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AVERAGE OPERATORS ON RECTANGULAR HERZ SPACES

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ABSTRACT. We introduce a family of Herz type spaces considering rectangles instead of balls and we study continuity properties of some average operators acting on them.

1. Introduction

Herz spaces have been studied for many years. The roots of this subject lie on the pioneering work of N. Wiener [11], A. Beurling [2] and C. Herz [9]. Later, these spaces were generalized by other mathematicians in order to study continuity properties of classical operators in harmonic analysis, as well as to develop local versions of Hardy spaces and bounded mean oscillation spaces.

There are several definitions of Herz space. The following is classical and corresponds to the inhomogeneous setting: a measurable function f belongs to the Herz space $K^{\alpha}_{p,q}(\mathbb{R}^n)$, $1 \le p,\, q < \infty$, $\alpha \in \mathbb{R}$ if

$$\|f\|_{K^{\alpha}_{p,q}} := \left(\sum_{k=0}^{\infty} 2^{nk\alpha q} \|f\chi_{C_k}\|_p^q\right)^{1/q} < \infty\,, \tag{1}$$

and for $q = \infty$,

$$\|f\|_{K^{\alpha}_{p,\infty}} := \sup_{k\ge 0}\left(2^{nk\alpha}\|f\chi_{C_k}\|_p\right) < \infty\,. \tag{2}$$

Here, C_0 is the open unit ball $B_1(0)$ and $C_k = B_{2^k}(0) \setminus B_{2^{k-1}}(0)$, $k \in \mathbb{N}$.

Setting $\alpha = -1/p$ in (2), we obtain the space $B^p(\mathbb{R}^n)$ that also can be characterized by mean of the condition [5], [7]

$$\sup_{R\ge 1}\left(\frac{1}{|B_R(0)|}\int_{B_R(0)} |f(x)|^p\,\mathrm{d}x\right)^{1/p} < \infty \tag{3}$$

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and the quantity on the left hand side of (3) defines an equivalent norm to $\|f\|_{K_{p,\infty}^{-1/p}}$ that is usually denoted by $\|f\|_{B^p}$. With any of these norms, $B^p(\mathbb{R}^n)$ turns out to be a Banach space. Moreover, for $1 \leq p_1 < p_2 < \infty$ we have the inclusions $B^{p_2}(\mathbb{R}^n) \subset B^{p_1}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n) \subset B^p(\mathbb{R}^n)$ for every p.

In this work we will restrict to the context of the space $B^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. Our aim is to explore what happens when we consider rectangles with sides parallel to the coordinate axes instead of balls in (3). As we will see below, although we obtain a smaller space than $B^p(\mathbb{R}^n)$, it is still appropriate to study continuity properties of some classical operators. In the context of the present paper, we study continuity properties of some discrete and continuous versions of the classical Hardy average operator. This operator has been extensively studied by many authors on different function spaces. We restrict ourself to consider the most simple versions of this operator in order to make the reading of the present paper easy.

The manuscript is organized as follows: the second section is devoted to introduce the rectangular Herz spaces and to give some examples. In the third section we introduce average operators to be considered and we prove the continuity of these averages on our spaces.

We will employ standard notation along this work and we will also adopt the convention to denote a constant that could be changing line by line by C.

2. Rectangular Herz spaces

For $1 \leq p < \infty$, we define the following space

$$\mathcal{B}^p(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{\mathcal{B}^p} < \infty \right\},$$

where

$$\|f\|_{\mathcal{B}^p} := \sup_{\substack{R_j \geq 1 \\ j=1,\ldots,n}} \left[\frac{1}{R_1 \cdots R_n} \int_{[-R_1,R_1]\times\cdots\times[-R_n,R_n]} |f(x)|^p \, \mathrm{d}x \right]^{1/p}. \tag{4}$$

If the context does not cause confusion, we will simply write $\mathcal{B}^p$. Notice that for $n = 1$, the spaces $\mathcal{B}^p(\mathbb{R})$ and $B^p(\mathbb{R})$ coincide.

Standard arguments (see [1], for example) allow us to see that $(\mathcal{B}^p, \|\cdot\|_{\mathcal{B}^p})$ is a Banach space. Moreover, it is clear that $\mathcal{B}^p \subset B^p$ and $\|\cdot\|_{B^p} \leq \|\cdot\|_{\mathcal{B}^p}$ since Lebesgue measure of balls and cubes are comparable.

