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ON SOME SETS OF ALMOST CONTINUOUS FUNCTIONS WHICH LOCALLY APPROXIMATE A FIXED FUNCTION

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ABSTRACT. In this paper, we investigate topological properties of the sets of almost continuous functions $f\colon [0,1] \to [0,1]$ which locally approximate a fixed function. This approximation is associated with the certain value of entropy of a function at a fixed point of the primal function.

1. Introduction and notation

The almost continuous (in the sense of Stallings) functions are an important element of research in the real functions theory. In recent times, they are also under consideration in the context of discrete dynamical systems ([9], [17]) including chaos theory ([16]). The concept of chaos, in particular this one related to functions mapping the unit interval into itself, is connected with many definitions, often non-equivalent. Synthetic information on this subject can be found in [2], [3], [12]. In [11], one can read that it is commonly accepted that positivity of topological entropy is an evidence of chaos. For this reason, instead of chaos, we will consider entropy of functions as a measure of chaos.

Although the concept of an entropy of a function is global, one can notice that sometimes an entropy of a function is focused at a single point (cf. Lemma 2.4). In this paper, we will consider an entropy of a function at a point which conforms to the definitions contained in [9] and [17]. So far, it has not been decided which functions have entropy points (i.e., such points that an entropy of a function at this point is equal to the "global" entropy of a function). However, there are known many results showing that a function with a fixed point can be approximated, in different ways, by functions having a "suitable" entropy point at this

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point. Most of these approximations were associated with strong entropy points and ∞-entropy points ([9], [10], [17]). In contrast to previous papers, in this one, we will consider for the first time an approximation of a function by functions with an entropy at a given point lower or higher than the entropy of the primal function.

A thorough analysis of the approximations signaled above (we will concentrate on the topology of the uniform convergence) shows that, in some cases, a function can be approximated even by functions from an open set of functions (each function "lying near" a function having infinite entropy at a given point also has this property ([16]). At the same time, Proposition 3.2 (a) in [9] shows that such a situation cannot often happen, e.g., for continuous functions.

In this situation, it seemed to be interesting to combine both signaled issues: analysis of a structure of sets of functions which approximate a given function in the context of a value of an entropy at a point. Unfortunately, not all the questions that could be posed with referring to this issue are already answered. Hence, some open problems will be indicated.

Throughout the paper, the closed unit interval will be denoted by $\mathbb{I}$. Moreover, we will consider the natural topology in $\mathbb{I}$. From now on, if it will not be stated otherwise, we will consider only the functions from $\mathbb{I}$ into $\mathbb{I}$. The symbol ρ_u will stand for the metric of uniform convergence, so

$$\rho_u(f,g) = \sup\{|f(x) - g(x)| : x \in \mathbb{I}\}$$

for functions f and g. If we will write about a space X of functions, we will consider the set X with the metric ρ_u. The interior (closure) of a set $A \subset X$ in the space (X, ρ_u) will be denoted by $\mathrm{int}_X(A)$ ($\mathrm{cl}_X(A)$). Moreover, $B_u(g,r)$ will denote an open ball with a center at g and a radius $r > 0$ in such a space. The symbol $\mathrm{card}(A)$ will stand for the cardinality of a set A. By $\exp(X)$ we will denote the set of all subsets of X.

Let (X, ρ) be a metric space. We shall say that a set $U \subset X$ is *X-dense at a point* $g \in X$ if one can find an open (in (X,ρ)) set V such that $g \in V \subset \mathrm{cl}_\rho(U)$ (where $\mathrm{cl}_\rho(U)$ denotes the closure of the set U in (X,ρ)).

If f is a function then the set of all fixed points of f will be denoted as $\mathrm{Fix}(f)$. Moreover, we will write $\mathrm{Fix}_c(f)$ for the set of all fixed points of f being its continuity points. We shall say that a set $U \subset \mathbb{I}$ is an *f-invariant set* if $f(U) \subset U$. Moreover, the restriction of f to the set U will be denoted by $f \restriction U$.

In our considerations, we will focus on functions which are almost continuous in the sense of Stallings. Let (X, T_X) and (Y, T_Y) be topological spaces. We shall say that $f \colon X \to Y$ is *almost continuous in the sense of Stallings* if for every open set $U \subset X \times Y$ containing the graph of the function f, the set U contains a graph of a continuous function $g \colon X \to Y$. The family of all almost continuous in the sense of Stalling functions $f \colon \mathbb{I} \to \mathbb{I}$ will be denoted by $\mathcal{A}$.

An entropy of a function will also play an important role in our considerations. Now, we shortly recall definition of a topological entropy (we will formulate this definition for functions considered in this paper, i.e., functions mapping $\mathbb{I}$ into itself). It is worth adding that this definition given first for continuous function by R. Bowen [5] and E. Dinaburg [7] was extended to an arbitrary function by Čiklová [6].

A *topological entropy of a function* f is the number

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left[\frac{1}{n} \log\bigl(s_n(\varepsilon)\bigr) \right],$$

where

$$s_n(\varepsilon) = \max\bigl\{\mathrm{card}(M) : M \subset \mathbb{I} \text{ is } (n,\varepsilon)\text{-separated set}\bigr\}$$

and a set $M \subset \mathbb{I}$ is (n,ε)-*separated* for $\varepsilon > 0$ and $n \in \mathbb{N}$ if, for each $x, y \in M$, $x \neq y$, there is $0 \leqslant i < n$ such that $|f^i(x) - f^i(y)| > \varepsilon$.

As in [1], we shall say that $f : [a,b] \to [c,d]$ is *piecewise linear* (*piecewise monotone*) if there exists a finite partition of interval $[a,b]$ into closed subintervals $\{P_i\}_{i=1}^n$ such that, for any $i \in \{1,\dots,n\}$, the function $f \restriction P_i$ is linear (monotone).

Let f be a continuous and piecewise monotone. We shall say that f *has a constant slop* s if on each of its pieces of monotonicity it is affine with the slope coefficient of absolute value s (see [1]).

The following lemma is a modified form of Corollary 4.3.13 in [1].

Lemma 1.1. *Let P be a non-degenerate closed interval and $f\colon P \to P$ be a continuous piecewise linear function with constant slop equal to s. Then, $h(f) = \max\{0, \log s\}$.* □

We end this section with a simple remark which will be useful in the next part of the paper.

Remark 1.2. If $P \subset \mathbb{I}$ is a non-degenerate closed interval then the family $\mathcal{F}_P$ of all almost continuous functions from P to P has cardinality $2^{\mathfrak{c}}$. □

2. An entropy of a function at a point

We start this section with recalling the notion of an entropy of an f-bundle introduced in [17].

Let $f : \mathbb{I} \to \mathbb{I}$. A pair $(\mathcal{F}, J) = B_f$, where $\mathcal{F}$ is a family of pairwise disjoint (nonsingletons) continuums in $\mathbb{I}$ and $J \subset \mathbb{I}$ is a connected set such that $J \subset f(A)$ for any $A \in \mathcal{F}$, is called an f-*bundle*. If we additionally assume that $A \subset J$ for all $A \in \mathcal{F}$ then such an f-bundle is called an f-*bundle with dominating fiber*.

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. A set $M \subset \bigcup \mathcal{F}$ is *(B_f, n, ε)-separated* if for each $x, y \in M$, $x \neq y$ there is $0 \leqslant i < n$ such that $f^i(x), f^i(y) \in J$ and $|f^i(x) - f^i(y)| > \varepsilon$.

