DE GRUYTER
OPEN

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.1515/tmmp-2016-0012
Tatra Mt. Math. Publ. **65** (2016), 143–149

FUBINI PROPERTY FOR MICROSCOPIC SETS

Adam Paszkiewicz* — Elżbieta Wagner-Bojakowska

ABSTRACT. In 2000, I. Recław and P. Zakrzewski introduced the notion of Fubini Property for the pair $(\mathcal{I},\mathcal{J})$ of two σ-ideals in the following way. Let $\mathcal{I}$ and $\mathcal{J}$ be two σ-ideals on Polish spaces X and Y, respectively. The pair $(\mathcal{I},\mathcal{J})$ has the Fubini Property (FP) if for every Borel subset B of $X\times Y$ such that all its vertical sections $B_x=\{y\in Y:(x,y)\in B\}$ are in $\mathcal{J}$, then the set of all $y\in Y$, for which horizontal section $B^y=\{x\in X:(x,y)\in B\}$ does not belong to $\mathcal{I}$, is a set from $\mathcal{J}$, i.e.,

$$\{y\in Y: B^y\notin\mathcal{I}\}\in\mathcal{J}.$$

The Fubini property for the σ-ideal $\mathcal{M}$ of microscopic sets is considered and the proof that the pair $(\mathcal{M},\mathcal{M})$ does not satisfy (FP) is given.

In measure theory or in functional analysis there is often proved that some property holds "almost everywhere", i.e., except for some set of Lebesgue measure zero or "nearly everywhere", it means except for some set of the first category. The families $\mathcal{N}$ of nullsets and $\mathcal{K}$ of sets of the first category on the real line, or generally in $\mathbb{R}^n$, form σ-ideals, which are orthogonal: there exist two sets A and B such that

$$\mathbb{R}=A\cup B$$

where A is a set of the first category and B is a nullset (see [6, Theorem 1.6]).

One of the famous result containing a phrase "almost everywhere" is Fubini Theorem for sets of planar measure zero. This theorem gives a close connection between two-dimensional measure of a measurable subset of $\mathbb{R}^2$ and the linear measure of its sections perpendicular to an axis. It says that if $E\subset[0,1]^2$ is a measurable subset of the plane such that two-dimensional Lebesgue measure of E is equal to zero, then $E_x=\{y\in Y:(x,y)\in E\}$ is a set of linear measure zero for $x\in[0,1]$ except for a set of linear measure zero (see [6, Theorem 14.2]). The converse of Fubini Theorem is true in the sense that if almost all sections of a measurable subset E of the plane are nullsets on the real line, then E is a nullset on the plane.

2010 Mathematics Subject Classification: 28A05, 28A75.
Keywords: sets of Lebesgue measure zero, sets of the first category, microscopic sets on the plane, Fubini Theorem, Fubini property.
*Supported by the grant 2011/01/B/ST1/03994.

In category case the situation is similar, as Fubini Theorem has a category analogue. In 1932, K. Kuratowski and S. Ulam proved that if E is a subset of the plane of the first category, then E_x is a linear set of the first category for all x except for a set of the first category (compare [3, Corollary 1a, p. 247]; see also [6, Theorem 15.1]). Also, a partial converse of Kuratowski-Ulam Theorem is true: if E is a subset of the plane having the Baire property and E_x is of the first category for all x except for a set of the first category, then E is a set of the first category on the plane.

The analogous considerations where carried out for microscopic sets by A. Karasińska and E. Wagner-Bojakowska in [5].

The notion of a microscopic set on the real line was introduced by J. Appell, E. D'Aniello and M. Väth in [1]. A set $A \subset \mathbb{R}$ is microscopic if for each $\varepsilon > 0$ there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of intervals such that $A \subset \bigcup_{n\in\mathbb{N}} I_n$ and $m(I_n) < \varepsilon^n$ for $n \in \mathbb{N}$, where m denotes one-dimensional Lebesgue measure. Clearly, the family of microscopic sets is a σ-ideal situated between countable sets and sets of Lebesgue measure zero, which is different from each of these families. There exists a microscopic set which is residual (so, uncountable), and the classical Cantor set is a nullset, which is not microscopic.

In [5], the authors introduced the notion of microscopic and strongly microscopic sets on the plane. A set $A \subset \mathbb{R}^2$ is microscopic if for each $\varepsilon > 0$ there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of intervals, i.e., rectangles with sides parallel to coordinate axes, such that $A \subset \bigcup_{n\in\mathbb{N}} I_n$ and $m_2(I_n) < \varepsilon^n$ for each $n \in \mathbb{N}$, where m_2 denotes two-dimensional Lebesgue measure. In the definition of strongly microscopic set on the plane, intervals are replaced with squares with sides parallel to coordinate axes.

In [5], it is proved, among other facts, that these families are σ-ideals of subsets of the plane situated between countable sets $\mathcal{C}_2$ and sets of Lebesgue measure zero $\mathcal{N}_2$ and essentially different from each of these families. Additionally, these σ-ideals are also orthogonal to the σ-ideal of sets of the first category on the plane. One of the main result in the paper mentioned above is the Fubini type theorem for microscopic sets. It is also proved that the converse of the Fubini type theorem for microscopic sets does not hold.