Proposition 1. *The space $\mathcal{B}^p(\mathbb{R}^n)$ is properly contained in $B^p(\mathbb{R}^n)$ when $n \geq 2$.*

P r o o f. For the sake of clarity, we will consider the case $n = 2$.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined as follows:

$$f(x) = \begin{cases} 0 & \text{if} \quad x \notin \big([-1,1]\times\mathbb{R}\big) \cup \big(\mathbb{R}\times[-1,1]\big), \\ 1 & \text{if} \quad x \in \big([-1,1]\times[-1,1]\big), \\ 2^{1/p} & \text{if} \quad x \in \big([-1,1]\times(1,2]\big) \\ & \qquad \cup \big([-1,1]\times[-2,-1)\big) \\ & \qquad \cup \big((1,2]\times[-1,1]\big) \\ & \qquad \cup \big([-2,-1)\times[-1,1]\big), \\ & \vdots \\ n^{1/p} & \text{if} \quad x \in \big([-1,1]\times(n-1,n]\big) \\ & \qquad \cup \big([-1,1]\times[-n,-n+1)\big) \\ & \qquad \cup \big((n-1,n]\times[-1,1]\big) \\ & \qquad \cup \big([-n,-n+1)\times[-1,1]\big), \qquad n \geq 2. \end{cases}$$

Take $R \geq 1$. We can find $k \in \mathbb{N}$ such that $k \leq R < k+1$ and thus

$$\begin{aligned} \frac{1}{\left|[-R,R]^2\right|} \int\limits_{[-R,R]^2} |f(x)|^p \,\mathrm{d}x &\leq \frac{1}{4k^2} \int\limits_{[-(k+1),k+1]^2} |f(x)|^p \,\mathrm{d}x \\ &= \frac{1}{4k^2}\left[1\cdot 2^2 + 2\cdot 2^3 + 3\cdot 2^3 + \cdots + (k+1)\cdot 2^3\right] \\ &\leq \frac{2}{k^2}\left[1+2+\cdots+(k+1)\right] \\ &= \frac{(k+1)(k+2)}{k^2} \leq 6 \end{aligned}$$

which shows that $f \in B^p\left(\mathbb{R}^2\right)$. However, if we now consider rectangles of the form $[-1,1]\times[-L,L]$ for $L \geq 2$, we can pick $m \in \mathbb{N}$ such that $m \leq L < m+1$ and therefore

$$\begin{aligned} \frac{1}{|[-1,1]\times[-L,L]|} \int\limits_{[-1,1]\times[-L,L]} |f(x)|^p \,\mathrm{d}x &= \frac{1}{4L} \int\limits_{[-1,1]\times[-L,L]} |f(x)|^p \,\mathrm{d}x \\ &\geq \frac{1}{4(m+1)} \int\limits_{[-1,1]\times[-m,m]} |f(x)|^p \,\mathrm{d}x \\ &= \frac{1}{4(m+1)}\left[1\cdot 2^2 + 2\cdot 2^2 + \cdots + m\cdot 2^2\right] \\ &= m/2 \to \infty \qquad \text{if } m \to \infty, \end{aligned}$$

that is, $f \notin \mathcal{B}^p\left(\mathbb{R}^2\right)$. □

Using the idea of the previous example, we can get a characterization of the space $\mathcal{B}^p(\mathbb{R}^n)$. To this end, consider the following subsets of $\mathbb{R}^n$:

$$C_{j_1,j_2,\ldots,j_n} = C_{j_1} \times C_{j_2} \times \cdots \times C_{j_n}$$

where

$$C_0 = [-1,1] \quad \text{and} \quad C_j = \left\{x \in \mathbb{R} : 2^{j-1} < |x| \leq 2^j\right\} \qquad \text{for} \quad j \in \mathbb{N}.$$

For $1 \leq p < \infty$ and $f \in L^p_{loc}(\mathbb{R}^n)$ define

$$\|f\|^*_{\mathcal{B}^p} := \sup_{\substack{j_i \geq 0 \\ i=1,2,\ldots,n}} 2^{-\frac{(j_1+j_2+\cdots+j_n)}{p}} \left\|f\chi_{C_{j_1,j_2,\ldots,j_n}}\right\|_p.$$

Now, we can state the following characterization.

