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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.1515/tmmp-2016-0001
Tatra Mt. Math. Publ. **65** (2016), 1–21

LIMIT THEOREMS FOR k-SUBADDITIVE LATTICE GROUP-VALUED CAPACITIES IN THE FILTER CONVERGENCE SETTING

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ABSTRACT. We investigate some properties of lattice group-valued positive, monotone and k-subadditive set functions, and in particular, we give some comparisons between regularity and continuity from above. Moreover, we prove different kinds of limit theorems with respect to filter convergence. Furthermore, some open problems are posed.

1. Introduction

Limit theorems for lattice group-valued set functions have been an object of several recent investigations. A comprehensive survey can be found in [6]. There are also several versions of theorems of this kind for finitely or countably additive measures in the setting of filter convergence (for a related literature, see also [5–9], [14]). In [7], some Brooks-Jewett, Nikodým and Vitali-Hahn-Saks-type theorems are proved for positive and finitely additive lattice group-valued measures with respect to filter convergence, in which the pointwise convergence of the involved measures is required, not necessarily with respect to a single order sequence or regulator. Here, we deal with set functions, which are not necessarily finitely additive. There have been several studies about limit theorems in the non-additive context, for instance measuroids, k-triangular, quasi-triangular, null-additive set functions (for a literature related to these topics, see for instance [13], [15–17], [20–23], [25], [26]).

2010 Mathematics Subject Classification: Primary: 26E50, 28A12, 28A33, 28B10, 28B15, 40A35, 46G10, 54A20, 54A40; Secondary: 06F15, 06F20, 06F30, 22A10, 28A05, 40G15, 46G12, 54H11, 54H12.
Keywords: lattice group, (diagonal) filter, filter (D)-convergence, filter (O)-convergence, k-subadditive capacity, continuous capacity, regular capacity, (s)-bounded capacity, Fremlin's lemma, Maeda-Ogasawara-Vulikh theorem, limit theorem, Brooks-Jewett theorem, Vitali-Hahn-Saks theorem, Nikodým theorem, Dieudonné theorem.

In this paper, we deal with the non-additive case in the setting of lattice groups and filter convergence. We consider positive and k-subadditive set functions. Some examples are the so-called "M-measures", namely monotone set functions m with $m(\emptyset) = 0$, continuous and compatible with respect to supremum and infimum, which have several applications, for example to intuitionistic fuzzy events, which are pairs $A = (\mu_A, \nu_A)$ of measurable $[0,1]$-valued functions with $\mu_A + \nu_A \leq 1$ (see also [1], [11]). In the literature, it is often dealt with the so-called *observables*, namely set functions defined on the family of all Borel subsets of a nonempty set G with values in the space R of all intuitionistic fuzzy sets (see also [24]). Every element of R can be identified with a pair of functions $X = (X^\flat, 1 - X^\sharp)$, where X is a suitable observable. The set functions $X^\flat$, $X^\sharp$ are examples of M-measures (see also [11]). We investigate some basic properties of lattice group-valued k-subadditive capacities and give some comparison results between regularity and continuity from above. Moreover, we prove different kinds of limit theorems in the filter convergence setting, in which it is supposed that the involved filter is diagonal. An example of a diagonal filter is a family of subsets of $\mathbb{N}$ having asymptotic density one (see [6]). Since the involved set functions are positive and monotone, then our techniques, which are similar to those in [7], allow to consider only diagonal filters. The used argument is to fix a decreasing/disjoint sequence of sets, to apply the Maeda-Ogasawara-Vulikh representation theorem of Dedekind complete lattice groups as subgroups of suitable continuous extended-real valued functions (see also [3]) and to apply the results obtained for real-valued set functions in [23] to the σ-algebra generated by the chosen sequence. Moreover, for technical reasons, in our context we deal with (D)-convergence, because it is possible to use the Fremlin's lemma which allows to replace a series of (D)-sequences with a single regulator. In [5], [6], [8], [9], [14], some limit theorems are proved for finitely additive and not necessarily positive lattice group-valued measures, for diagonal filters which satisfy some additional properties. Finally, we pose some open problems.

2. Filters and lattice groups

We begin with recalling the following basic notions on filters (see also [6], [9]).

Definitions 2.1.

(a) A *filter* $\mathcal{F}$ of $\mathbb{N}$ is a nonempty collection of subsets of $\mathbb{N}$ with $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever A, $B \in \mathcal{F}$, such that for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$.

(b) A filter of $\mathbb{N}$ is said to be *free* if and only if it contains the filter $\mathcal{F}_{\mathrm{cofin}}$ of all cofinite subsets of $\mathbb{N}$.

(c) Given a free filter $\mathcal{F}$ of $\mathbb{N}$, we say that a subset of $\mathbb{N}$ is *$\mathcal{F}$-stationary* if and only if it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of $\mathbb{N}$.

(d) A free filter $\mathcal{F}$ of $\mathbb{N}$ is said to be *diagonal* if and only if for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that $J \setminus A_n$ is finite for all $n \in \mathbb{N}$ (see also [6], [9]).

Remark 2.2. Observe that the filter $\mathcal{F}_{\rm st}$ of $\mathbb{N}$ consisting of all subsets of $\mathbb{N}$ having asymptotic density one is diagonal (see also [6]). Moreover, if $(\Delta_n)_n$ is a partition of $\mathbb{N}$ into infinite subsets and $(A_n)_n$ is a sequence with $A_n \subset \Delta_n$ for each $n \in \mathbb{N}$, then the filter generated by the set $\bigcup_{n=1}^{\infty}(\Delta_n \setminus A_n)$ is diagonal (see also [2]). Some other examples of diagonal and non-diagonal filters can be found in [2], [6].

We now recall the following fundamental concepts on lattice groups (see also [6], [12]).

Definitions 2.3.

(a) A Dedekind complete lattice group R is said to be *super Dedekind complete* if and only if for every nonempty set $A \subset R$ bounded from above, there is a finite or countable subset A' having the same supremum as A.

(b) A sequence $(\sigma_p)_p$ in R is an *(O)-sequence* if and only if it is decreasing and $\bigwedge_{p\in\mathbb{N}} \sigma_p = 0$.

(c) A bounded double sequence $(a_{t,l})_{t,l}$ in R is a *(D)-sequence* or a *regulator* if and only if $(a_{t,l})_l$ is an (O)-sequence for any $t \in \mathbb{N}$.

(d) A lattice group R is said to be *weakly σ-distributive* if and only if

$$\bigwedge_{\varphi\in\mathbb{N}^{\mathbb{N}}} \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \right) = 0$$

for every (D)-sequence $(a_{t,l})_{t,l}$ in R.

(e) A sequence $(x_n)_n$ in R is said to be *order convergent* (or *(O)-convergent*) to x if and only if there exists an (O)-sequence $(\sigma_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x| \le \sigma_p$ for each $n \ge n_0$, and in this case, we write $(O)\lim_n x_n = x$.

(f) A sequence $(x_n)_n$ in R is *(D)-convergent* to x if and only if there is a (D)-sequence $(a_{t,l})_{t,l}$ in R such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n^* \in \mathbb{N}$ with $|x_n - x| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ whenever $n \ge n^*$, and we write $(D)\lim_n x_n = x$.

(g) We call *sum* of a series $\sum_{n=1}^{\infty} x_n$ in R the limit $(O)\lim_n \sum_{r=1}^{n} x_r$, if it exists in R.

