$ where $A_0 = \emptyset$ and $A_i = \{p_1, \ldots, p_i\}$ for $i = 1, \ldots, n$. Since

$$j(x + \mathbf{1}_{A_0}) = j(x) > n > j(x+b) = j\bigl(x + \mathbf{1}_{A_n}\bigr),$$

there is $t \in \{0, \ldots, n-1\}$ such that

$$j(x + \mathbf{1}_{A_t}) \geq n > j\left(x + \mathbf{1}_{A_{t+1}}\right).$$

We will define a sequence $c$ for which

$$j(c) \leq n = j(x+c)\,. \tag{1}$$

If $j(x + \mathbf{1}_{A_t}) = n$, we put $c := \mathbf{1}_{A_t}$. Suppose that $j(x + \mathbf{1}_{A_t}) > n$. Adding $\mathbf{1}_{p_{t+1}}$ to $\mathbf{1}_{x+A_t}$, we reduce a number of ones from $j(x + \mathbf{1}_{A_t})$ to $j(x + \mathbf{1}_{A_{t+1}})$. Hence, in the sequence $x + \mathbf{1}_{A_t}$, there is a series of $j(x + \mathbf{1}_{A_t}) - j(x + \mathbf{1}_{A_{t+1}})$ ones immediately before $p_{t+1}$. It is easy to see that for $u := p_{t+1} - \bigl(n - j(x + \mathbf{1}_{A_{t+1}})\bigr)$, the sequence $c := \mathbf{1}_{A_t} + \mathbf{1}_u$ satisfies (1). Thus, the assertion of lemma follows from (II).

(IV) $j(b) < n$, $j(x+b) < n$ and $j(b) \neq j(x+b)$.

Using (II) or (III) for $n' := \max\bigl\{j(b), j(x+b)\bigr\}$, we find a sequence $c$ such that $j(c) = j(x+c) = n'$. By (I), we get the assertion. □

In Lemma 2, we have proved that the assertion of Theorem 1 holds when $j(x) \leq n$. The next lemma shows that, if $j(x) = n + k$, then it is sufficient to find a subinterval of $[2, m]$ which contains at least $k+1$ ones from a sequence $x$, or two disjoint subintervals which together have at least $k+2$ ones from $x$.

**Lemma 3.** *Suppose that $n \le \frac{1}{2}m$, $x \in X_m^0$ and $j(x) = n + k$ for some $k \in \mathbb{N}$. If there is an interval $U \subset [2, m]$ such that*

$$j_U(x) \ge k+1 \quad \text{and} \quad z_U(x) \le n-1, \tag{*}$$

*or there are disjoint intervals $U, V \subset [2, m]$ such that*

$$j_U(x) + j_V(x) \ge k+2 \quad \text{and} \quad z_U(x) + z_V(x) \le n-2, \tag{**}$$

*then there is a sequence $a \in X_m^0$ satisfying $j(a) = j(x+a) = n$.*

P r o o f. We will find a sequence $b \in X_m^0$ such that $j(b) \le n$ and $j(x+b) \le n$. Let us suppose that there exists an interval $U$ satisfying (*). Write

$$u := \min J_U(x), \; v := \max J_U(x), \; B := Z_{[u,v]}(x) \cup \{v\} \; \text{ and } \; b := 1_B.$$

It is easily seen that

$$j(b) = z_{[u,v]}(x) + 1 \le z_U(x) + 1 \le n$$

and

$$j(x+b) \le j(x) - j_U(x) + 1 \le (n+k) - (k+1) + 1 = n.$$

Thus, the assertion follows from Lemma 2. The proof in the case (**) is similar. □

In the proof of Theorem 1, we will use Lemma 3 several times. To find intervals satisfying conditions (*) or (**), we will often need the following easy combinatorical property, similar to the pigeonhole principle.