Proposition 2. *$f \in \mathcal{B}^p(\mathbb{R}^n)$ if and only if $\|f\|^*_{\mathcal{B}^p} < \infty$. Moreover, $\|f\|_{\mathcal{B}^p}$ and $\|f\|^*_{\mathcal{B}^p}$ are equivalent norms.*

Proof. Assume that $\|f\|^*_{\mathcal{B}^p} < \infty$. For $i = 1, \ldots, n$, let $R_i > 1$ and choose $j_i \in \mathbb{N}$ such that

$$2^{j_i - 1} < R_i \leq 2^{j_i}.$$

We have that

$$\begin{aligned}
\int_{\prod_{i=1}^n [-R_i,R_i]} |f(x)|^p \,\mathrm{d}x &\leq \sum_{k_1=0}^{j_1} \sum_{k_2=0}^{j_2} \cdots \sum_{k_n=0}^{j_n} \int_{C_{k_1,k_2,\ldots,k_n}} |f(x)|^p \,\mathrm{d}x \\
&\leq \sum_{k_1=0}^{j_1} \sum_{k_2=0}^{j_2} \cdots \sum_{k_n=0}^{j_n} 2^{k_1+k_2+\cdots+k_n} \left(\|f\|^*_{\mathcal{B}^p}\right)^p \\
&\leq C 2^{j_1+j_2+\cdots+j_n} \left(\|f\|^*_{\mathcal{B}^p}\right)^p \\
&\leq C R_1 R_2 \cdots R_n \left(\|f\|^*_{\mathcal{B}^p}\right)^p.
\end{aligned}$$

Hence, $f \in \mathcal{B}^p(\mathbb{R}^n)$ and $\|f\|_{\mathcal{B}^p} \leq C \|f\|^*_{\mathcal{B}^p}$.

Conversely, if $f \in \mathcal{B}^p(\mathbb{R}^n)$, $i = 1, \ldots, n$ and $j_i \geq 0$,

$$\begin{aligned}
\left\|f\chi_{C_{j_1,j_2,\ldots,j_n}}\right\|_p^p &= \int_{\prod_{i=1}^n \left[-2^{j_i}, 2^{j_i}\right]} |f(x)|^p \,\mathrm{d}x \\
&\leq C \|f\|^p_{\mathcal{B}^p} 2^{j_1+j_2+\cdots+j_n}
\end{aligned}$$

which implies that

$$\|f\|^*_{\mathcal{B}^p} = \sup_{\substack{j_i \geq 0 \\ i=1,2,\ldots,n}} 2^{-\frac{(j_1+j_2+\cdots+j_n)}{p}} \left\|f\chi_{C_{j_1,j_2,\ldots,j_n}}\right\|_p \leq C \|f\|_{\mathcal{B}^p}.$$

This concludes the proof. □

3. Continuity of average operators

Average integral operators were considered by Hardy, Littlewood and Pólya in [8]. They proved the following classical inequality:

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^1 f^p(x)\,\mathrm{d}x,$$

where $1 < p < \infty$, $F(x) = \int_0^x f(t)\,\mathrm{d}t$, $f \ge 0$ and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible.

The operator H_φ introduced by Carton-Lebrun and Fosset in [3] and by Xiao in [10] is closely related to this operator, which is pointwisely defined as follows:

$$H_\varphi f(x) := \int_0^1 f(tx)\,\varphi(t)\,\mathrm{d}t. \tag{5}$$

Xiao in [10] proved continuity of H_φ under appropriate conditions on φ on $L^p(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ for $1 \le p \le \infty$. It is our goal to prove continuity of this and other related operators in our rectangular Herz spaces.

We will start by considering the following discrete version of (5).

Let $\{r_k\}_{k=1}^\infty$ be a sequence in $(0,1]$ which is strictly decreasing and $\lim_{k\to\infty} r_k = 0$. If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a Lebesgue measurable function and $\varphi\colon \{r_k : k \in \mathbb{N}\} \longrightarrow (0,\infty)$ is any function, consider the operator H_φ^d formally defined as

$$H_\varphi^d f(x) = \sum_{k=1}^\infty \varphi(r_k)\, f(r_k x).$$

Now, notice that a necessary and sufficient condition for the existence of H_φ^d as a bounded operator on $L^p(\mathbb{R}^n)$ is that