We now recall the Fremlin's lemma, which has a fundamental importance in the setting of (D)-convergence, because it allows us to replace a series of regulators with a single (D)-sequence.

Lemma 2.4 (see also [19, Lemma 1C], [24, Theorem 3.2.3])**.** *Let R be any Dedekind complete lattice group and let $\left(a_{t,l}^{(n)}\right)_{t,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geq 0$, there is a (D)-sequence $(a_{t,l})_{t,l}$ in R with*

$$u \wedge \left(\sum_{n=1}^{q} \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \right) \right) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \qquad \text{for every} \quad q \in \mathbb{N} \quad \text{and} \quad \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Remark 2.5. Observe that in every Dedekind complete lattice group R any (O)-convergent sequence is (D)-convergent too, while the converse is true if and only if R is weakly σ-distributive. Moreover, using the following theorem, one obtains that in any Dedekind complete lattice group order convergence of arbitrary families implies (D)-convergence, while, if the involved lattice group is super Dedekind complete and weakly σ-distributive, then we also have the converse implication (see also [4], [12]).

The following result links (O)-sequences and regulators and will be useful to investigate some properties of lattice group-valued measures.

Theorem 2.6 (see also [4, Theorem 3.1])**.** *Given any Dedekind complete lattice group R and any (O)-sequence $(\sigma_l)_l$ in R, the double sequence defined by $a_{t,l} := \sigma_l$, t, $l \in \mathbb{N}$, is a (D)-sequence, and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$,*

$$\sigma_l \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \tag{1}$$

where $l = \varphi(1)$. Conversely, if R is super Dedekind complete and weakly σ-distributive, then for any regulator $(a_{t,l})_{t,l}$ in R there exists an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is $\varphi_p \in \mathbb{N}^{\mathbb{N}}$ with

$$\bigvee_{t=1}^{\infty} a_{t,\varphi_p(t)} \leq \sigma_p. \tag{2}$$

We now recall the Maeda-Ogasawara-Vulikh theorem, which gives a representation of lattice groups as subsets of continuous extended real-valued functions defined on suitable topological spaces (see also [3], [6], [12]). From now on, we denote the supremum and infimum in R by the symbols $\vee$ and $\wedge$ and the pointwise supremum and infimum by sup and inf, respectively.

Theorem 2.7. *Given a Dedekind complete lattice group R, there exists a compact extremely disconnected topological space Ω, unique up to homeomorphisms,* *such that R can be embedded isomorphically as a subgroup of*

$$C_\infty(\Omega) \\ = \Big\{ f \in \widetilde{\mathbb{R}}^\Omega \colon f \textit{ is continuous, and } \big\{\omega \colon |f(\omega)| = +\infty\big\} \textit{ is nowhere dense in } \Omega \Big\}.$$

Moreover, if we set $\widehat{a}$ to be an element of $C_\infty(\Omega)$ which corresponds to $a \in R$ under the above isomorphism, then, for any family $(a_\lambda)_{\lambda \in \Lambda}$ of elements of R with $R \ni a_0 = \bigvee_\lambda a_\lambda$ (where the supremum is taken with respect to R), $\widehat{a_0} = \bigvee_\lambda \widehat{a_\lambda}$ with respect to $C_\infty(\Omega)$, and we get $\widehat{a_0}(\omega) = \sup_\lambda \widehat{a_\lambda}(\omega)$ in the complement of a meager subset of Ω. The same is true for $\bigwedge_\lambda a_\lambda$.

Now, we recall the concepts of order and (D)-filter convergence in the lattice group setting.

Definitions 2.8.

(a) Let $\mathcal{F}$ be a filter of $\mathbb{N}$. A sequence $(x_n)_n$ in R $(O\mathcal{F})$*-converges to* $x \in R$ if and only if there exists an (O)-sequence $(\sigma_p)_p$ such that

$$\{n \in \mathbb{N} : |x_n - x| \le \sigma_p\} \in \mathcal{F} \qquad \text{for any} \quad p \in \mathbb{N}\,.$$

(b) A sequence $(x_n)_n$ in R $(D\mathcal{F})$*-converges to* $x \in R$ if and only if there is a (D)-sequence $(\alpha_{t,r})_{t,r}$ with

$$\left\{ n \in \mathbb{N} : |x_n - x| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)} \right\} \in \mathcal{F} \qquad \text{for all} \quad \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Remarks 2.9.

(a) Observe that, when $R = \mathbb{R}$, the $(O\mathcal{F})$- and $(D\mathcal{F})$-convergence coincide with the usual filter convergence. Moreover, when $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, $(O\mathcal{F})$- and $(D\mathcal{F})$--convergence are equivalent to (O)- and (D)-convergence, respectively.

(b) In any Dedekind complete lattice group R, $(O\mathcal{F})$-convergence implies $(D\mathcal{F})$-convergence, and, when R is super Dedekind complete and weakly σ-distributive, the converse implication holds too (see [9, Theorem 2.3]).

The following technical lemma links order filter convergence with usual (O)-convergence in the setting of diagonal filters and lattice groups, and will be useful to prove our main limit theorems.

Lemma 2.10 (see also [6, Lemma II.2.23], [9, Lemma 2.2])**.** *Let R be any Dedekind complete lattice group, let $(a_{j,n})_{j,n}$ be a double sequence in R, and let $\mathcal{F}$ be a diagonal filter of $\mathbb{N}$. If*

$$(O\mathcal{F}) \lim_{j \in \mathbb{N}} a_{j,n} = 0 \quad \textit{for each} \quad n \in \mathbb{N} \tag{3}$$

with respect to an (O)*-sequence* $\left(\sigma_p^{(n)}\right)_p$ *(depending on* n*), then for every* $I \in \mathcal{F}^*$ *there exists* $J \in \mathcal{F}^*$*,* $J \subset I$*, with* $(O)\lim_{j\in J} a_{j,n} = 0$ *for any* $n \in \mathbb{N}$ *with respect to* $\left(\sigma_p^{(n)}\right)_p$*.*

Proof. For each $n \in \mathbb{N}$, let $\left(\sigma_p^{(n)}\right)_p$ be an (O)-sequence associated with (3). By hypothesis, for each n, $p \in \mathbb{N}$ we have

$$A_{n,p} := \left\{ j \in \mathbb{N} : |a_{j,n}| \le \sigma_p^{(n)} \right\} \in \mathcal{F}.$$

As $\mathcal{F}$ is diagonal, for any $I \in \mathcal{F}^*$ there is $J \in \mathcal{F}^*$, $J \subset I$, such that for every n, $p \in \mathbb{N}$ the set $J \setminus A_{n,p}$ is finite. We have

$$B_{n,p} := \left\{ j \in J : |a_{j,n}| \not\le \sigma_p^{(n)} \right\} = J \setminus A_{n,p},$$

and hence $B_{n,p}$ is finite too. So, for each n, $p \in \mathbb{N}$ there is $\overline{j} \in \mathbb{N}$ (without loss of generality $\overline{j} \in J$) with $|a_{j,n}| \le \sigma_p^{(n)}$ whenever $j \in J$, $j \ge \overline{j}$. Thus, we get the assertion. $\square$

Remark 2.11. Note that the set J in Lemma 2.10 depends only on I and not on n.

3. Lattice group-valued capacities

We now recall the basic concepts and some main properties of lattice group--valued capacities (see also [22], [23]). From now on, R denotes a Dedekind complete lattice group, G is an infinite set, Σ is a σ-algebra of subsets of G, and k is a fixed positive integer.

Definitions 3.1.