**Lemma 4.** *Suppose that $s \in \mathbb{N}$, $T$ is an interval in $\mathbb{N}$, $A \cup B = T$ and $A \cap B = \emptyset$. If $|T| \ge 2s$ and $|A| - |B| \ge s+1$, then there exists an interval $U \subset T$ such that*

$$|U| = 2s \quad \text{and} \quad |U \cap A| \ge s+1.$$

P r o o f. We can assume that $T = [1, k]$. Suppose, contrary to our claim, that $|A \cap U| \le s$ for any interval $U \subset T$ with $2s$ elements. In particular, we have $|A \cap U| \le |B \cap U|$. Let $p$ be a positive integer satisfying $2sp < k \le 2s(p+1)$. Let us consider a partition of $T$ into subintervals:

$$[1, 2s], [2s+1, 4s], \ldots, \big[2s(p-1)+1, 2sp\big], [2sp+1, k],$$

and let $A_i, B_i$ denote intersections of these intervals with $A$ and $B$ ($i = 0, \ldots p$). Then $|A_i| \le |B_i|$ for $i = 0, \ldots, p-1$ and $|A_p| \le s$ and consequently

$$|A| - |B| \le |A_p| - |B_p| \le |A_p| \le s$$

contrary to our assumptions. □

We are ready to prove Theorem 1. In the proof, we will often use a partition of interval $[2, 4n-1]$ into two or three subintervals: $L$ (left), $R$ (right) and $M$ (middle).

P r o o f o f T h e o r e m 1. In the beginning, assume that

$$m = 4n - 1 \quad \text{and} \quad x \in X^0_{4n-1}.$$

If $n = 1$, the proof is obvious. Suppose that $n \geq 2$ and fix a sequence $x \in X^0_{4n-1}$. We consider four cases.

(I) $j(x) \leq n$.

(II) $n + 1 \leq j(x) \leq 2n - 2$.

(III) $2n - 1 \leq j(x) \leq 3n - 2$.

(IV) $3n - 1 \leq j(x)$.

**Ad (I)** The assertion follows from Lemma 1.

**Ad (IV)** Let $p := \max J(x)$, $A := \big(Z(x) \setminus \{1\}\big) \cup \{p\}$ and $a := \mathbf{1}_A$. Then $j(a) = z(x) \leq n$, $x + a = \big(1, 0, \dots, \underset{p}{0}, 1, \dots, 1\big)$, and consequently, $j(x+a) = 4n - p \leq n$. This completes the proof by Lemma 2.

**Ad (II)** $j(x) = n + k$ where $1 \leq k \leq n-2$. Let us divide the interval $[2, 4n-1]$ into three subintervals such that each of the left and the right of them has $n+k$ elements (and the middle interval has $2n - 2k - 2$ elements):

$$L := [2, n+k+1]\,, \quad M := [n+k+2, 3n-k-1] \quad \text{and} \quad R := [3n-k, 4n-1]\,.$$

(II.1) $j_L(x) \geq k+1$ or $j_R(x) \geq k+1$.

We can assume that $j_L(x) \geq k+1$. Since $z_L(x) = n + k - j_L(x) \leq n - 1$, $L$ fulfills condition (*) from Lemma 3, which finishes the proof of the case (II.1).

(II.2) $j_L(x) \leq k$, $j_R(x) \leq k$ and $1 \leq k < \frac{n}{2}$.

The assertion follows again from Lemma 3, because $M$ fulfills condition (*):

$$j_M(x) = j(x) - j_L(x) - j_R(x) \geq (n+k) - 2k = n - k > k,$$
$$z_M(x) = |M| - j_M(x) \leq (2n - 2k - 2) - (n-k) = n - k - 2 < n - 1.$$

(II.3) $j_L(x) \leq k$, $j_R(x) \leq k$ and $\frac{n}{2} \leq k \leq n-2$.

Note that

$$j_{L\cup M}(x) = j(x) - j_R(x) \geq n = (k+2) + (n-k-2)\,,$$
$$z_{L\cup M}(x) = |L \cup M| - j_{L\cup M}(x) \leq (3n - k - 2) - n = (n-2) + (n-k)\,.$$

From $L \cup M$, we will remove a subinterval $U$ with $2n - 2k - 2$ elements, which has more zeros than ones. In this way, we will obtain a set which fulfills one of conditions (**) or (*) from Lemma 3.