The *entropy of an f-bundle B_f* is defined in the following way:

$$h(B_f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left[\frac{1}{n} \log \left(s_n^{B_f}(\varepsilon) \right) \right],$$

where

$$s_n^{B_f}(\varepsilon) = \max\bigl\{\mathrm{card}(M) : M \subset \mathbb{I} \text{ is } (B_f, n, \varepsilon)\text{-separated set}\bigr\}.$$

A sequence of f-bundles $\{B_f^k\}_{k \in \mathbb{N}}$, where $B_f^k = (\mathcal{F}_k, J_k)$ for $k \in \mathbb{N}$, *converges to a point* x_0 ($B_f^k \underset{k \to \infty}{\longrightarrow} x_0$) if for any $\varepsilon > 0$ there exists $k_0 \in N$ such that $\bigcup \mathcal{F}_k \subset B(x_0, \varepsilon)$ and $B\bigl(f(x_0), \varepsilon\bigr) \cap J_k \neq \emptyset$ for any $k \geqslant k_0$. Put

$$E_f(x_0) = \left\{ \limsup_{n \to \infty} h(B_f^n) : B_f^n \underset{n \to \infty}{\longrightarrow} x_0 \right\}.$$

It is easy to prove the following fact.

Lemma 2.1. *Let f be a function and $[a, b] \subset \mathbb{I}$ ($a < b$) be an f-invariant set. If $x_0 \in (a, b)$, then $E_f(x_0) = E_{f\restriction[a,b]}(x_0)$.* □

An *entropy of a function f at a point* $x_0 \in \mathbb{I}$ is the number $e_f(x_0) \in [0, +\infty]$ defined in the following way

$$e_f(x_0) = \sup E_f(x_0).$$

Taking into account the properties of the function $e_f : \mathbb{I} \to [0, +\infty]$ presented in [17] and the above terminology, we shall say that $x_0 \in \mathbb{I}$ is an *entropy point* (a *strong entropy point*) of f if $e_f(x_0) = h(f)$ ($e_f(x_0) = h(f)$ and $x_0 \in \mathrm{Fix}(f)$). Obviously, these definitions agree with the definitions of an entropy point and a strong entropy point introduced in [17]. The set of all entropy (strong entropy) points of f will be denoted by $H(f)$ ($H_s(f)$).

In order to ensure the readability of further considerations, we signal two lemmas. The first one can be proved similarly to Theorem 12 in [17].

Lemma 2.2. *Let P be a non-degenerate closed interval and $\varphi : \mathbb{I} \to P$ be a homeomorphism. Let $f : \mathbb{I} \to \mathbb{I}$ and $g : P \to P$ be functions such that $g = \varphi \circ f \circ \varphi^{-1}$. If $x_0 \in H_s(f)$, then $\varphi(x_0) \in H_s(g)$.* □

Moreover, we see at once

Lemma 2.3. *Let f be a function. If there exists a non-degenerate closed interval $[a, b] \subset \mathbb{I}$ such that $f(x) = x$ for $x \in [a, b]$, then $e_f(x) = 0$ for any $x \in (a, b)$.* □

The next lemma is used in the proof of Theorem 3.4 (a), and it simultaneously illustrates the situation when the entropy of a function is focused on one point of the domain.

Lemma 2.4. *There exists a continuous function f such that $h(f) = \infty$, $H_s(f) = \{0\}$ and $H(f) \cap (0,1] = \emptyset$.*

Proof. Put $f(0) = 0$ and $f(1) = 1$. Moreover, if $n \in \mathbb{N}$, then let $f\big(\frac{1}{4^n}(1 + \frac{k}{2^{n+1}-1})\big) = \frac{2}{4^n}$ for $k \in \big\{1, 3, \ldots, 2^{n+1} - 1\big\}$ and $f\big(\frac{1}{4^n}(1 + \frac{k}{2^{n+1}-1})\big) = \frac{1}{4^n}$ for $k \in \big\{0, 1, \ldots, 2^{n+1}-2\big\}$. The function f is defined as an affine function otherwise.

Clearly, the function f is continuous. We will show that $0 \in H_s(f)$. Obviously, $0 \in \mathrm{Fix}(f)$. Put $a_k^n = \frac{1}{4^n}\big(1 + \frac{k}{2^{n+1}-1}\big)$ for $n \in \mathbb{N}$ and $k \in \{0, 1, 2, \ldots, 2^{n+1} - 1\}$. For any $n \in \mathbb{N}$ we define the family $\mathcal{F}_n$ and the set J_n in the following way: $\mathcal{F}_n = \big\{[a_k^n, a_{k+1}^n] : k \in \{0, 2, \ldots, 2^{n+1} - 2\}\big\}$ and $J_n = \big[\frac{1}{4^n}, \frac{2}{4^n}\big]$.

Of course, $\mathrm{card}(\mathcal{F}_n) = 2^n$ for $n \in \mathbb{N}$. Moreover, we have $J_n \subset f\big([a_k^n, a_{k+1}^n]\big)$ and $\big[a_k^n, a_{k+1}^n\big] \subset J_n$ for $n \in \mathbb{N}$ and $k \in \{0, 2, \ldots, 2^{n+1} - 2\}$. Putting $B_f^n = (\mathcal{F}_n, J_n)$ for $n \in \mathbb{N}$, we obtain f-bundles with dominating fiber. Lemma 3.1 [17] implies that $h(B_f^n) \geq \log 2^n = n$ for $n \in \mathbb{N}$ so,

$$\limsup_{n\to\infty} h(B_f^n) \geq \lim_{n\to\infty} n = \infty. \tag{1}$$

It is easy to show that $B_f^n \underset{n\to\infty}{\longrightarrow} 0$. Hence, and from (1), we obtain that $\infty \in E_f(0)$. Lemma 3.6 [17] gives that $h(f) = \infty$, and, in consequence, we have $e_f(0) = h(f)$. Finally, $0 \in H_s(f)$.

Now, we will show that

$$\text{if } x \in (0,1], \quad \text{then } x \notin H(f). \tag{2}$$

We start with proving the following fact

$$\text{for any } n \in \mathbb{N} \text{ we have } h\left(f \restriction \left[\frac{2}{4^n}, 1\right]\right) < \infty. \tag{3}$$

Fix $n \in \mathbb{N}$ and consider the function $g = f \restriction \big[\frac{2}{4^n}, 1\big] : \big[\frac{2}{4^n}, 1\big] \to \big[\frac{2}{4^n}, 1\big]$. Obviously,

$$\left[\frac{2}{4^n}, 1\right] = \bigcup_{i=1}^{n-1} \left[\frac{1}{4^i}, \frac{2}{4^i}\right] \cup \bigcup_{i=1}^{n} \left[\frac{2}{4^i}, \frac{1}{4^{i-1}}\right],$$

and, for any $i \in \{1, \ldots, n-1\}$, the sets $\big[\frac{2}{4^i}, \frac{1}{4^{i-1}}\big]$ and $\big[\frac{1}{4^i}, \frac{2}{4^i}\big]$ are g-invariant. The set $\big[\frac{2}{4^n}, \frac{1}{4^{n-1}}\big]$ is also g-invariant. It is obvious (see, e.g., Lemma 4.1.10 [1]) that

$$h(g) = \max\left\{\left\{h\left(g \restriction \left[\frac{1}{4^i}, \frac{2}{4^i}\right]\right) : i \in \{1, 2, \ldots, n-1\}\right\} \cup \left\{h\left(g \restriction \left[\frac{2}{4^i}, \frac{1}{4^{i-1}}\right]\right) : i \in \{1, 2, \ldots, n\}\right\}\right\}. \tag{4}$$

Since for any $i \in \{1, 2, \dots, n\}$ the functions $g \restriction \left[\frac{2}{4^i}, \frac{1}{4^{i-1}}\right]$ and $g \restriction \left[\frac{1}{4^i}, \frac{2}{4^i}\right]$ are piecewise linear and have a constant slop, so, by Lemma 1.1, we obtain $h\big(g \restriction \left[\frac{2}{4^i}, \frac{1}{4^{i-1}}\right]\big) = 0$ and $h\big(g \restriction \left[\frac{1}{4^i}, \frac{2}{4^i}\right]\big) = \max\big\{0, \log(2^{i+1} - 1)\big\} < \log(2^{i+1}) = i+1$ for $i \in \{1, 2, \dots, n\}$. From this and (4) we conclude that $h(g) \le n+1 < \infty$. So, (3) has been proved.