(a) A *capacity* $m\colon \Sigma \to R$ is a set function increasing with respect to the inclusion and such that $m(\emptyset) = 0$.

(b) A capacity m is said to be k*-subadditive* on Σ if and only if

$$m(A \cup B) \le m(A) + k\, m(B) \quad \text{whenever} \quad A, B \in \Sigma,\ A \cap B = \emptyset. \tag{4}$$

(c) When $R = \mathbb{R}$, a 1-subadditive capacity is called also a *submeasure* (see also [5], [16]–[18]).

The following result holds.

Proposition 3.2. *A capacity m is k-subadditive on Σ if and only if for every $n \in \mathbb{N}$, $n \geq 2$, and for each $E_1, E_2, \dots, E_n \in \Sigma$ we get*

$$m\left(\bigcup_{q=1}^{n} E_q\right) \leq m(E_1) + k \sum_{q=2}^{n} m(E_q). \tag{5}$$

P r o o f. We prove the "only if" part only, since the "if" part is trivial.

We first prove (5) when $n = 2$. If $E_1, E_2 \in \Sigma$, from (4) and monotonicity of m, we get

$$\begin{aligned} m(E_1 \cup E_2) &= m\big(E_1 \cup (E_2 \setminus E_1)\big) \\ &\leq m(E_1) + k\, m(E_2 \setminus E_1) \\ &\leq m(E_1) + k\, m(E_2). \end{aligned} \tag{6}$$

In the general case, taking into account (6), proceeding by induction, supposed that (5) holds for $n-1$, we obtain

$$\begin{aligned} m\left(\bigcup_{q=1}^{n} E_q\right) &= m\left(\left(\bigcup_{q=1}^{n-1} E_q\right) \cup E_n\right) \leq m\left(\bigcup_{q=1}^{n-1} E_q\right) + k\, m(E_n) \\ &\leq m(E_1) + k \sum_{q=2}^{n-1} m(E_q) + k\, m(E_n) = m(E_1) + k \sum_{q=2}^{n} m(E_q). \end{aligned}$$

□

Definitions 3.3.

(a) We say that a capacity m is *continuous from above at $\emptyset$* if and only if

$$(O) \lim_n m(H_n) = \bigwedge_n m(H_n) = 0$$

whenever $(H_n)_n$ is a decreasing sequence in Σ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$.

(b) The capacities $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are said to be *uniformly continuous from above at $\emptyset$* if and only if

$$(O) \lim_n \left(\bigvee_j m_j(H_n)\right) = \bigwedge_n \left(\bigvee_j m_j(H_n)\right) = 0$$

for each decreasing sequence $(H_n)_n$ of elements of Σ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$.

(c) A capacity m is *k-σ-subadditive* on Σ if and only if

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq m(E_1) + k \sum_{n=2}^{\infty} m(E_n) \tag{7}$$

for any sequence $(E_n)_n$ from Σ.

The following property will be useful in the sequel.

Proposition 3.4. *Let $m\colon \Sigma \to R$ be a k-subadditive capacity continuous from above at $\emptyset$. Then, m is k-σ-subadditive.*

Proof. First of all, we prove (7) when $(E_n)_n$ is a disjoint sequence of elements of Σ. Since m is continuous from above at $\emptyset$, we get

$$(O)\lim_n m\left(\bigcup_{q=n+1}^{\infty} E_q\right) = \bigwedge_n m\left(\bigcup_{q=n+1}^{\infty} E_q\right) = 0. \tag{8}$$

Taking (5) into account, for every $n \geq 2$ we have

$$\begin{aligned} m\left(\bigcup_{q=1}^{\infty} E_q\right) &\leq m\left(\bigcup_{q=1}^{n} E_q\right) + k\, m\left(\bigcup_{q=n+1}^{\infty} E_q\right) \\ &\leq m(E_1) + k\sum_{q=2}^{n} m(E_q) + k\, m\left(\bigcup_{q=n+1}^{\infty} E_q\right). \end{aligned} \tag{9}$$

Letting n tend to $+\infty$, taking (8) into account, and using monotonicity at the right place, from (9) we obtain (7), at least when the E_n's are pairwise disjoint. In the general case, let $(E_n)_n$ be any sequence of elements of Σ and put $A_1 := E_1$, $A_n := E_n \setminus \left(\bigcup_{q=1}^{n-1} E_q\right)$. It is not difficult to check that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n$ and the A_n's are pairwise disjoint. Thus we get

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} A_n\right) \leq m(A_1) + k\sum_{n=2}^{\infty} m(A_n) \leq m(E_1) + k\sum_{n=2}^{\infty} m(E_n).$$

□

Definitions 3.5.

(a) A capacity $m : \Sigma \to R$ is (s)-*bounded* on Σ if and only if there exists an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ and for every disjoint sequence $(C_h)_h$ in Σ there is a positive integer h_0 with $m(C_h) \leq \sigma_p$ whenever $h \geq h_0$.

(b) We say that the capacities $m_j\colon \Sigma \to R$ are *uniformly* (s)-*bounded* on Σ if and only if there is an (O)-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ and for any disjoint sequence $(C_h)_h$ in Σ there exists $h_0 \in \mathbb{N}$ with $m_j(C_h) \leq \sigma_p$ for all $j \in \mathbb{N}$ and $h \geq h_0$.

(c) The sequence of capacities $m_j : \Sigma \to R$, $j \in \mathbb{N}$, is said to be *equibounded* on Σ if and only if there is $u \in R$ with $m_j(A) \leq u$ for each $A \in \Sigma$ and $j \in \mathbb{N}$.

Definition 3.6. A topology τ on Σ is a *Fréchet-Nikodým topology* if and only if the functions $(A, B) \mapsto A\Delta B$ and $(A, B) \mapsto A \cap B$ from $\Sigma \times \Sigma$ (endowed with the product topology) to Σ are continuous, and for any τ-neighborhood V of $\emptyset$

in Σ there exists a τ-neighborhood U of $\emptyset$ in Σ such that, if $E \in \Sigma$ is contained in some suitable element of U, then $E \in V$ (see also [18]).

Remark 3.7. Observe that a topology τ on Σ is a Fréchet-Nikodým topology if and only if there exists a family of submeasures $\mathcal{Z} := \{\eta_i : i \in \Lambda\}$, such that a basis of τ-neighborhoods of $\emptyset$ in Σ is given by

$$\mathcal{D} := \Big\{ U_{\varepsilon,J} := \big\{ A \in \Sigma : \eta_i(A) < \varepsilon \text{ for all } i \in J \big\} : \varepsilon \in \mathbb{R}^+,\, J \subset \Lambda \text{ is finite} \Big\}$$

(see also [5], [18]).

Definitions 3.8.

(a) Let τ be a Fréchet-Nikodým topology on Σ. A capacity $m : \Sigma \to R$ is said to be *τ-continuous* on Σ if and only if for each decreasing sequence $(H_n)_n$ in Σ, with $\tau\text{-}\lim_n H_n = \emptyset$, we get

$$(O) \lim_n m(H_n) = \bigwedge_n m(H_n) = 0.$$

(b) Let $\mathcal{G}, \mathcal{H} \subset \Sigma$ be two lattices, such that $\mathcal{G}$ is closed under countable unions, and the complement of every element of $\mathcal{H}$ belongs to $\mathcal{G}$. We say that a capacity $m \colon \Sigma \to R$ is *$\mathcal{G}$-$\mathcal{H}$-regular* if and only if it satisfies the following condition:

(R1) for every $E \in \Sigma$ there are two sequences $(F_n)_n$ in $\mathcal{H}$ and $(G_n)_n$ in $\mathcal{G}$, with

$$F_n \subset F_{n+1} \subset E \subset G_{n+1} \subset G_n \qquad \text{for any} \quad n, \tag{10}$$

and $(O) \lim_n m(G_n \setminus F_n) = \bigwedge_n m(G_n \setminus F_n) = 0$.