$$\sum_{k=1}^\infty r_k^{-n/p} \varphi(r_k) < \infty. \tag{6}$$

Indeed, assuming the convergence of the series in (6), given $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and using Minkowski inequality, we obtain

$$\begin{aligned}\left\|H_\varphi^d f\right\|_p &\le \sum_{k=1}^\infty \varphi(r_k) \left(\int_{\mathbb{R}^n} |f(r_k x)|^p\,\mathrm{d}x\right)^{1/p} \\ &= \|f\|_p \sum_{k=1}^\infty r_k^{-n/p} \varphi(r_k),\end{aligned}$$

which implies that

$$\left\|H_\varphi^d\right\|_{L^p \to L^p} \le \sum_{k=1}^\infty r_k^{-n/p} \varphi(r_k).$$

Conversely, if H_φ^d is bounded on $L^p(\mathbb{R}^n)$, as Xiao in [10], we can consider the function

$$f_\varepsilon(x) = |x|^{-\frac{n}{p}-\varepsilon}\chi_{\{|x|>1\}},$$

where $0<\varepsilon<1$. It turns out that $\|f_\varepsilon\|_p = \frac{C_n}{p\varepsilon}$, C_n an n-dimensional constant and

$$H_\varphi^d f_\varepsilon(x) = \left(\sum_{k=1}^{\infty} r_k^{-\frac{n}{p}-\varepsilon}\varphi(r_k)\right)|x|^{-\frac{n}{p}-\varepsilon}\chi_{\{|x|>1\}}.$$

Thus, the same procedure as done in [10] shows that

$$\left\|H_\varphi^d\right\|_{L^p\to L^p}^p \|f_\varepsilon\|_p^p \geq \left[\varepsilon^\varepsilon \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}-\varepsilon}\varphi(r_k)\right]^p \|f_\varepsilon\|_p^p$$

and therefore,

$$\left\|H_\varphi^d\right\|_{L^p\to L^p} \geq \left[\varepsilon^\varepsilon \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}-\varepsilon}\varphi(r_k)\right] \geq \varepsilon^\varepsilon \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}}\varphi(r_k)$$

for any $0<\varepsilon<1$. Now, letting $\varepsilon\to 0$, we obtain

$$\left\|H_\varphi^d\right\|_{L^p\to L^p} \geq \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}}\varphi(r_k).$$

We have proved the following result.

Theorem 3. *The operator H_φ^d is a bounded operator on $L^p(\mathbb{R}^n)$, $1\leq p<\infty$, if and only if $\sum_{k=1}^{\infty} r_k^{-\frac{n}{p}}\varphi(r_k)<\infty$. In such a case,*

$$\left\|H_\varphi^d\right\|_{L^p\to L^p} = \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}}\varphi(r_k).$$

We can also consider the following generalization of the operator H_φ^d.

Let $\Phi\colon \left\{r_{k_1}^{(1)}\colon k_1\in\mathbb{N}\right\}\times\cdots\times\left\{r_{k_n}^{(n)}\colon k_n\in\mathbb{N}\right\}\longrightarrow(0,\infty)$ be any function where, for every $j=1,\ldots,n$, the sequence $\left\{r_{k_j}^{(j)}\right\}_{k_j=1}^{\infty}\subset(0,1]$ is strictly decreasing and $\lim_{k_j\to\infty} r_{k_j}^{(j)}=0$. For a Lebesgue measurable function $f:\mathbb{R}^n\longrightarrow\mathbb{R}$, define formally

$$\mathbb{H}_\Phi^d f(x) = \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\Phi\left(r_{k_1}^{(1)},\ldots,r_{k_n}^{(n)}\right) f\left(r_{k_1}^{(1)}x_1,\ldots,r_{k_n}^{(n)}x_n\right). \tag{7}$$

With the same proof as in Theorem 3, we can show:

Theorem 4. *The operator $\mathbb{H}_\Phi^d$ is a bounded operator on $L^p(\mathbb{R}^n)$, $1\leq p<\infty$, if and only if*

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\Phi\left(r_{k_1}^{(1)},\ldots,r_{k_n}^{(n)}\right)\left(r_{k_1}^{(1)}\right)^{-1/p}\cdots\left(r_{k_n}^{(n)}\right)^{-1/p}<\infty.$$

In such a case,

$$\left\|\mathbb{H}_{\Phi}^{d}\right\|_{L^p\to L^p}=\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\Phi\left(r_{k_1}^{(1)},\ldots,r_{k_n}^{(n)}\right)\left(r_{k_1}^{(1)}\right)^{-1/p}\cdots\left(r_{k_n}^{(n)}\right)^{-1/p}.$$

Now, we will study the action of the operator $\mathbb{H}_{\Phi}^{d}$ on our rectangular Herz spaces defined in the previous section.