Let us now turn to the basic part of the proof of (2). Suppose, contrary to our claim, that there is $x_0 \in (0, 1]$ such that $x_0 \in H(f)$. It means that there exists $n_0 \in \mathbb{N}$ such that $\frac{2}{4^{n_0}} < x_0$ and $\infty = h(f) \in E_f(x_0)$. Thus, one can find a sequence of f-bundles $B_f^k = (\mathcal{F}_k, J_k)$ such that $B_f^k \underset{k\to\infty}{\longrightarrow} x_0$ and $\limsup_{k\to\infty} h(B_f^k) = \infty$. Without loss of generality, we can assume that $\bigcup_{k\in\mathbb{N}}(\bigcup \mathcal{F}_k) \subset \left[\frac{2}{4^{n_0}}, 1\right]$.

Consider $g = f \restriction \left[\frac{2}{4^{n_0}}, 1\right]$. It is easy to see that for any $k \in \mathbb{N}$ the pair $(\mathcal{F}_k, J_k)$ is a g-bundle. Moreover, for any $k \in \mathbb{N}$ and $x \in \bigcup \mathcal{F}_k$ we have

$$f^i(x) \in \left[\frac{2}{4^{n_0}}, 1\right] \quad \text{and} \quad g^i(x) = f^i(x) \qquad \text{for} \quad i \in \mathbb{N}. \tag{5}$$

Therefore, for any $k, l \in \mathbb{N}$ and $\varepsilon > 0$,

$$M \text{ is } (B_f^k, l, \varepsilon)\text{-separated set } \Leftrightarrow M \text{ is } (B_g^k, l, \varepsilon)\text{-separated set.} \tag{6}$$

Thus, for any $k \in \mathbb{N}$, we obtain that $s_l^{B_f^k}(\varepsilon) = s_l^{B_g^k}(\varepsilon)$ for any $l \in \mathbb{N}$ and $\varepsilon > 0$, which gives that $h(B_f^k) = h(B_g^k)$. Hence,

$$\limsup_{k\to\infty} h(B_g^k) = \limsup_{k\to\infty} h(B_f^k) = \infty.$$

Lemma 3.6 [17] gives that $h(g) = \infty$. On the other hand, by (3), we obtain $h(g) < \infty$. This contradiction implies that (2) is true, and, in consequence, f has the required properties. $\square$

3. A local approximation of a function

In many papers dealing with the issues discussed here, an approximation of a function had a global nature. However, due to the fact that we are studying mainly a “local” entropy, it seems appropriate to consider also a local approximation. So, in this section, we will concentrate on a local approximation of a function.

Let $x_0 \in [0, 1]$ and $\varepsilon > 0$. We shall say that a function f is *ε-approximated at a point x_0 by a function g* if $g(x) = f(x)$ for $x \in [0, 1] \setminus (x_0 - \varepsilon, x_0 + \varepsilon)$ and $\rho_u(f, g) < \varepsilon$. Clearly, if $\varepsilon^* \in (0, \varepsilon)$ and f is ε^*-approximated at a point x_0 by g, then f is also ε-approximated at a point x_0 by g.

Let $f \in \mathcal{A}$, $x_0 \in \mathbb{I}$ and $\varepsilon > 0$. We will consider the following families of functions:

- $\mathcal{A}(f,\varepsilon,x_0)$ – the family of all functions $g\in\mathcal{A}$ which ε-approximate a function f at a point x_0.
- $\mathcal{A}(f,\varepsilon)$ – the family of all functions $g\in\mathcal{A}$ such that there exists $x_0\in\mathbb{I}$ such that $g\in\mathcal{A}(f,\varepsilon,x_0)$.
- $\mathcal{A}_E(f,\varepsilon,x_0)$ – the family of all functions $g\in\mathcal{A}(f,\varepsilon,x_0)$ such that $e_g(x_0)=h(f)$.
- $\mathcal{A}_L(f,\varepsilon,x_0)$ – the family of all functions $g\in\mathcal{A}(f,\varepsilon,x_0)$ such that $e_g(x_0)<h(f)$ or $e_g(x_0)=h(f)=0$.
- $\mathcal{A}_G(f,\varepsilon,x_0)$ – the family of all functions $g\in\mathcal{A}(f,\varepsilon,x_0)$ such that $e_g(x_0)>h(f)$ or $e_g(x_0)=h(f)=\infty$.

Clearly, for any $\varepsilon>0$, $x_0\in\mathbb{I}$ and $f\in\mathcal{A}$ we have $\mathcal{A}(f,\varepsilon,x_0)\subset\mathcal{A}(f,\varepsilon)$. Moreover, $\mathcal{A}_L(f,\varepsilon,x_0)\cap\mathcal{A}_G(f,\varepsilon,x_0)=\emptyset$. If $h(f)=0$ then $\mathcal{A}_E(f,\varepsilon,x_0)=\mathcal{A}_L(f,\varepsilon,x_0)$. It is obvious that $\mathcal{A}_E(f,\varepsilon,x_0)=\mathcal{A}_G(f,\varepsilon,x_0)$ if $h(f)=\infty$. Obviously, if $h(f)\in(0,\infty)$, then $\mathcal{A}_E(f,\varepsilon,x_0)\cap\mathcal{A}_L(f,\varepsilon,x_0)=\emptyset$ and $\mathcal{A}_E(f,\varepsilon,x_0)\cap\mathcal{A}_G(f,\varepsilon,x_0)=\emptyset$.

One can ask how "big" or "small" these sets are? We will start the answer to this question from the set $\mathcal{A}(f,\varepsilon)$ considered in the space $(\mathcal{A},\rho_u)$.

Theorem 3.1. *For any $f\in\mathcal{A}$ and any $\varepsilon\in\left(0,\frac{1}{2}\right]$ the set $\mathcal{A}(f,\varepsilon)$ is nowhere dense in $(\mathcal{A},\rho_u)$.*

Proof. Fix $h\in\mathcal{A}$ and $\alpha>0$, and choose $h_0\in\mathcal{A}$ such that $\rho_u(h,h_0)<\frac{\alpha}{2}$ and $h_0(x)\neq f(x)$ for $x\in\{0,1\}$. Let $\alpha_0=\min\left\{|f(0)-h_0(0)|,|f(1)-h_0(1)|,\frac{\alpha}{2}\right\}$. Then, $B_u(h_0,\alpha_0)\subset B_u(h,\alpha)\setminus\mathcal{A}(f,\varepsilon)$, because for any $x_0\in\mathbb{I}$ at least one end-point of $\mathbb{I}$ does not belong to $(x_0-\varepsilon,x_0+\varepsilon)$. □

Notice that the above property gives that for any $f\in\mathcal{A}$ and $x_0\in\mathbb{I}$ there is $\varepsilon>0$ such that each of the sets mentioned above is nowhere dense in $(\mathcal{A},\rho_u)$.