(c) We say that the capacities $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are *$\mathcal{G}$-$\mathcal{H}$-uniformly regular* if and only if for any $E \in \Sigma$ there exist two sequences $(F_n)_n$ in $\mathcal{H}$ and $(G_n)_n$ in $\mathcal{G}$ satisfying (10) and with

$$(O) \lim_n \left(\bigvee_j m_j(G_n \setminus F_n) \right) = \bigwedge_n \left(\bigvee_j m_j(G_n \setminus F_n) \right) = 0\,.$$

The following result holds.

Proposition 3.9 (see also [8, Proposition 3.5])**.** *Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of $\mathcal{G}$-$\mathcal{H}$-regular capacities. Then we get:*

(R2) *for every $E \in \Sigma$ there are two sequences $(F_n)_n$ in $\mathcal{H}$ and $(G_n)_n$ in $\mathcal{G}$ fulfilling* (10) *and with*

$$(O) \lim_n m_j(G_n \setminus F_n) = \bigwedge_n m_j(G_n \setminus F_n) = 0 \qquad \textit{for every} \quad j \in \mathbb{N}\,.$$

Now, we give a comparison result between lattice group-valued continuous from above at $\emptyset$ and $\mathcal{G}$-$\mathcal{H}$-regular capacities, extending [6, Theorem II.3.1] and [10, Theorem 2.2], which were formulated for σ-additive and $\mathcal{G}$-$\mathcal{H}$-regular measures. For technical reasons, we use the tool of (D)-convergence since we deal with series of regulators, and so, it is possible to apply Lemma 2.4.

Theorem 3.10. *Let R be a Dedekind complete weakly σ-distributive lattice group, (G, d) a compact metric space, Σ the σ-algebra of all Borel sets of G, and let $\mathcal{G}$ and $\mathcal{H}$ be the lattices of all open and all compact subsets of G, respectively. Then every k-subadditive $\mathcal{G}$-$\mathcal{H}$-regular capacity $m\colon \Sigma \to R$ is continuous from above at $\emptyset$.*

Conversely, if R is also super Dedekind complete, then every k-subadditive capacity $m\colon \Sigma \to R$, continuous from above at $\emptyset$ is $\mathcal{G}$-$\mathcal{H}$-regular.

Proof. We begin with the first part. Arbitrarily choose a decreasing sequence $(H_n)_n$ in Σ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$. By $\mathcal{G}$-$\mathcal{H}$-regularity of m and Theorem 2.6, (1), for each $n \in \mathbb{N}$, in correspondence with H_n, there is a (D)-sequence $\big(a_{t,l}^{(n)}\big)_{t,l}$, such that the (O)-limit in (R1) is a (D)-limit with respect to it. Arbitrarily choose $\varphi \in \mathbb{N}^{\mathbb{N}}$: then there are $D_n \in \mathcal{H}$ and $U_n \in \mathcal{G}$ (depending on n and φ) with $D_n \subset H_n \subset U_n$ and

$$m(U_n \setminus D_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)}, \ \ m(U_n \setminus D_n) \leq m(G) \qquad \text{for each} \quad n \in \mathbb{N}\,. \tag{11}$$

In correspondence with $u := m(G)$, by virtue of Lemma 2.4, there is a regulator $(a_{t,l})_{t,l}$ with

$$u \wedge \left(\sum_{n=1}^{q} \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \right) \right) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \qquad \text{for all} \quad q \in \mathbb{N} \ \text{ and } \ \varphi \in \mathbb{N}^{\mathbb{N}}. \tag{12}$$

Set $C_q = \bigcap_{n=1}^{q} D_n$, $q \in \mathbb{N}$: then $C_q \in \mathcal{H}$, $C_q \subset D_q \subset H_q$ for any q, the sequence $(C_q)_q$ is decreasing and $\bigcap_{q=1}^{\infty} C_q = \emptyset$. Since the sets C_q are compact, the finite intersection property gives existence to an integer $\overline{n}$ such that $C_q = \emptyset$ for each $q \geq \overline{n}$. Thanks to k-subadditivity of m, Proposition 3.2, (11) and (12), we have

$$\begin{aligned} &m(H_q) - m(C_q) = m(H_q) \leq u \wedge \left(m \left(\bigcup_{n=1}^{q} (H_n \setminus D_n) \right) \right) \leq \\ &k \left(u \wedge \left(\sum_{n=1}^{q} m(H_n \setminus D_n) \right) \right) \leq k \left(u \wedge \left(\sum_{n=1}^{q} m(U_n \setminus D_n) \right) \right) \leq k \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \end{aligned} \tag{13}$$

for any $q \geq \overline{n}$, where φ is as in (11) (note that $\overline{n}$ depends on φ). This finishes the proof of the first part.

We now turn to the last assertion. Set

$$\mathcal{T} := \Big\{ A \in \Sigma\colon \text{ there is a } (D)\text{-sequence } (a_{t,l})_{t,l} \text{ such that for every } \varphi \in \mathbb{N}^{\mathbb{N}} \\ \text{there are } D \in \mathcal{G},\ F \in \mathcal{H} \text{ with } F \subset A \subset D \text{ and } m(D \setminus F) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big\}. \quad (14)$$

Observe that $\mathcal{H} \subset \mathcal{T}$. Indeed, arbitrarily choose $W \in \mathcal{H}$, and for every $h \in \mathbb{N}$ set $D_h := \{x \in G : d(x, W) < 1/h\}$, $W_h := D_h \setminus W$. The sequence $(W_h)_h$ is decreasing, and $\bigcap_{h=1}^{\infty} W_h = \emptyset$. By continuity from above at $\emptyset$ of m and Theorem 2.6, (1), we find a regulator $(a_{t,l})_{t,l}$, such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $k_0 \in \mathbb{N}$, with

$$m(D_{k_0} \setminus W) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$

As $D_{k_0} \in \mathcal{G}$, $W \in \mathcal{H}$ and $W \subset D_{k_0}$, then $W \in \mathcal{T}$.

We now prove that $\mathcal{T}$ is a σ-algebra. It is not difficult to check that if $A \in \mathcal{T}$, then $G \setminus A \in \mathcal{T}$. Let now $(A_n)_n$ be a disjoint sequence of elements of $\mathcal{T}$, with $A := \bigcup_{n=1}^{\infty} A_n$. We claim that $A \in \mathcal{T}$.

For each $n \in \mathbb{N}$ there exist a (D)-sequence $\big(a_{t,l}^{(n)}\big)_{t,l}$ and two sets $D_n \in \mathcal{G}$, $F_n \in \mathcal{H}$, with $F_n \subset A_n \subset D_n$ and

$$m(D_n \setminus F_n) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)}.$$

Let $u := m(G)$. Thanks to the Fremlin's lemma, there exists a regulator $(a_{t,l})_{t,l}$ in R, with

$$u \wedge \left(\sum_{n=1}^{q} \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \right) \right) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for each $q \in \mathbb{N}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$.