Let $s := n - k - 1$. Since

$$z_L(x) - j_L(x) = |L| - 2j_L(x) \geq (n+k) - 2k = n - k = s + 1,$$
$$|L| = n + k > n \geq n + (n - 2k) > 2s,$$

Lemma 4 shows that there is an interval $U \subset L$ such that

$$z_U(x) \geq s + 1 = n - k,$$
$$j_U(x) \leq s - 1 = n - k - 2.$$

The set $A := L \cup M \setminus U$ is an interval or a union of two disjoint intervals, and

$$j_A(x) = j_{L \cup M}(x) - j_U(x) \geq k + 2,$$
$$z_A(x) = z_{L \cup M}(x) - z_U(x) \leq n - 2.$$

Thus, the assertion follows from Lemma 3.

**Ad (III)** $j(x) = 2n + h$ where $-1 \leq h \leq n - 2$.

Let us divide the interval $[2, 4n-1]$ into two subintervals each of which has $2n - 1$ elements:

$$L := [2, 2n] \quad \text{and} \quad R := [2n+1, 4n-1].$$

(III.1) $j_L(x) = j_R(x)$. Thus, $h = 2p$ for some $p \geq 0$, and $j_L(x) = j_R(x) = n + p$.

(III.1.1) $\{2n, 2n+1\} \cap J(x) = \emptyset$. Taking $L' := [2, 2n-1]$ and $R' := [2n+2, 4n-1]$, we obtain

$$j_{L'}(x) = j_{R'}(x) = n + p,$$
$$z_{L'}(x) = z_{R'}(x) = 2n - 2 - (n+p) = n - p - 2,$$

and consequently,

$$j_{R'}(x) - z_{R'}(x) - 1 = 2p + 1 \geq p + 1 \geq 1,$$
$$|R'| = 2n - 2 \geq 2(h+2) - 2 = 4p + 2 \geq 2(p+1).$$

By Lemma 4, there exists an interval $U \subset R'$ such that

$$j_U(x) \geq p + 2 \quad \text{and} \quad z_U(x) \leq p.$$

Since $L'$ and $U$ are disjoint intervals satisfying

$$j_{L'}(x) + j_U(x) \geq (n+p) + (p+2) = (n+h) + 2,$$
$$z_{L'}(x) + z_U(x) \leq (n - p - 2) + p = n - 2,$$

it is sufficient to use Lemma 3 for $k = n + h$.

(III.1.2) $\{2n, 2n+1\} \cap J(x) \neq \emptyset$. We can assume that $2n + 1 \in J(x)$. Writing $L' := [2, 2n+1]$ and $R' := [2n+2, 4n-1]$ we get

$$j_{L'}(x) = n + p + 1 \quad \text{and} \quad z_{L'}(x) = n - p - 1,$$
$$j_{R'}(x) = n + p - 1 \quad \text{and} \quad z_{R'}(x) = n - p - 1.$$

If $p = 0$, then $j(x) = n + n$, $j_{L'}(x) = n + 1$, $z_{L'}(x) = n - 1$, and the assertion follows from Lemma 3 (for $k = n$). Thus, we can require that $p \geq 1$. Since

$$j_{R'}(x) - z_{R'}(x) - 1 = 2p - 1 \geq p \quad \text{and} \quad |R'| = 2n - 2 \geq 2h + 2 > 2p,$$

Lemma 4 guarantees that there is an interval $U \subset R'$ such that

$$j_U(x) \geq p + 1 \quad \text{and} \quad z_U(x) \leq p - 1.$$

Thus $L'$ and $U$ are disjoint intervals satisfying

$$\begin{aligned} j_{L'}(x) + j_U(x) &\geq (n + p + 1) + (p + 1) = n + h + 2, \\ z_{L'}(x) + z_U(x) &\leq (n - p - 1) + (p - 1) = n - 2, \end{aligned}$$

and we obtain the assertion using Lemma 3 again (for $k = n + h$).
(III.2) $j_L(x) \neq j_R(x)$. We can assume that $j_L(x) > j_R(x)$. Taking $p := j_L(x) - n$, we obtain $p \geq 0$,

$$\begin{aligned} j_L(x) &= n + p \quad &&\text{and} \quad z_L(x) = n - p - 1, \\ j_R(x) &= n - p + h \quad &&\text{and} \quad z_R(x) = n + p - h - 1. \end{aligned}$$