Moreover, it is easy to prove that if $\varepsilon>\frac{1}{2}$, then the interior (in $(\mathcal{A},\rho_u)$) of the set $\mathcal{A}(f,\varepsilon)$ is non-empty.

As previously announced, let us turn now to the analysis of the local approximation, i.e., we will consider the space $\left(\mathcal{A}(f,\varepsilon,x_0),\rho_u\right)$. To simplify further notation, let us denote the set of all $\xi\in\mathcal{A}_G(f,\varepsilon,x_0)$ ($\xi\in\mathcal{A}(f,\varepsilon,x_0)$) such that $x_0\in\mathrm{Fix}_c(\xi)$ by $\mathcal{A}^*_G(f,\varepsilon,x_0)$ ($\mathcal{A}^*(f,\varepsilon,x_0)$).

Theorem 3.2. *Let $f\in\mathcal{A}$ and $x_0\in\mathbb{I}$. If $x_0\in\mathrm{Fix}_c(f)$ then for any $\varepsilon>0$ the set $\mathcal{A}^*(f,\varepsilon,x_0)$ is nowhere dense in $(\mathcal{A}(f,\varepsilon,x_0),\rho_u)$.*

Proof. Observe first that the set $\mathcal{A}^*(f,\varepsilon,x_0)$ is closed in $(\mathcal{A}(f,\varepsilon,x_0),\rho_u)$, so it suffices to show that $\mathrm{int}_{\mathcal{A}(f,\varepsilon,x_0)}\left(\mathcal{A}^*(f,\varepsilon,x_0)\right)=\emptyset$. Fix $g\in\mathcal{A}^*(f,\varepsilon,x_0)$ and $\alpha>0$. We may assume that $\alpha<\frac{\varepsilon}{2}$. Since f and g are continuous at x_0 and $f(x_0)=x_0=g(x_0)$, there is $\delta\in(0,\varepsilon)$ with $|f(x)-g(x)|<\frac{\varepsilon}{2}$ for $x\in(x_0-\delta,x_0+\delta)$. Let $h:\mathbb{I}\to[0,\alpha)$ be a continuous function such that $h(x)=0$

for $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$ and $h(x_0) \neq 0$. If $x_0 \neq 1$ then easily $\min\{g+h, 1\} \in \big(\mathcal{A}(f,\varepsilon,x_0) \setminus \mathcal{A}^*(f,\varepsilon,x_0)\big) \cap B_u(g,\alpha)$. If $x_0 = 1$ then it is easy to see that $\max\{g-h, 0\} \in \mathcal{A}(f,\varepsilon,x_0) \setminus \mathcal{A}^*(f,\varepsilon,x_0) \cap B_u(g,\alpha)$. □

Previously we indicated that in this paper we will consider sets of approximating functions with an entropy at a given point different from an entropy of "primal function" for the first time. Clearly, from the above theorem, we obtain immediately

Corollary 3.3. *Let $f \in \mathcal{A}$ and $x_0 \in \mathbb{I}$. If $x_0 \in \mathrm{Fix}_c(f)$ then for any $\varepsilon > 0$ the sets $\mathcal{A}^*_G(f,\varepsilon,x_0)$ and $\mathcal{A}^*_L(f,\varepsilon,x_0)$ are nowhere dense in $\big(\mathcal{A}(f,\varepsilon,x_0),\rho_u\big)$.* □

Now, we will present the statement describing the structure of sets $\mathcal{A}^*_G(f,\varepsilon,x_0)$ and $\mathcal{A}_G(f,\varepsilon,x_0)$ in spaces $\mathcal{A}^*(f,\varepsilon,x_0)$ and $\mathcal{A}(f,\varepsilon,x_0)$, respectively.

Theorem 3.4. *Let $f \in \mathcal{A}$ and $x_0 \in \mathrm{Fix}_c(f)$. Then, for any $\varepsilon > 0$,*

(a) $\mathrm{card}\big(\mathcal{A}^*_G(f,\varepsilon,x_0)\big) = 2^{\mathfrak{c}}$, $\mathrm{int}_{\mathcal{A}^*(f,\varepsilon,x_0)}\big(\mathcal{A}^*_G(f,\varepsilon,x_0)\big) = \emptyset$ *and the set* $\big(\mathcal{A}^*_G(f,\varepsilon,x_0),\rho_u\big)$ *is dense in* $\big(\mathcal{A}^*(f,\varepsilon,x_0),\rho_u\big)$ *(and consequently, it is not nowhere dense in* $\big(\mathcal{A}^*(f,\varepsilon,x_0),\rho_u\big)$*);*

(b) $\mathrm{card}\big(\mathcal{A}_G(f,\varepsilon,x_0)\big) = 2^{\mathfrak{c}}$ *and* $\mathrm{int}_{\mathcal{A}(f,\varepsilon,x_0)}\big(\mathcal{A}_G(f,\varepsilon,x_0)\big) \neq \emptyset$. *Moreover,*
- $\mathrm{int}_{\mathcal{A}(f,\varepsilon,x_0)}(\mathcal{A}_G(f,\varepsilon,x_0)) \cap B_u(f,\sigma) \neq \emptyset$ *for any $\sigma > 0$;*
- *there is an open (in $(\mathcal{A}(f,\varepsilon,x_0),\rho_u)$) set U such that $U \subset \mathcal{A}_G(f,\varepsilon,x_0)$ and $x_0 \in H(g)$ for any $g \in U$.*

Proof. Let $f \in \mathcal{A}$ and $x_0 \in \mathrm{Fix}_c(f)$. We will only consider the case $x_0 \in (0,1)$. The proofs in other cases run similarly.
(a) First, we will show that

$$\mathrm{card}\big(\mathcal{A}^*_G(f,\varepsilon,x_0)\big) = 2^{\mathfrak{c}}. \tag{7}$$

Let $\varepsilon > 0$ and $f_1 : \mathbb{I} \to \mathbb{I}$ be a continuous function such that $h(f_1) = \infty$, $H_s(f_1) = \{0\}$ and $H(f_1) \cap (0,1] = \emptyset$ (the existence of such a function follows from Lemma 2.4).

Since $x_0 \in \mathrm{Fix}_c(f)$, there is $\delta \in \big(0, \frac{\varepsilon}{4}\big)$ such that $|f(x) - x_0| < \frac{\varepsilon}{4}$ whenever $x \in [x_0 - \delta, x_0 + \delta]$.

Let $\varphi : [0,1] \to \big[x_0, x_0 + \frac{\delta}{2}\big]$ be a linear function such that $\varphi(0) = x_0$ and $\varphi(1) = x_0 + \frac{\delta}{2}$. Obviously, the function $t = \varphi \circ f_1 \circ \varphi^{-1} \colon \big[x_0, x_0 + \frac{\delta}{2}\big] \to \big[x_0, x_0 + \frac{\delta}{2}\big]$ is continuous and $h(t) = h(f_1) = \infty$. Moreover, Lemma 2.2 implies that $\varphi(0) = x_0 \in H_s(t)$.

Let $\mathcal{F}_P$ be a family of functions from Remark 1.2 for $P = \big[x_0 - \frac{\delta}{2}, x_0 - \frac{\delta}{4}\big]$, and let $\beta \in \mathcal{F}_P$.