Since the sequence $(F_n)_n$ is disjoint, by continuity from above at $\emptyset$ of m and Theorem 2.6, (1), there exists a regulator $(b_{t,l})_{t,l}$ such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a positive integer n_0 with

$$m\left(\left(\bigcup_{n=1}^{\infty} F_n \right) \setminus \left(\bigcup_{n=1}^{n_0} F_n \right) \right) = m\left(\bigcup_{n=n_0+1}^{\infty} F_n \right) \le \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}\,.$$

Set $c_{t,l} := 2\,k\,(a_{t,l} + b_{t,l})$, t, $l \in \mathbb{N}$,

$$D := \bigcup_{n=1}^{\infty} D_n \quad \text{and} \quad F := \bigcup_{n=1}^{n_0} F_n.$$

Observe that $(c_{t,l})_{t,l}$ is a (D)-sequence, $F \subset A \subset D$, $D \in \mathcal{G}$, $F \in \mathcal{H}$, and taking k-σ-subadditivity of m into account, thanks to Proposition 3.4, we get

$$\begin{aligned} m(D \setminus F) &\leq m\left(D \setminus \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + k\, m\left(\left(\bigcup_{n=1}^{\infty} F_n\right) \setminus F\right) \\ &\leq m\left(\bigcup_{n=1}^{\infty} (D_n \setminus F_n)\right) + k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \\ &\leq k \sum_{n=1}^{\infty} m(D_n \setminus F_n) + k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \\ &\leq k \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \\ &\leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}. \end{aligned}$$

From this it follows that $A \in \mathcal{T}$, that is the claim. Therefore, $\mathcal{T}$ is a σ-algebra. Since $\mathcal{T} \supset \mathcal{H}$, then $\mathcal{T} = \Sigma$.

Now, arbitrarily pick $A \in \Sigma$. Since $\Sigma = \mathcal{T}$, there exists a (D)-sequence $(a_{t,l})_{t,l}$ satisfying (14). Since R is super Dedekind complete and weakly σ-distributive, by Theorem 2.6, (2), in correspondence with $(a_{t,l})_{t,l}$, we find a sequence $(\varphi_n)_n$ in $\mathbb{N}^{\mathbb{N}}$ and an (O)-sequence $(b_n)_n$ such that, for each $n \in \mathbb{N}$, $\bigvee_{t=1}^{\infty} a_{t,\varphi_n(t)} \leq b_n$, and thus, there exist $D_n^* \in \mathcal{G}$, $F_n^* \in \mathcal{H}$, with $F_n^* \subset A \subset D_n^*$ and $m(D_n^* \setminus F_n^*) \leq b_n$. Put $D_n := \bigcap_{r=1}^{n} D_r^*$, $F_n := \bigcup_{r=1}^{n} F_r^*$. We get $F_n \subset F_{n+1} \subset A \subset D_{n+1} \subset D_n$, $D_n \in \mathcal{G}$, $F_n \in \mathcal{H}$, $m(D_n \setminus F_n) \leq m(D_n^* \setminus F_n^*) \leq b_n$, and so, $\bigwedge_n m(D_n \setminus F_n) = (O) \lim_n m(D_n \setminus F_n) = 0$. Thus, m is $\mathcal{G}$-$\mathcal{H}$-regular on Σ. This proves the last part. □

Remarks 3.11.

(a) Observe that, arguing similarly as above, it is possible to prove that, under the same hypotheses as in Theorem 3.10, given an equibounded sequence $m_j : \Sigma \to R$, $j \in \mathbb{N}$ of k-subadditive capacities, the m_j's are uniformly $\mathcal{G}$-$\mathcal{H}$-regular if and only if they are uniformly continuous from above at $\emptyset$.

(b) Note that, in general, even when $R = \mathbb{R}$, the concepts of regularity and continuity from above at $\emptyset$ are not equivalent. Indeed, if $\mathcal{G} = \mathcal{H} = \Sigma$, then every capacity is obviously $\mathcal{G}$-$\mathcal{H}$-regular, but not necessarily continuous from above at $\emptyset$. Conversely, if $\mathcal{G} = \mathcal{H} = \{\emptyset, G\}$, and $m : \Sigma \to \mathbb{R}$ is any capacity continuous from above at $\emptyset$, and such that $m(\emptyset) = 0$, $m(G) = 1$ and there are $E \in \Sigma$ and $a \in (0,1)$ with $m(E) = a$, then it is easy to see that m is not $\mathcal{G}$-$\mathcal{H}$-regular.

4. The main results

Before proving our main limit theorems, we recall the following result proved by E. P a p, which we will use in our setting.

Theorem 4.1 ([22, Theorem 1], [23, Theorem 11.10]). *Let $m_j : \Sigma \to \mathbb{R}$, $j \in \mathbb{N}$, be a sequence of (s)-bounded k-subadditive set functions such that*

$$m_j(A) - k\, m_j(B) \leq m_j(A \cup B) \tag{15}$$

*whenever $A, B \in \Sigma$, $A \cap B = \emptyset$, and $j \in \mathbb{N}$ (such set functions are called **k-triangular**). Suppose that for every $E \in \Sigma$, in $\mathbb{R}$ there exists the limit $m_0(E) := \lim_j m_j(E)$ for each $E \in \Sigma$, and that m_0 is (s)-bounded.*

Then, the set functions $m_j : \Sigma \to \mathbb{R}$, $j \in \mathbb{R}$, are uniformly (s)-bounded.

The following proposition will be useful in the sequel.

Proposition 4.2. *Let R be any Dedekind complete lattice group, $\mathcal{F}$ a free filter of $\mathbb{N}$, and let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of set functions, such that the limit $m_0(E) := (O\mathcal{F}) \lim_j m_j(E)$ exists in R for every $E \in \Sigma$.*

If m_j is a capacity for every j, then m_0 is also a capacity. If m_j is k-subadditive for any j, then m_0 is k-subadditive too.

P r o o f. First of all, it is easy to check that if $m_j(\emptyset) = 0$ for every $j \in \mathbb{N}$, then $m_0(\emptyset) = 0$.

Now, fix arbitrarily A, $B \in \Sigma$, and let $\big(\sigma_p^{(A)}\big)_p$, $\big(\sigma_p^{(B)}\big)_p$, $\big(\sigma_p^{(A\cup B)}\big)_p$ be three (O)-sequences related to $(O\mathcal{F})$-convergence of $(m_j)_j$ to m_0 at the sets A, B, $A \cup B$, respectively. Choose arbitrarily $p \in \mathbb{N}$. In correspondence with p, there are three elements F_A, F_B, $F_{A\cup B}$ of $\mathcal{F}$ with

$$\begin{aligned} |m_j(A) - m_0(A)| &\leq \sigma_p^{(A)}, \\ |m_j(B) - m_0(B)| &\leq \sigma_p^{(B)}, \\ |m_j(A \cup B) - m_0(A \cup B)| &\leq \sigma_p^{(A\cup B)} \qquad \text{whenever} \quad j \in F_A \cap F_B \cap F_{A\cup B}. \end{aligned}$$

Note that $F_A \cap F_B$ and $F_A \cap F_B \cap F_{A\cap B}$ belong to $\mathcal{F}$. Let $j_* = \min F_A \cap F_B$ and $j_0 = \min F_A \cap F_B \cap F_{A\cap B}$. If $A \subset B$ and all the m_j's are monotone, then we get

$$m_0(A) \leq \sigma_p^{(A)} + m_{j_*}(A) \leq \sigma_p^{(A)} + m_{j_*}(B) \leq \sigma_p^{(A)} + m_0(B) + \sigma_p^{(B)}.$$

By arbitrariness of p we get $m_0(A) \leq m_0(B)$ and hence monotonicity of m_0, thanks to arbitrariness of A and B.