For these spaces, the proof of the continuity of the operator $\mathbb{H}_{\Phi}^{d}$ is even easier. We provide it for the sake of completeness.

Theorem 5. *The operator $\mathbb{H}_{\Phi}^{d}$ is a bounded operator on $\mathcal{B}^p(\mathbb{R}^n)$, $1\le p<\infty$, if and only if*

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\Phi\left(r_{k_1}^{(1)},\ldots,r_{k_n}^{(n)}\right)<\infty\,. \tag{8}$$

In such a case,

$$\left\|\mathbb{H}_{\Phi}^{d}\right\|_{\mathcal{B}^p\to\mathcal{B}^p}=\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\Phi\left(r_{k_1}^{(1)},\ldots,r_{k_n}^{(n)}\right).$$

P r o o f. Assuming condition (8), taking $R_j>1$, $j=1,\ldots,n$, and using Minkowski inequality, we can see that

$$\left[\frac{1}{R_1\cdots R_n}\int\limits_{[-R_1,R_1]\times\cdots\times[-R_n,R_n]}\left|\mathbb{H}_{\Phi}^{d}f(x)\right|^p\mathrm{d}x\right]^{1/p}$$

$$\le\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\Phi\left(r_{k_1}^{(1)},\ldots,r_{k_n}^{(n)}\right)\left[\frac{1}{R_1\cdots R_n}\int\limits_{[-R_1,R_1]\times\cdots\times[-R_n,R_n]}\left|f\left(r_{k_1}^{(1)}x_1,\ldots,r_{k_n}^{(n)}x_n\right)\right|^p\mathrm{d}x\right]^{1/p}$$

$$\le\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\Phi\left(r_{k_1}^{(1)},\ldots,r_{k_n}^{(n)}\right)\|f\|_{\mathcal{B}^p},$$

and hence,

$$\left\|\mathbb{H}_{\Phi}^{d}\right\|_{\mathcal{B}^p}\le\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\Phi\left(r_{k_1}^{(1)},\ldots,r_{k_n}^{(n)}\right).$$

Now, if the operator $\mathbb{H}_{\Phi}^{d}$ is bounded on $\mathcal{B}^p(\mathbb{R}^n)$, it is susfficient to consider the function $f_0\equiv 1$ because, in such a case, we easily obtain the required reverse inequality. □

Our next goal is to generalize the operator given by (7). Before doing this, we will define another class of rectangular spaces closely related to $\mathcal{B}^p$.

Definition 6. For $1 \leq p < \infty$, we define

$$\mathcal{CMO}^p(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{\mathcal{CMO}^p} < \infty \right\},$$

where

$$\|f\|_{\mathcal{CMO}^p} := \sup_{\substack{R_j \geq 1 \\ j=1,\ldots,n}} \left[\frac{1}{R_1 \cdots R_n} \int_{[-R_1,R_1]\times\cdots\times[-R_n,R_n]} |f(x) - f_{R_1\cdots R_n}|^p \, \mathrm{d}x \right]^{1/p}, \quad (9)$$

and $f_{R_1\ldots R_n}$ is the average of f on $[-R_1, R_1] \times \cdots \times [-R_n, R_n]$.

It is not difficult to show that $(\mathcal{CMO}^p, \|\cdot\|_{\mathcal{CMO}^p})$ is a Banach space if we identify functions that differ by a constant almost everywhere on $\mathbb{R}^n$. Also, we obtain an equivalent norm to $\|\cdot\|_{\mathcal{CMO}^p}$ if we consider the quantity

$$\|f\|^*_{\mathcal{CMO}^p} := \sup_{\substack{R_j \geq 1 \\ j=1,\ldots,n}} \inf_{a\in\mathbb{R}} \left[\frac{1}{R_1 \cdots R_n} \int_{[-R_1,R_1]\times\cdots\times[-R_n,R_n]} |f(x) - a|^p \, \mathrm{d}x \right]^{1/p}.$$

This space is the rectangular version of the space CMO^p [4], [7] whose elements satisfy the condition

$$\sup_{R\geq 1} \left[\frac{1}{|Q(0,R)|} \int_{Q(0,R)} |f(x) - f_{Q(0,R)}|^p \mathrm{d}x \right]^{1/p} < \infty .$$

Here, $Q(0,R)$ denotes the cube centered at 0 with side length equal to R. Clearly, $\mathcal{B}^p \subset \mathcal{CMO}^p \subset CMO^p$.