Note that $1 \leq j_L(x) - j_R(x) = 2p - h$. Moreover, $p = 0$ yields $h = -1 < p$, and consequently, $h \geq p$ gives $p \geq 1$.
(III.2.1) $h < p$. We have

$$j_L(x) \geq (n + h) + 1 \quad \text{and} \quad z_L(x) \leq n - h - 2 \leq n - 1,$$

which implies the assertion by Lemma 3 (with $k = n + h$).
(III.2.2) $h > p \geq 1$. Setting $s := h - p + 1$, we get

$$j_R(x) - z_R(x) - 1 = 2h - 2p \geq s \geq 1 \quad \text{and} \quad |R| = 2n - 1 \geq 2(h + 2) - 1 > 2s.$$

By Lemma 4, there exists an interval $U \subset R$ such that

$$j_U(x) \geq s + 1 = h - p + 2 \quad \text{and} \quad z_U(x) \leq s - 1 = h - p.$$

Since $L$ and $U$ are disjoint intervals satisfying

$$\begin{aligned} j_L(x) + j_U(x) &\geq (n + p) + (h - p + 2) = n + h + 2, \\ z_L(x) + z_U(x) &\leq (n - p - 1) + (h - p) = n + h - 2p - 1 \leq n - 2, \end{aligned}$$

it suffices to use Lemma 3 for $k = n + h$.
(III.2.3) $h = p \geq 1$. We have

$$\begin{aligned} j_L(x) &= n + h \quad &&\text{and} \quad z_L(x) = n - h - 1, \\ j_R(x) &= n \quad &&\text{and} \quad z_R(x) = n - 1. \end{aligned}$$

If $2n + 1 \in J(x)$, then the interval $L' := [2, 2n + 1]$ satisfies

$$j_{L'}(x) = (n + h) + 1 \quad \text{and} \quad z_{L'}(x) = n - h - 1 < n - 1,$$

and we obtain the assertion using Lemma 3 again.

If $2n+1 \notin J(x)$, then the interval $R' := [2n+2, 4n-1]$ satisfies

$$j_{R'}(x) - z_{R'}(x) = 2.$$

Thus, there is a number $i$ such that $i, i+1 \in R' \cap J(x)$. Since, $L$ and $[i, i+1]$ are disjoint intervals and

$$j_L(x) + j_{[i,i+1]}(x) = (n+h) + 2,$$
$$z_L(x) + z_{[i,i+1]}(x) = n - h - 1 \leq n - 2.$$

Using Lemma 3 once more, we complete the proof in the case $m = 4n-1$.

Now, assume that
$$\frac{1}{4}m < n \leq \frac{1}{2}m \quad \text{and} \quad x \in X_m^0.$$

We know that for $y := (x_1, \ldots, x_m, 0, \ldots, 0) \in X_{4n-1}^0$ there exists $d \in X_{4n-1}^0$ satisfying $j(d) = j(y+d) = n$. Writing $c := (d_1, \ldots, d_m)$, we have $j(c) \leq j(d) = n$ and $j(x+c) \leq j(y+d) = n$. Thus, by Lemma 2, there is a sequence $a \in X_m^0$ such that $j(a) = j(x+a) = n$. By Remark 2, we obtain the assertion in the general case. □

**Remark 3.** If $n \leq \frac{1}{4}m$, then the sequence $x := (0, 1, 0, 1, \ldots) \in X_m$ cannot be written as $x = b - a$ where $j(a) = j(b) = n$. Indeed, if $j(a) = n$, then $j(x+a) \geq j(x) - j(a) + 1 \geq 2n - n + 1 > n$. By Remark 2, the same is true when $n \geq \frac{3}{4}m$.