Now, let $A \cap B = \emptyset$ and suppose that the m_j's are k-subadditive. We have

$$\begin{aligned} m_0(A \cup B) &\leq \sigma_p^{(A \cup B)} + m_{j_0}(A \cup B) \\ &\leq \sigma_p^{(A \cup B)} + m_{j_0}(A) + k\, m_{j_0}(B) \\ &\leq \sigma_p^{(A \cup B)} + m_0(A) + k\, m_0(B) + \sigma_p^{(A)} + \sigma_p^{(B)}. \end{aligned} \tag{16}$$

From arbitrariness of p and (16), we get

$$m_0(A \cup B) \leq m_0(A) + k\, m_0(B),$$

and so, we obtain k-subadditivity of m_0, by arbitrariness of A and B. □

We are in position to prove a Brooks-Jewett-type theorem for non-additive capacities, which extends [7, Theorem 2.5].

Theorem 4.3. *Let R be any Dedekind complete lattice group, $\mathcal{F}$ a diagonal filter of $\mathbb{N}$, and let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be an equibounded sequence of k-subadditive capacities, such that $m_0(E) := (O\mathcal{F}) \lim_j m_j(E)$ exists in R for every $E \in \Sigma$, m_0 is continuous from above at $\emptyset$ and m_j is (s)-bounded on Σ for every $j \geq 0$.*

If $R \subset C_\infty(\Omega)$ is as in Theorem 2.7, then for every $I \in \mathcal{F}^*$ and for each disjoint sequence $(C_h)_h$ in Σ there exist a set $J \subset I$, $J \in \mathcal{F}^*$, and a meager set $N \subset \Omega$ with

$$(O) \lim_h \left(\bigvee_{j \in J} m_j(C_h) \right) = 0 \tag{17}$$

and

$$\lim_h \left(\sup_{j \in J} m_j(C_h)(\omega) \right) = 0 \qquad \text{for each} \quad \omega \in \Omega \setminus N. \tag{18}$$

P r o o f. Let $R \subset C_\infty(\Omega)$ be as in Theorem 2.7, $m_j : \Sigma \to R$, $j \geq 0$, as in the hypotheses, and let $\left(\sigma_p^{(j)}\right)_p$ be associated with m_j as in Definition 3.5 (a). Pick arbitrarily $j \geq 0$. By Theorem 2.7, there exists a meager set $N_*^{(j)} \subset \Omega$ such that $m_j(\cdot)(\omega)$ is real-valued on Σ, and

$$\lim_p \sigma_p^{(j)}(\omega) = \inf_p \sigma_p^{(j)}(\omega) = 0 \qquad \text{for every} \quad \omega \in \Omega \setminus N_*^{(j)}. \tag{19}$$

Thus, for every $\varepsilon > 0$ and $\omega \in \Omega \setminus N_*^{(j)}$, there is $p_0 \in \mathbb{N}$ with $\sigma_p^{(j)}(\omega) \leq \varepsilon$ for each $p \geq p_0$. If $(C_h)_h$ is any arbitrary disjoint sequence from Σ, then for every $j \geq 0$, in correspondence with p_0, there exists $h_0 \in \mathbb{N}$ with

$$m_j(C_h)(\omega) \leq \sigma_{p_0}^{(j)}(\omega) \leq \varepsilon \qquad \text{for all} \quad h \geq h_0\,. \tag{20}$$

If $N_0 := \bigcup_{j=0}^\infty N_*^{(j)}$, then N_0 is a meager subset of Ω, and from (20), it follows that the real-valued capacities $m_j(\cdot)(\omega)$, $j \geq 0$, $\omega \in \Omega \setminus N_0$, are (s)-bounded on Σ.

Moreover, since m_j, $j \in \mathbb{N}$, is k-subadditive, from Proposition 4.2, it follows that m_0 is k-subadditive. From this and from Theorem 2.7, it follows that for every $j \geq 0$ and $\omega \in \Omega \setminus N_0$ the capacity $m_j(\cdot)(\omega)$ is k-subadditive. By monotonicity of m_j and $m_j(\cdot)(\omega)$, we also get

$$m_j(A) - k\, m_j(B) \leq m_j(A) \leq m_j(A \cup B)$$

and

$$m_j(A)(\omega) - k\, m_j(B)(\omega) \leq m_j(A)(\omega) \leq m_j(A \cup B)(\omega)$$

whenever $j \geq 0$, $\omega \in \Omega \setminus N_0$, A, $B \in \Sigma$, $A \cap B = \emptyset$. Thus, for any $\omega \in \Omega \setminus N_0$, the capacities $m_j(\cdot)(\omega)$, $j \geq 0$, satisfy (15).

Now, let $\mathcal{K}$ be the σ-algebra generated by the sets C_h, $h \in \mathbb{N}$, in the set $\bigcup_{h=1}^{\infty} C_h$. For each $B \in \mathcal{K}$ there is a set $P \subset \mathbb{N}$ with $B = \bigcup_{h \in P} C_h$. Let $\mathcal{B}$ be a countable family whose elements are all finite and cofinite unions of the C_h's in $\bigcup_{h=1}^{\infty} C_h$. By hypothesis, we get

$$m_0(E) = (O\mathcal{F}) \lim_j m_j(E) \qquad \text{for every} \quad E \in \mathcal{B}\,. \tag{21}$$

Since $\mathcal{B}$ is countable, by virtue of (21) and Lemma 2.10, for every set $I \in \mathcal{F}^*$ there is a set $J \in \mathcal{F}^*$, $J \subset I$, depending on I and $(C_h)_h$, with

$$(O) \lim_{j \in J} m_j(E) = m_0(E) \qquad \text{for each} \quad E \in \mathcal{B}.$$

By Theorem 2.7, we find a meager set $N' \subset \Omega$, without loss of generality $N' \supset N_0$, depending on I and $(C_h)_h$, with

$$\lim_{j \in J} m_j(E)(\omega) = m_0(E)(\omega) \qquad \text{for every} \quad \omega \in \Omega \setminus N' \text{ and } E \in \mathcal{B}.$$

Moreover, since m_0 is continuous from above at $\emptyset$, again thanks to Theorem 2.7, there exists a meager set $N \subset \Omega$, without loss of generality $N \supset N'$, with

$$\lim_n \left[m_0 \left(\bigcup_{h \geq n} C_h \right) (\omega) \right] = 0 \qquad \text{for every} \quad \omega \in \Omega \setminus N.$$

For each $\varepsilon > 0$ and $\omega \in \Omega \setminus N$ there is a positive integer $h_0(\varepsilon, \omega)$ with

$$m_0 \left(\bigcup_{h \geq h_0} C_h \right) (\omega) \leq \varepsilon,$$

and hence, by monotonicity of m_0 (which follows from Proposition 4.2), we have

$$m_0 \left(\bigcup_{\substack{h \geq h_0 \\ h \in P}} C_h \right) (\omega) \leq \varepsilon$$

whenever $P \subset \mathbb{N}$. Moreover, there is $j_0 \in J$, $j_0 = j_0(\varepsilon, \omega, h_0)$ such that for any $j \in J$, $j \geq j_0$, we get

$$\left| m_j\left(\bigcup_{\substack{h\leq h_0\\ h\in P}} C_h\right)(\omega) - m_0\left(\bigcup_{\substack{h\leq h_0\\ h\in P}} C_h\right)(\omega)\right| \leq \varepsilon$$

and

$$\left| m_j\left(\bigcup_{h> h_0} C_h\right)(\omega) - m_0\left(\bigcup_{h> h_0} C_h\right)(\omega)\right| \leq \varepsilon.$$