Now, we consider the following operator:

For Lebesgue measurable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, and $\phi : [0,1]^n \longrightarrow (0,\infty)$, we define

$$\mathbb{H}_\phi f(x) := \int_{[0,1]^n} f(t_1x_1, \ldots, t_nx_n)\, \phi(t_1, \ldots, t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n . \quad (10)$$

Observe that the same proof as given by Xiao in [10] shows that $\mathbb{H}_\phi$ is a bounded operator on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, if and only if

$$\int_{[0,1]^n} t_1^{-1/p} \cdots t_n^{-1/p} \phi(t_1, \ldots, t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n < \infty .$$

We will give equivalent conditions for the boundedness of the operator $\mathbb{H}_\phi$ on the spaces $\mathcal{B}^p$ and $\mathcal{CMO}^p$.

Theorem 7. *The operator $\mathbb{H}_\phi$ is a bounded operator on $\mathcal{B}^p(\mathbb{R}^n)$ and $\mathcal{CMO}^p(\mathbb{R}^n)$, $1 \le p < \infty$, if and only if*

$$\int\limits_{[0,1]^n} \phi\left(t_1, \ldots, t_n\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n < \infty\,.$$

Moreover,

$$\|\mathbb{H}_\phi\|_{\mathcal{B}^p \to \mathcal{B}^p} = \|\mathbb{H}_\phi\|_{\mathcal{CMO}^p \to \mathcal{CMO}^p} = \int\limits_{[0,1]^n} \phi\left(t_1, \ldots, t_n\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n\,. \tag{11}$$

P r o o f. Just for illustration, we prove the equivalence for the space $\mathcal{CMO}^p\left(\mathbb{R}^n\right)$.

Suppose that the integral in (11) is finite. Then, for $R_j > 1$, $j = 1, \ldots, n$ and $f \in \mathcal{CMO}^p\left(\mathbb{R}^n\right)$, we can easily see that

$$\left(\mathbb{H}_\phi f\right)_{R_1 \cdots R_n} = \int\limits_{[0,1]^n} f_{t_1 R_1 \cdots t_n R_n} \phi\left(t_1, \ldots, t_n\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n\,.$$

Now, by Minkowski inequality and an appropriate change of variable, we have that

$$\begin{aligned}
&\left[\frac{1}{R_1 \cdots R_n} \int\limits_{[-R_1,R_1]\times\cdots\times[-R_n,R_n]} \left|\mathbb{H}_\phi f\left(x\right) - \left(\mathbb{H}_\phi f\right)_{R_1 \cdots R_n}\right|^p \mathrm{d}x\right]^{1/p} \\
&\le \int\limits_{[0,1]^n} \left(\frac{1}{R_1 \cdots R_n} \int\limits_{[-R_1,R_1]\times\cdots\times[-R_n,R_n]} \left|f\left(t_1 x_1, \ldots, t_n x_n\right) - f_{t_1 R_1 \cdots t_n R_n}\right|^p \mathrm{d}x\right)^{1/p} \\
&\quad \times \phi\left(t_1, \ldots, t_n\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n \\
&\le \|f\|_{\mathcal{CMO}^p} \int\limits_{[0,1]^n} \phi\left(t_1, \ldots, t_n\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n\,,
\end{aligned}$$

which implies that

$$\|\mathbb{H}_\phi\|_{\mathcal{CMO}^p \to \mathcal{CMO}^p} \le \int\limits_{[0,1]^n} \phi\left(t_1, \ldots, t_n\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n\,.$$

For the converse, it suffices to consider the function $f_0\left(x\right) \equiv 1$. □

Finally, it should be remarked that Theorems 5 and 7 remain true if we consider homogeneous versions of the spaces $\mathcal{B}^p$ and $\mathcal{CMO}^p$, that is, those defined by taking $R_j > 0$ for every $j = 1, \ldots, n$ in (4) and (9).

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