In some applications, it is convenient to use the following obvious consequence of Theorem 1.

**Corollary 1.** *Suppose that $\frac{1}{4}m < n < \frac{3}{4}m$. For any sequence $x \in X_m$, there exist sequences $a, b \in X_m$ such that $x = b - a$ and $n = j(a) \leq j(b) \leq n+1$.*

## 3. Consequences

The main reason for proving Theorem 1 is a problem connected with sets $A_p$ mentioned in Introduction. Recall that $A_p$ is the set of numbers from the interval $[0,1)$ which binary expansion (with infinitely many zero terms) has density of ones equal to $p$. Using Theorem 1, it can be shown (see [2]) that

$$A_p - A_p \supset [0,1) \qquad \text{for} \quad p \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

i.e., that any $x$ from $[0,1)$ can be written as a difference $x = b - a$ of two numbers from $A_p$. To prove it, the binary expansion of $x$ is devided into finite sequences $x^1, x^2, x^3, \ldots$ with lengths converging to $\infty$, and such that each sequence $x^i$ starts from zero. Then each $x^i$ is presented as a difference $x^i = b^i - a^i$ of sequences with the density of ones close to $p$. Gluing $a^i$ together, we obtain $a$, and from $b^i$, we obtain $b$.

Note that all $A_p$ are Borel sets, and that $\operatorname{int}(A_p + A_p + A_p) = \emptyset$ for $p < \frac{1}{3}$ (see [2]). Hence, for any $p \in \left(\frac{1}{4}, \frac{1}{3}\right)$, the difference $A_p - A_p$ is much bigger than 3-sum of $A_p$.

Theorem 1 can also be applied to show that for infinitely many $p$ there is a set $E \subset \mathbb{Z}(p)$ with $E - E = \mathbb{Z}(p)$ and $E + E + E \neq \mathbb{Z}(p)$. Let us consider the set

$$E := \{x \in X_7 : j(x) \leq 2\}.$$

Due to Theorem 1, $E - E = X_7$. On the other hand, for any $x, y, z \in E$, we have $j(x+y+z) \leq j(x)+j(y)+j(z) \leq 6$, and consequently, $(1, \dots, 1) \notin E+E+E$. Using notation $\mathbb{Z}\left(2^7\right)$ instead of $X_7$, we see that the set

$$E := \{x \in \mathbb{Z}(128) : j(x) \leq 2\} = \{2^k + 2^l : 0 \leq k \leq l \leq 6\} \cup \{2^0\}$$

satisfies $E - E = \mathbb{Z}(128)$ and $E + E + E \neq \mathbb{Z}(128)$.

We can also find analogous sets for $m \geq 7$.

**Proposition 1.** *For any $m \geq 7$, there is a set $E \subset X_m$ such that*

$$E - E = X_m \quad \textit{and} \quad E + E + E \neq X_m.$$

P r o o f. It is easy to check that the set

$$E := \{x \in X_m : j_{[m-6,m]}(x) \leq 2\}$$

satisfies the required conditions. □

It is worth noting that Theorem 1 is not necessary in the proof of Proposition 1. The equality $E - E = X_7$ may be easy verified by direct calculations. However, using methods from the proof of Theorem 1, one can generalize Proposition 1. It can be shown that for $k \in \left[2^{m-1}, 2^m\right)$ with $j(k) \geq \frac{3}{4}m + 9$, there is a set $E \subset \mathbb{Z}(k)$ such that $E - E = \mathbb{Z}(k)$ and $E + E + E \neq \mathbb{Z}(k)$.

If $n \leq \frac{1}{4}m$, then the set $E := \{x \in X_m : j(x) \leq n\}$ satisfies $E - E \neq X_m$ (see Remark 2). Thus, using our method, we cannot indicate a set $E \subset X_m$ with $E - E = X_m$ and $E + E + E + E \neq X_m$.

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