Let now $B \in \mathcal{K}$ and $\omega \in \Omega \setminus N$. Taking account of k-subadditivity, positivity and monotonicity of m_j, for every $j \in J$, $j \geq j_0$, we get:

$$\begin{aligned}
0 &\leq |m_j(B)(\omega) - m_0(B)(\omega)| \\
&= \left| m_j\left(\bigcup_{h\in P} C_h\right)(\omega) - m_0\left(\bigcup_{h\in P} C_h\right)(\omega)\right| \\
&\leq \left| m_j\left(\bigcup_{\substack{h\leq h_0\\ h\in P}} C_h\right)(\omega) - m_0\left(\bigcup_{\substack{h\leq h_0\\ h\in P}} C_h\right)(\omega)\right| + k\, m_0\left(\bigcup_{\substack{h> h_0\\ h\in P}} C_h\right)(\omega) + k\, m_j\left(\bigcup_{\substack{h> h_0\\ h\in P}} C_h\right)(\omega) \\
&\leq \left| m_j\left(\bigcup_{\substack{h\leq h_0\\ h\in P}} C_h\right)(\omega) - m_0\left(\bigcup_{\substack{h\leq h_0\\ h\in P}} C_h\right)(\omega)\right| + k\, m_0\left(\bigcup_{h> h_0} C_h\right)(\omega) + k\, m_j\left(\bigcup_{h> h_0} C_h\right)(\omega) \\
&\leq \left| m_j\left(\bigcup_{\substack{h\leq h_0\\ h\in P}} C_h\right)(\omega) - m_0\left(\bigcup_{\substack{h\leq h_0\\ h\in P}} C_h\right)(\omega)\right| + k\left| m_j\left(\bigcup_{h> h_0} C_h\right)(\omega) - m_0\left(\bigcup_{h> h_0} C_h\right)(\omega)\right| \\
&\quad + 2\, k\, m_0\left(\bigcup_{h> h_0} C_h\right)(\omega) \\
&\leq (3\,k+1)\,\varepsilon\,.
\end{aligned}$$

So,

$$\lim_{j\in J} m_j(B)(\omega) = m_0(B)(\omega) \qquad \text{for every} \quad \omega \in \Omega \setminus N \text{ and } B \in \mathcal{K}.$$

Therefore, for each $\omega \in \Omega \setminus N$, the finitely additive real-valued capacities $m_j(\cdot)(\omega)$, $j \in J$, satisfy the hypotheses of Theorem 4.1 on $\mathcal{K}$, and so, they are uniformly (s)-bounded on $\mathcal{K}$. Thus we get

$$\lim_h \left(\sup_{j\in J} m_j(C_h)(\omega)\right) = 0 \qquad \text{for all} \qquad \omega \in \Omega \setminus N,$$

that is (18). From this, since N is meager and taking Theorem 2.7 into account, we obtain

$$(O)\lim_h \left(\bigvee_{j\in J} m_j(C_h)\right) = 0\,,$$

that is (17). This finishes the proof. □

The following result extends [8, Theorem 3.8] to our context.

Theorem 4.4. *Let R, Ω, $\mathcal{F}$ be as in Theorem 4.3 and let $m_j : \Sigma \to R$, $j \in \mathbb{N}$ be an equibounded sequence of k-subadditive capacities. Assume that $m_0(E) := (O\mathcal{F})\lim_j m_j(E)$ exists in R for every $E \in \Sigma$.*

Then for every $I \in \mathcal{F}^*$ and for each decreasing sequence $(H_n)_n$ in Σ with

$$(O)\lim_n m_j(H_n) = \bigwedge_n m_j(H_n) = 0 \qquad \text{for every} \quad j \geq 0 \tag{22}$$

there are a set $J \subset I$, $J \in \mathcal{F}^*$, and a meager set $N^* \subset \Omega$ with

$$\lim_n \left(\sup_{j\in J} m_j(H_n)(\omega)\right) = \inf_n \left(\sup_{j\in J} m_j(H_n)(\omega)\right) = 0 \tag{23}$$

and

$$(O)\lim_n \left(\bigvee_{j\in J} m_j(H_n)\right) = \bigwedge_n \left(\bigvee_{j\in J} m_j(H_n)\right) = 0\,. \tag{24}$$

Proof. Let $(H_n)_n$ be as in (22), set $C_h = H_h \setminus H_{h+1}$, $h \in \mathbb{N}$, and $C_0 := \bigcap_{n=1}^{\infty} H_n$. Note that, by monotonicity, we get $m_j(C_0) = 0$ for every $j \geq 0$. Moreover, thanks to monotonicity and k-subadditivity, we get

$$m_j(H_n \setminus C_0) \leq m_j(H_n) \leq m_j(H_n \setminus C_0) + k\, m_j(C_0) = m_j(H_n \setminus C_0),$$

and hence

$$m_j(H_n \setminus C_0) = m_j(H_n) \qquad \text{for all} \quad j, n \in \mathbb{N}\,. \tag{25}$$

By virtue of Theorem 2.7 and equiboundedness of $(m_j)_j$, there exists a meager set $N_0 \subset \Omega$ such that the capacities $m_j(\cdot)(\omega)$ are real-valued and

$$\lim_n m_j(H_n)(\omega) = \inf_n m_j(H_n)(\omega) = 0 \tag{26}$$

for every $j \geq 0$ and $\omega \in \Omega \setminus N_0$.

Let $\mathcal{K}$ be the σ-algebra generated by the C_h's, $h \geq 0$, in H_1. Let $(K_s)_s$ be any decreasing sequence in $\mathcal{K}$ with $\bigcap_{s=1}^{\infty} K_s = \emptyset$. There exists a decreasing sequence $(P_s)_s$ of subsets of $\mathbb{N}$ with $K_s = \bigcup_{h\in P_s} C_h$. If $p_s = \min P_s$, $s \in \mathbb{N}$, then $(p_s)_s$

is an increasing sequence of positive integers and $K_s \subset H_{p_s}$ for every $s \in \mathbb{N}$. By monotonicity of the m_j's, we get

$$(O) \lim_s m_j(K_s) = \bigwedge_s m_j(K_s) = 0$$

and

$$\lim_s m_j(K_s)(\omega) = \inf_s m_j(K_s)(\omega) = 0 \tag{27}$$

for every $j \geq 0$ and $\omega \in \Omega \setminus N_0$. Moreover, if $(D_n)_n$ is any disjoint sequence in $\mathcal{K}$, then we have

$$0 \leq m_j(D_n) \leq m_j\left(\bigcup_{r=n}^{\infty} D_r\right) \qquad \text{for each} \quad n \in \mathbb{N} \quad \text{and} \quad j \geq 0\,. \tag{28}$$

Hence, from (27), (28) and monotonicity of m_j, it follows that

$$\lim_n m_j(D_n)(\omega) = \inf_n m_j(D_n)(\omega) = 0$$

whenever $j \geq 0$ and $\omega \in \Omega \setminus N_0$, and thus, the capacities $m_j(\cdot)(\omega)$, $j \geq 0$, $\omega \in \Omega \setminus N_0$, are (s)-bounded on $\mathcal{K}$.