We define the function $g_\beta : \mathbb{I} \to \mathbb{I}$ in the following way: $g_\beta(x) = f(x)$ if $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$, $g_\beta(x) = t(x)$ if $x \in \big[x_0, x_0 + \frac{\delta}{2}\big]$, $g_\beta(x) = \beta(x)$ if $x \in P$. Moreover, g_β is an affine function otherwise.

It is easy to prove that $g_\beta \in \mathcal{A}$, $x_0 \in \mathrm{Fix}_c(g_\beta)$, $e_{g_\beta}(x_0)=\infty$ and f is ε-approximated at the point x_0 by g_β. Therefore, $g_\beta \in \mathcal{A}^*_G(f,\varepsilon,x_0)$. Moreover, we see at once that if $\beta_1,\beta_2 \in \mathcal{F}_P$ and $\beta_1 \neq \beta_2$, then $g_{\beta_1} \neq g_{\beta_2}$. Thus,

$$2^{\mathfrak{c}} = \mathrm{card}(\mathcal{F}_P) = \mathrm{card}\big(\{g_\beta : \beta \in \mathcal{F}_P\}\big) \leq \mathrm{card}\big(\mathcal{A}^*_G(f,\varepsilon,x_0)\big),$$

and the proof of (7) is complete.

Now, we will show that $\mathrm{int}_{\mathcal{A}^*(f,\varepsilon,x_0)}\big(\mathcal{A}^*_G(f,\varepsilon,x_0)\big) = \emptyset$. Suppose, contrary to our claim, that $\mathrm{int}_{\mathcal{A}^*(f,\varepsilon,x_0)}\big(\mathcal{A}^*_G(f,\varepsilon,x_0)\big) \neq \emptyset$. It means that there are $g_0 \in \mathcal{A}^*_G(f,\varepsilon,x_0)$ and $\alpha > 0$ such that

$$B_u(g_0,\alpha) \cap \mathcal{A}^*(f,\varepsilon,x_0) \subset \mathcal{A}^*_G(f,\varepsilon,x_0). \tag{8}$$

Obviously, $g_0 \in B_u(g_0,\alpha) \cap \mathcal{A}^*(f,\varepsilon,x_0)$. Moreover, $x_0 \in \mathrm{Fix}_c(g_0) \cap \mathrm{Fix}_c(f)$. Thus one can find $\gamma \in \big(0, \min\{\frac{\alpha}{3},\frac{\varepsilon}{2}\}\big)$ such that $|g_0(x) - x_0| < \min\big\{\frac{\alpha}{3},\frac{\varepsilon}{4}\big\}$ and $|f(x) - x_0| < \frac{\varepsilon}{4}$ if $|x - x_0| \leq \gamma$. Therefore, $g_0(x_0-\gamma), g_0(x_0+\gamma) \in \big(x_0 - \frac{\alpha}{3}, x_0 + \frac{\alpha}{3}\big)$.

Let us consider the function $g_1 : \mathbb{I} \to \mathbb{I}$ defined in the following way: $g_1(x) = g_0(x)$ if $x \in \mathbb{I} \setminus (x_0-\gamma, x_0+\gamma)$ and $g_1(x) = x$ if $x \in \big[x_0 - \frac{\gamma}{2}, x_0 + \frac{\gamma}{2}\big]$. Moreover, g_1 is an affine function otherwise. Clearly, $g_1 \in \mathcal{A}$. Moreover,

$$g_1 \in B_u(g_0,\alpha) \cap \mathcal{A}^*(f,\varepsilon,x_0). \tag{9}$$

Indeed. It is easy to see that $|g_0(x) - g_1(x)| < \frac{2}{3}\alpha$ for $x \in \mathbb{I}$. Hence, $\rho_u(g_1,g_0) < \alpha$, which means that $g_1 \in B_u(g_0,\alpha)$. Similarly, we show that $\rho_u(g_1,f) < \varepsilon$.

Moreover, $g_1(x) = g_0(x) = f(x)$ for $x \in \mathbb{I} \setminus (x_0-\varepsilon, x_0+\varepsilon)$ and $x_0 \in \mathrm{Fix}_c(g_1)$. Thus, $g_1 \in \mathcal{A}^*(f,\varepsilon,x_0)$, and (9) is proved.

On the other hand, Lemma 2.3 implies that $e_{g_1}(x_0) = 0$, so $g_1 \notin \mathcal{A}^*_G(f,\varepsilon,x_0)$, contrary to (8). This contradiction gives that $\mathrm{int}_{\mathcal{A}^*(f,\varepsilon,x_0)}\big(\mathcal{A}^*_G(f,\varepsilon,x_0)\big) = \emptyset$.

Now, we will prove that $\mathcal{A}^*_G(f,\varepsilon,x_0)$ is dense in $\big(\mathcal{A}^*(f,\varepsilon,x_0),\rho_u\big)$. For this reason, we will show that $B_u(g,\alpha) \cap \mathcal{A}^*_G(f,\varepsilon,x_0) \neq \emptyset$ for any $g \in \mathcal{A}^*(f,\varepsilon,x_0)$ and $\alpha > 0$. So, let $g \in \mathcal{A}^*(f,\varepsilon,x_0)$ and $\alpha > 0$.

Since $x_0 \in \mathrm{Fix}_c(f) \cap \mathrm{Fix}_c(g)$, there is $\delta \in \big(0, \min\{\frac{\alpha}{4},\frac{\varepsilon}{4}\}\big)$ such that $|f(x) - x_0| < \frac{\varepsilon}{4}$ and $|g(x) - x_0| < \min\{\frac{\alpha}{4},\frac{\varepsilon}{4}\}$ for $x \in [x_0-\delta, x_0+\delta]$.

Let $\varphi : [0,1] \to \big[x_0, x_0 + \frac{\delta}{2}\big]$ be a linear function such that $\varphi(0) = x_0$ and $\varphi(1) = x_0 + \frac{\delta}{2}$. Obviously, the function $t = \varphi \circ f_1 \circ \varphi^{-1} : \big[x_0, x_0+\frac{\delta}{2}\big] \to \big[x_0, x_0+\frac{\delta}{2}\big]$ is continuous and $h(t) = h(f_1) = \infty$. Moreover, by Lemma 2.2, we obtain that $\varphi(0) = x_0 \in H_s(t)$.

Put $g_*(x) = g(x)$ for $x \in \mathbb{I} \setminus (x_0-\delta, x_0+\delta)$ and $g_*(x) = t(x)$ for $x \in \big[x_0, x_0+\frac{\delta}{2}\big]$. Moreover, let g_* be an affine function otherwise. Clearly, $g_* : \mathbb{I} \to \mathbb{I}$, $g_* \in \mathcal{A}$ and $x_0 \in \mathrm{Fix}_c(g_*)$. Moreover, $|g(x) - g_*(x)| < \frac{\alpha}{2}$ for $x \in \mathbb{I}$. Hence, $\rho_u(g_*,g) < \alpha$, which gives that $g_* \in B_u(g,\alpha)$.