Arguing similarly as in Theorem 4.3, we get that for any $j \geq 0$ and $\omega \in \Omega \setminus N$ the capacity $m_j(\cdot)(\omega)$ is k-subadditive and satisfies (15). Furthermore, it is possible to prove that for every $I \in \mathcal{F}^*$ there exist a set $J \subset I$, $J \in \mathcal{F}^*$ and a meager set $N \subset \Omega$ (depending on I and $(C_h)_h$), without loss of generality $N \supset N_0$, with

$$\lim_{j \in J} m_j(B)(\omega) = m_0(B)(\omega) \qquad \text{for each} \quad B \in \mathcal{K} \quad \text{and} \quad \omega \in \Omega \setminus N.$$

Therefore, for each $\omega \in \Omega \setminus N$, the real-valued capacities $m_j(\cdot)(\omega)$, $j \in J$, satisfy the hypotheses of Theorem 4.1 on $\mathcal{K}$, and so, they are uniformly (s)-bounded on $\mathcal{K}$.

By monotonicity and k-subadditivity of the m_j's, for every j, $n \in \mathbb{N}$ and $q \geq n$ we get

$$m_j(H_n \setminus H_q) \leq m_j(H_n), \quad m_j(H_n) \leq m_j(H_n \setminus H_q) + k\, m_j(H_n \cap H_q),$$

and hence

$$|m_j(H_n) - m_j(H_n \setminus H_q)| \leq k\, m_j(H_n \cap H_q). \tag{29}$$

By (29) and (22), taking (25) and Theorem 2.7 into account, there exists a meager set $N^* \subset \Omega$, without loss of generality $N^* \supset N$, depending only on I and $(H_n)_n$, with

$$\lim_q m_j(H_n \setminus H_q)(\omega) = m_j(H_n \setminus C_0)(\omega) = m_j(H_n)(\omega) \tag{30}$$

for all $\omega \in \Omega \setminus N^*$, $j, n \in \mathbb{N}$. We now prove that N^* satisfies (23). If (23) is not true, then there are $\varepsilon > 0$ and $\omega \in \Omega \setminus N^*$ such that for every $p \in \mathbb{N}$ there are $n \in \mathbb{N}$, $n > p$ and $j \in J$ with $m_j(H_n)(\omega) > \varepsilon$, and hence, thanks to (30),

$$m_j(H_n \setminus H_q)(\omega) > \varepsilon \tag{31}$$

for q large enough. At the first step, in correspondence with $p = 1$, there exist three integers $n_1 > 1$, $j_1 \in J$ and $q_1 > \max\{n_1, j_1\}$, with $m_{j_1}(H_{n_1})(\omega) > \varepsilon$ and $m_{j_1}(H_{n_1} \setminus H_{q_1})(\omega) > \varepsilon$. By (26), in correspondence with $j = 1, 2, \ldots, j_1$, we find an integer $h_1 > q_1$ with

$$m_j(H_n)(\omega) \leq \varepsilon \tag{32}$$

whenever $n \geq h_1$. At the second step, there exist three integers $n_2 > h_1$, $j_2 \in J$ and $q_2 > \max\{n_2, j_2\}$, with

$$m_{j_2}(H_{n_2})(\omega) > \varepsilon; \quad m_{j_2}(H_{n_2} \setminus H_{q_2})(\omega) > \varepsilon. \tag{33}$$

From (32) and (33) it follows that $j_2 > j_1$. Proceeding by induction, we find two strictly increasing sequences $(n_h)_h$, $(q_h)_h$ in $\mathbb{N}$ and a strictly increasing sequence $(j_h)_h$ in J with $q_h > n_h > q_{k-1}$ for every $k \geq 2$, $q_h > j_h$ and $m_{j_h}(H_{n_h} \setminus H_{q_h})(\omega) > \varepsilon$ for each $h \in \mathbb{N}$. But this is impossible, since the sets $H_{n_h} \setminus H_{q_h}$, $k \in \mathbb{N}$, are pairwise disjoint elements of Σ and the capacities $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$ are uniformly (s)-bounded on Σ for each $\omega \in \Omega \setminus N^*$. This proves (23).

Since N^* is meager, from (23) and Theorem 2.7, we obtain (24). This finishes the proof. □

As consequences of Theorem 4.4, we give the following Vitali-Hahn-Saks-(resp. Nikodým-)type theorem, which extends [7, Theorem 2.6].

Theorem 4.5. *Let $\mathcal{F}$, R, Ω be as in Theorem 4.4, let τ be a Fréchet-Nikodým topology on Σ, and let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be an equibounded sequence of τ-continuous (resp. continuous from above at $\emptyset$) k-subadditive capacities. Let $m_0(E) := (O\mathcal{F}) \lim_j m_j(E)$ exist in R for every $E \in \Sigma$, and suppose that m_0 is τ-continuous (resp. continuous from above at $\emptyset$) on Σ.*

Then for every $I \in \mathcal{F}^*$ and for each decreasing sequence $(H_n)_n$ in Σ with $\tau - \lim_n H_n = \emptyset$ (resp. $\bigcap_{n=1}^{\infty} H_n = \emptyset$), there exist a set $J \subset I$, $J \in \mathcal{F}^*$, and a meager set $N \subset \Omega$ satisfying (23) *and* (24).

Now, we give a Dieudonné-type theorem for regular lattice group-valued capacities with respect to filter convergence, extending [8, Theorem 3.10].

Theorem 4.6. *Let $\mathcal{F}$, R, Ω be as in Theorem 4.4, and let $\mathcal{G}$, $\mathcal{H} \subset \Sigma$ be two lattices as in Definition 3.8 (c). Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be an equibounded sequence of $\mathcal{G}$-$\mathcal{H}$-regular k-subadditive capacities such that $m_0(E) = (O\mathcal{F}) \lim_j m_j(E)$ for any $E \in \Sigma$ and m_0 is $\mathcal{G}$-$\mathcal{H}$-regular. Then we get*

(R3) *for every $E \in \Sigma$ and $I \in \mathcal{F}^*$ there are $J \in \mathcal{F}^*$, $J \subset I$, and two sequences $(F_n)_n$ in $\mathcal{H}$, $(G_n)_n$ in $\mathcal{G}$ satisfying* (10) *and with*

$$(O) \lim_n \left(\bigvee_{j \in J} m_j(G_n \setminus F_n) \right) = \bigwedge_n \left(\bigvee_{j \in J} m_j(G_n \setminus F_n) \right) = 0\,.$$

Furthermore, there exists a meager set $N \subset \Omega$ with

$$(O) \lim_n \left(\sup_{j \in J} m_j (G_n \setminus F_n)(\omega) \right) = \inf_n \left(\sup_{j \in J} m_j (G_n \setminus F_n)(\omega) \right) = 0$$

for each $\omega \in \Omega \setminus N$.

Proof. Given $E \in \Sigma$, pick two sequences $(F_n)_n$ and $(G_n)_n$ fulfilling (R2). The m_j's satisfy the hypotheses of Theorem 4.4, and the sequence $(G_n \setminus F_n)_n$ fulfils (22). Thus we get (R3). □

Acknowledgements. Our thanks to the referee for his/her suggestions, which improved the exposition of the paper.

Open problems:

(a) Prove some theorems in which it is possible to obtain τ-continuity, continuity from above or regularity of the limit measure in the conclusions.

(b) Prove similar results for other classes of filters and/or for measures, which are not necessarily positive.

(c) Prove some results on filter weak compactness/weak convergence of non--additive measures.

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Received October 20, 2014

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