Since $x_0 \in H_s(t)$, it follows that x_0 is a strong entropy point of the function $g_* \restriction \left[x_0, x_0 + \frac{\delta}{2}\right]$. Therefore, there exists a sequence of $g_* \restriction \left[x_0, x_0 + \frac{\delta}{2}\right]$--bundles $B^k_{g_*\restriction[x_0,x_0+\frac{\delta}{2}]} = (\mathcal{F}_k, J_k)$ (for $k \in \mathbb{N}$) converging to x_0 and such that $\infty = h\big(g_* \restriction \left[x_0, x_0 + \frac{\delta}{2}\right]\big) = \limsup_{k\to\infty} h\big(B^k_{g_*\restriction[x_0,x_0+\frac{\delta}{2}]}\big)$. Obviously, $\bigcup \mathcal{F}_k \subset \left[x_0, x_0+\frac{\delta}{2}\right]$. Since the set $\left[x_0, x_0+\frac{\delta}{2}\right]$ is g_*-invariant and $h\big(g_* \restriction \left[x_0, x_0+\frac{\delta}{2}\right]\big) = h(g_*)$, we have that for any $k \in \mathbb{N}$ the pair $(\mathcal{F}_k, J_k)$ is a g_*-bundle and $e_{g_*}(x_0) \geq \limsup_{k\to\infty} h((\mathcal{F}_k, J_k)) = \infty$. Finally, we get that $e_{g_*}(x_0) = \infty$.

Let $\varepsilon_1 \in (0, \varepsilon)$ be such that $\rho_u(f, g) < \varepsilon_1 < \varepsilon$. Obviously, $f(x) = g(x) = g_*(x)$ for $x \in \mathbb{I} \setminus (x_0 - \varepsilon, x_0 + \varepsilon)$, so $|f(x) - g_*(x)| < \frac{\varepsilon}{2}$. If $x \in \mathbb{I} \setminus (x_0 - \varepsilon, x_0 - \delta] \cup [x_0 + \delta, x_0 + \varepsilon)$ then $g_*(x) = g(x)$, thus $|f(x) - g_*(x)| < \varepsilon_1$. For $x \in (x_0 - \delta, x_0 + \delta)$ we have $g_*(x), g(x) \in \left(x_0 - \frac{\varepsilon}{4}, x_0 + \frac{\varepsilon}{4}\right)$, which gives that $|g(x) - g_*(x)| < \frac{\varepsilon}{2}$. Finally, we have $\rho_u(f, g_*) \leq \min\left\{\varepsilon_1, \frac{\varepsilon}{2}\right\} < \varepsilon$. Therefore, g_* ε-approximates f at x_0.

From the above we obtain that $g_* \in B_u(g, \alpha) \cap \mathcal{A}^*_G(f, \varepsilon, x_0)$. Since g and α were arbitrary, we obtain that $\mathcal{A}^*_G(f, \varepsilon, x_0)$ is dense in $\big(\mathcal{A}^*(f, \varepsilon, x_0), \rho_u\big)$.

(b) Let $\varepsilon > 0$. Clearly, $\mathcal{A}^*_G(f, \varepsilon, x_0) \subset \mathcal{A}_G(f, \varepsilon, x_0)$, so condition (a) gives that $\operatorname{card}\big(\mathcal{A}_G(f, \varepsilon, x_0)\big) = 2^{\mathfrak{c}}$.

Fix $\sigma > 0$. Without loss of generality we may assume that $\sigma < \varepsilon$. Since $x_0 \in \operatorname{Fix}_c(f)$, we have that there exists $\delta \in \left(0, \frac{\sigma}{4}\right)$ such that $|f(x) - x_0| < \frac{\sigma}{4}$ for $x \in [x_0 - \delta, x_0 + \delta]$. Putting $g_\sigma(x) = f(x)$ for $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$, $g_\sigma(x) = \frac{\delta}{2} \sin\left(\frac{1}{x - x_0}\right) + x_0$ for $x \in \left(x_0, x_0 + \frac{\delta}{2}\right]$, $g_\sigma(x_0) = x_0$ and linear otherwise, we obtain that $g_\sigma \in \mathcal{A}$.

Set $V = B_u\left(g_\sigma, \frac{\delta}{4}\right)$. We will show that

$$V \subset B_u(f, \varepsilon) \cap B_u(f, \sigma). \tag{10}$$

Let $\tau \in V$. Thus, $\tau(x) \in \left[g_\sigma(x) - \frac{\delta}{4}, g_\sigma(x) + \frac{\delta}{4}\right]$ for $x \in \mathbb{I}$. First, we will prove that $\tau \in B_u(f, \sigma)$.

- If $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$ then $\tau(x) \in \left[f(x) - \frac{\delta}{4}, f(x) + \frac{\delta}{4}\right]$, so $|f(x) - \tau(x)| < \frac{\varepsilon}{16}$.
- If $x \in (x_0 - \delta, x_0]$ then $g_\sigma(x) \in \left[x_0 - \frac{\varepsilon}{4}, x_0 + \frac{\varepsilon}{4}\right]$ and, in consequence, $\tau(x) \in \left[x_0 - \frac{\varepsilon}{4} - \frac{\delta}{4}, x_0 + \frac{\varepsilon}{4} + \frac{\delta}{4}\right]$. Thus, $|f(x) - \tau(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{8}$.
- If $x \in \left(x_0, x_0 + \frac{\delta}{2}\right]$ then $g_\sigma(x) \in \left[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\right]$. It gives that $\tau(x) \in \left[x_0 - \frac{3}{4}\delta, x_0 + \frac{3}{4}\delta\right] \subset \left[x_0 - \frac{3}{16}\varepsilon, x_0 + \frac{3}{16}\varepsilon\right] \subset \left[x_0 - \frac{\varepsilon}{4}, x_0 + \frac{\varepsilon}{4}\right]$. Thus, $|f(x) - \tau(x)| \leq \frac{\varepsilon}{2}$.
- If $x \in \left(x_0 + \frac{\delta}{2}, x_0 + \delta\right]$ then $g_\sigma(x) \in \left[x_0 - \frac{\varepsilon}{4}, x_0 + \frac{\varepsilon}{4}\right]$, so $\tau(x) \in \left[x_0 - \frac{\varepsilon}{4} - \frac{\delta}{4}, x_0 + \frac{\varepsilon}{4} + \frac{\delta}{4}\right]$. Therefore, $|f(x) - \tau(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{8}$.

We finally get that $|f(x) - \tau(x)| < \frac{5}{8}\varepsilon$ for $x \in \mathbb{I}$, so $\tau \in B_u(f, \varepsilon)$. Similarly, we show that $\tau \in B_u(f, \sigma)$.

From (10), we obtain that $\rho_u(g_\sigma, f) < \varepsilon$. Moreover, $g_\sigma(x) = f(x)$ for $x \in \mathbb{I} \setminus (x_0 - \varepsilon, x_0 + \varepsilon)$. Thus,

$$g_\sigma \in V \cap \mathcal{A}(f, \varepsilon, x_0). \tag{11}$$

Now, we will show that

$$\text{if } t \in V \quad \text{then,} \quad x_0 \in H(t) \quad \text{and} \quad h(t) = \infty. \tag{12}$$

Fix $t \in V$. Clearly, $t(x) \in \left[g_\sigma(x) - \frac{\delta}{4}, g_\sigma(x) + \frac{\delta}{4}\right]$ for $x \in \mathbb{I}$. Let $k_0 \in \mathbb{N}$ be such that $\frac{2}{4k_0+1} < \frac{\delta}{4}$. Putting $\alpha_k = x_0 + \frac{2}{4k+4k_0-3}$ and $\beta_k = x_0 + \frac{2}{4k+4k_0-1}$ for $k \in \mathbb{N}$, we have that if $k \in \mathbb{N}$, then $\alpha_{k+1} < \beta_k < \alpha_k$, $\beta_k, \alpha_k \in \left(x_0, x_0 + \frac{\delta}{4}\right]$, $g_\sigma(\alpha_k) = x_0 + \frac{\delta}{2}$ and $g_\sigma(\beta_k) = x_0 - \frac{\delta}{2}$. Moreover,

$$\lim_{k \to \infty} \alpha_k = x_0. \tag{13}$$

Clearly, $t(x_0) \in \left[x_0 - \frac{\delta}{4}, x_0 + \frac{\delta}{4}\right]$, $t(\beta_k) \in \left[x_0 - \frac{3}{4}\delta, x_0 - \frac{1}{4}\delta\right]$ and $t(\alpha_k) \in \left[x_0 + \frac{1}{4}\delta, x_0 + \frac{3}{4}\delta\right]$ for $k \in \mathbb{N}$. Since for any $k \in \mathbb{N}$ we have $t(\beta_k) < t(\alpha_k)$ and $t \in \mathcal{A}$, one can find $\beta_k^*, \alpha_k^* \in [\beta_k, \alpha_k]$ such that $\beta_k^* < \alpha_k^*$, $t(\beta_k^*) = x_0 - \frac{\delta}{4}$ and $t(\alpha_k^*) = x_0 + \frac{\delta}{4}$. Of course,

$$\left[x_0 - \frac{\delta}{4}, x_0 + \frac{\delta}{4}\right] \subset t([\beta_k^*, \alpha_k^*]) \quad \text{and} \quad [\beta_k^*, \alpha_k^*] \subset \left[x_0, x_0 + \frac{\delta}{4}\right] \tag{14}$$

for any $k \in \mathbb{N}$. Put $\mathcal{F}_n = \left\{[\beta_k^*, \alpha_k^*] : k = n, n+1, \ldots\right\}$, $J_n = \left[x_0 - \frac{\delta}{4}, x_0 + \frac{\delta}{4}\right]$, and $B_t^n = (\mathcal{F}_n, J_n)$ for $n \in \mathbb{N}$. It is easy to see that $\{B_t^n\}_{n \in \mathbb{N}}$ is a sequence of t-bundles with dominating fiber converging to x_0.

Lemma 3.1. [17] gives that $h(B_t^n) = \infty$ for $n \in \mathbb{N}$. Thus, $\infty \in E_t(x_0)$, so $e_t(x_0) = \infty$. Moreover, Lemma 3.6 [17] implies that $h(t) = \infty$. Hence, $e_t(x_0) = h(t)$ which means that $x_0 \in H(t)$.

From (11), we get that $V \cap \mathcal{A}(f, \varepsilon, x_0) \neq \emptyset$. Moreover, the condition (12) implies that $V \cap \mathcal{A}(f, \varepsilon, x_0) \subset \mathcal{A}_G(f, \varepsilon, x_0)$. By (10), we obtain that $V \cap \mathcal{A}(f, \varepsilon, x_0) \subset B_u(f, \sigma)$, so $\operatorname{int}_{\mathcal{A}(f,\varepsilon,x_0)}\left(\mathcal{A}_G(f, \varepsilon, x_0)\right) \cap B_u(f, \sigma) \neq \emptyset$. Furthermore, (12) implies that x_0 is an entropy point of each function from $V \cap \mathcal{A}(f, \varepsilon, x_0)$. $\square$

We will now proceed to the analysis of sets of approximating functions with an entropy at a given point less or equal to an entropy of "primal function".

Theorem 3.5. *Let $f \in \mathcal{A}$ be such that $H_s(f) \neq \emptyset$. If there is a point $x_0 \in \mathrm{Fix}_c(f) \cap (0,1)$ then for any $\varepsilon > 0$ we have that*

(a) *the set $\mathcal{A}_L(f, \varepsilon, x_0)$ has cardinality $2^{\mathfrak{c}}$ and it is not $\mathcal{A}(f, \varepsilon, x_0)$-dense at f;*

(b) *the set $\mathcal{A}_E(f, \varepsilon, x_0)$ has cardinality $2^{\mathfrak{c}}$. Moreover, if $h(f) < \infty$, then it is not $\mathcal{A}(f, \varepsilon, x_0)$-dense at f.*

P r o o f. Let $y_0 \in H_s(f)$. It means that $y_0 \in \mathrm{Fix}(f)$ and $e_f(y_0) = h(f)$. Fix $\varepsilon > 0$.

(a) First, we will show that $\mathrm{card}\big(\mathcal{A}_L(f,\varepsilon,x_0)\big) = 2^{\mathfrak{c}}$. Since $x_0 \in \mathrm{Fix}_c(f) \cap (0,1)$, it follows that there exists $\delta \in \big(0, \frac{\varepsilon}{4}\big)$ such that for $x \in [x_0 - \delta, x_0 + \delta] \subset (0,1)$ we have $f(x) \in \big(x_0 - \frac{\varepsilon}{4}, x_0 + \frac{\varepsilon}{4}\big)$. In particular, $f(x_0 - \delta), f(x_0+\delta) \in \big(x_0 - \frac{\varepsilon}{4}, x_0 + \frac{\varepsilon}{4}\big)$. Let $P = \big[x_0 - \frac{3}{4}\delta, x_0 - \frac{2}{3}\delta\big]$ and $\mathcal{F}_P$ be as in Remark 1.2. Fix $\beta \in \mathcal{F}_P$ and put $f_\beta(x) = f(x)$ for $x \notin (x_0 - \delta, x_0 + \delta)$, $f_\beta(x) = x$ for $x \in \big[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\big]$, $f_\beta(x) = \beta(x)$ for $x \in P$. Moreover, let f_β be an affine function otherwise. It is easy to see that $f_\beta \in \mathcal{A}(f,\varepsilon,x_0)$. Furthermore, Lemma 2.3 gives that $e_{f_\beta}(x_0) = 0$. Thus, $f_\beta \in \mathcal{A}_L(f,\varepsilon,x_0)$. Since β was arbitrary, we have $\{f_\beta : \beta \in \mathcal{F}_P\} \subset \mathcal{A}_L(f,\varepsilon,x_0)$, which gives that $\mathrm{card}\big(\mathcal{A}_L(f,\varepsilon,x_0)\big) = 2^{\mathfrak{c}}$.

Suppose that $\mathcal{A}_L(f,\varepsilon,x_0)$ is $\mathcal{A}(f,\varepsilon,x_0)$-dense at f. Thus one can find $\sigma > 0$ such that $B_u(f,\sigma) \cap \mathcal{A}(f,\varepsilon,x_0) \subset \mathrm{cl}_{\mathcal{A}(f,\varepsilon,x_0)}\big(\mathcal{A}_L(f,\varepsilon,x_0)\big)$. Let W be open in $\big(\mathcal{A}(f,\varepsilon,x_0), \rho_u\big)$ set such that $W \cap B_u(f,\sigma) \cap \mathcal{A}(f,\varepsilon,x_0) \neq \emptyset$. Thus, $W \cap \mathcal{A}_L(f,\varepsilon,x_0) \cap B_u(f,\sigma) \cap \mathcal{A}(f,\varepsilon,x_0) \neq \emptyset$. Hence, $\mathrm{int}_{\mathcal{A}(f,\varepsilon,x_0)}\big(\mathcal{A}_G(f,\varepsilon,x_0)\big) \cap B_u(f,\sigma) = \emptyset$, in contradiction to Theorem 3.4 (b).

(b) First, we will show that $\mathrm{card}\big(\mathcal{A}_E(f,\varepsilon,x_0)\big) = 2^{\mathfrak{c}}$. Since $x_0 \in \mathrm{Fix}_c(f) \cap (0,1)$, there is $\delta \in \big(0, \frac{\varepsilon}{4}\big)$ such that $[x_0 - \delta, x_0 + \delta] \subset (0,1)$ and $|f(x) - x_0| < \frac{\varepsilon}{4}$ for $x \in [x_0 - \delta, x_0 + \delta]$. Assume that $y_0 = 0$ (for $y_0 \in (0,1]$ the proof runs in a similar way).

Let $\varphi \colon \mathbb{I} \to \big[x_0, x_0 + \frac{\delta}{2}\big]$ be a homeomorphism such that $\varphi(y_0) = x_0$. Let $P = \big[x_0 - \frac{3}{4}\delta, x_0 - \frac{2}{3}\delta\big]$ and $\mathcal{F}_P$ be as in Remark 1.2. Fix $\beta \in \mathcal{F}_P$.

Put $g_\beta(x) = f(x)$ for $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$, $g_\beta(x) = (\varphi \circ f \circ \varphi^{-1})(x)$ for $x \in \big[x_0, x_0 + \frac{\delta}{2}\big]$, $g_\beta(x) = x$ for $x \in \big[x_0 - \frac{\delta}{2}, x_0\big]$, and $g_\beta(x) = \beta(x)$ for $x \in P$. Moreover, let g_β be an affine function otherwise.

We see at once that $g_\beta \in \mathcal{A}$. Since the intervals $\big[x_0 - \frac{\delta}{2}, x_0\big]$ and $\big[x_0, x_0 + \frac{\delta}{2}\big]$ are g_β-invariant sets, Lemma 4.1.10 [1] implies that $h\big(g_\beta \restriction \big[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\big]\big) = \max\big\{h\big(g_\beta \restriction \big[x_0 - \frac{\delta}{2}, x_0\big]\big), h\big(g_\beta \restriction \big[x_0, x_0 + \frac{\delta}{2}\big]\big)\big\}$. Obviously, $h\big(g_\beta \restriction \big[x_0, x_0 + \frac{\delta}{2}\big]\big) = h(f)$. Lemma 1.1 gives that $h\big(g_\beta \restriction \big[x_0 - \frac{\delta}{2}, x_0\big]\big) = 0$. Therefore, $h\big(g_\beta \restriction \big[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\big]\big) = h(f)$. From the above and Lemma 3.6. [17], we obtain

$$e_{g_\beta \restriction [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]}(x_0) \leq h\left(g_\beta \restriction \left[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\right]\right) = h(f). \tag{15}$$

On the other hand, Lemma 2.2 gives that $\varphi(y_0) = x_0 \in H_s\big(g_\beta \restriction [x_0, x_0 + \frac{\delta}{2}]\big)$, so $e_{g_\beta \restriction [x_0, x_0 + \frac{\delta}{2}]}(x_0) = h\big(g_\beta \restriction [x_0, x_0 + \frac{\delta}{2}]\big) = h(f)$.

Let $\sigma > 0$. There exists a sequence of $g_\beta \restriction [x_0, x_0 + \frac{\delta}{2}]$-bundles $\big\{B^k_{g_\beta \restriction [x_0, x_0 + \frac{\delta}{2}]}\big\}_{k \in \mathbb{N}}$ converging to x_0 such that $\limsup_{k \to \infty} h\big(B^k_{g_\beta \restriction [x_0, x_0 + \frac{\delta}{2}]}\big) > h(f) - \sigma$. Clearly, for any $k \in \mathbb{N}$, the bundle $B^k_{g_\beta \restriction [x_0, x_0 + \frac{\delta}{2}]}$ is $g_\beta \restriction \big[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\big]$-bundle, so

$\limsup_{k\to\infty} h\big(B^k_{g_\beta\restriction[x_0,x_0+\frac{\delta}{2}]}\big) \in E_{g_\beta\restriction[x_0-\frac{\delta}{2},x_0+\frac{\delta}{2}]}(x_0)$. We finally have

$$e_{g_\beta\restriction[x_0-\frac{\delta}{2},x_0+\frac{\delta}{2}]}(x_0) \geq h(f). \tag{16}$$

By (15) and (16), we conclude that $e_{g_\beta\restriction[x_0-\frac{\delta}{2},x_0+\frac{\delta}{2}]}(x_0) = h(f)$. This, the fact that $\big[x_0-\frac{\delta}{2}, x_0+\frac{\delta}{2}\big]$ is g_β-invariant set, and Lemma 2.1 imply that

$$e_{g_\beta}(x_0) = \sup E_{g_\beta}(x_0) = \sup E_{g_\beta\restriction[x_0-\frac{\delta}{2},x_0+\frac{\delta}{2}]}(x_0) = e_{g_\beta\restriction[x_0-\frac{\delta}{2},x_0+\frac{\delta}{2}]}(x_0) = h(f).$$

Moreover, it is easy to see that $g_\beta \in \mathcal{A}(f,\varepsilon,x_0)$. In consequence, we obtain $g_\beta \in \mathcal{A}_E(f,\varepsilon,x_0)$. Since β was arbitrary, we have $\{f_\beta : \beta \in \mathcal{F}_P\} \subset \mathcal{A}_E(f,\varepsilon,x_0)$. Thus, $\operatorname{card}\big(\mathcal{A}_E(f,\varepsilon,x_0)\big) = 2^{\mathfrak{c}}$.

Assume now that $h(f) < \infty$ and suppose that $\mathcal{A}_E(f,\varepsilon,x_0)$ is $\mathcal{A}(f,\varepsilon,x_0)$--dense at f. Thus, $B_u(f,\sigma) \cap \mathcal{A}(f,\varepsilon,x_0) \subset \operatorname{cl}_{\mathcal{A}(f,\varepsilon,x_0)}\big(\mathcal{A}_E(f,\varepsilon,x_0)\big)$ for some $\sigma > 0$. Let W be an open in $\big(\mathcal{A}(f,\varepsilon,x_0),\rho_u\big)$ set such that $W \cap B_u(f,\sigma) \cap \mathcal{A}(f,\varepsilon,x_0) \neq \emptyset$. Thus, $W \cap \mathcal{A}_L(f,\varepsilon,x_0) \cap B_u(f,\sigma) \cap \mathcal{A}(f,\varepsilon,x_0) \neq \emptyset$. Hence, $\operatorname{int}_{\mathcal{A}(f,\varepsilon,x_0)}\big(\mathcal{A}_E(f,\varepsilon,x_0)\big) \cap B_u(f,\sigma) = \emptyset$, in contradiction to Theorem 3.4 (b). $\square$

Considerations connected with a structure of sets of functions which approximate a given function in conjunction with an entropy of a function at a point are in an introductory stage. Further studies related to the set of approximating functions which have an entropy at a given point exactly the same as an entropy of the "primal function" appears to be particularly interesting. It also seems to be interesting to examine these issues for functions $f : \mathbb{I}^n \to \mathbb{I}^n$ or, more generally, for functions defined on a locally Euclidean space (i.e., on such a space that there exists nonnegative integer n such that each point of the space has a neighbourhood which is homeomorphic with Euclidean space $\mathbb{R}^n$) or some compact manifold. In many issues, the basis of considerations dealing with discrete dynamical systems are functions from the unit interval into itself ([1], [4], [13]). Thus, it seems appropriate to start the further research from the issues presented in this paper.

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