In [8], the authors introduced the notion of Fubini Property for the pair $(\mathcal{I}, \mathcal{J})$ of two σ-ideals in the following way. Let $\mathcal{I}$ and $\mathcal{J}$ be two proper σ-ideals on Polish spaces X and Y, respectively. The pair $(\mathcal{I}, \mathcal{J})$ has the Fubini Property (FP) if for every Borel subset B of $X \times Y$ such that all its vertical sections $B_x = \{y \in Y : (x, y) \in B\}$ are in $\mathcal{J}$, then the set of all $y \in Y$, for which horizontal section $B^y = \{x \in X : (x, y) \in B\}$ does not belong to $\mathcal{I}$, is a set from $\mathcal{J}$, i.e.,

$$\{y \in Y : B^y \notin \mathcal{I}\} \in \mathcal{J}.$$

A Borel set $E \subset X \times Y$ is said to be a 0-1 counterexample to (FP) for the pair $(\mathcal{I}, \mathcal{J})$ if $E_x \in \mathcal{J}$ for each $x \in X$ and $X \setminus E^y \in \mathcal{I}$ for each $y \in Y$.

Now, let $X = Y = \mathbb{R}$. From Fubini Theorem and the converse of Fubini Theorem it follows that the pair $(\mathcal{N}, \mathcal{N})$ satisfies (FP). Analogously, Kuratowski-Ulam Theorem and partial converse of this theorem imply that the pair $(\mathcal{K}, \mathcal{K})$ satisfies (FP), as well.

On the other hand, neither $(\mathcal{K}, \mathcal{N})$ nor $(\mathcal{N}, \mathcal{K})$ satisfies (FP). Moreover, there exists a Borel set $E \subset \mathbb{R} \times \mathbb{R}$ which is a 0-1 counterexample to (FP) for the pair $(\mathcal{K}, \mathcal{N})$. It follows from the fact that the σ-ideals $\mathcal{N}$ and $\mathcal{K}$ are orthogonal. More precisely, there exists a nullset B of type G_δ such that $\mathbb{R} \setminus B \in \mathcal{K}$. Hence, a Borel set

$$E = \{(x, y) \in \mathbb{R}^2 \colon x + y \in B\}$$

is a 0-1 counterexample to Fubini Property for $(\mathcal{K}, \mathcal{N})$, and a set

$$F = \{(x, y) \in \mathbb{R}^2 \colon x + y \in \mathbb{R} \setminus B\}$$

is a 0-1 counterexample to (FP) for the pair $(\mathcal{N}, \mathcal{K})$ (see [8, Example 3.6]).

In [5], it is proved, using the fact that the σ-ideals $\mathcal{K}$ and $\mathcal{M}$ are orthogonal, that neither the pair $(\mathcal{K}, \mathcal{M})$ nor $(\mathcal{M}, \mathcal{K})$ satisfies (FP). For this purpose, the authors used a microscopic set M of type G_δ such that $\mathbb{R} \setminus M$ is a set of the first category (see also [4, Theorem 20.4]). Hence, Borel sets

$$\{(x, y) \in \mathbb{R}^2 \colon x + y \in M\}$$

and

$$\{(x, y) \in \mathbb{R}^2 \colon x + y \in \mathbb{R} \setminus M\}$$

are 0-1 counterexamples to (FP) for $(\mathcal{K}, \mathcal{M})$ and $(\mathcal{M}, \mathcal{K})$, respectively.

In the paper above mentioned, it is also proved that the pair $(\mathcal{M}, \mathcal{N})$ does not satisfy (FP). For this purpose, it is sufficient to consider a Borel set of the form

$$\{(x, y) \in \mathbb{R}^2 \colon x + y \in C\},$$

where C is the ternary Cantor set, as $C \in \mathcal{N} \setminus \mathcal{M}$.

The aim of this paper is to prove that the pair $(\mathcal{M}, \mathcal{M})$ does not satisfy (FP).

Our reasoning presented here is elementary and complete. The same result can be obtained using [2], [7] and the prior properties of microscopic sets ([4, Proposition 20.14]).

For each $x \in [0, 1]$, let $(0.\, x_1 x_2 \ldots)_2$ be a binary expansion of x, i.e., $x_i \in \{0, 1\}$ for each $i \in \mathbb{N}$ and

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}.$$

If x has two binary representations, we choose this one, which ends with the sequence of the figures one, i.e., $x = (0.\, x_1 x_2 \ldots x_n 111 \ldots)_2$. The set of all such sequences will be denoted by $\mathcal{F}$.

For every $x \in [0,1]$, if $x = (0.\,x_1x_2\ldots)_2$, $x_i \in \{0,1\}$ for $i \in \mathbb{N}$, put

$$\phi(x) = \sum_{i=1}^{\infty} \frac{2x_i}{5^i}. \tag{1}$$

Lemma 1. *The function ϕ is in Baire class one.*

P r o o f. Put $E = \bigcup_{n=1}^{\infty}\{\frac{k}{2^n} : k = 0,1,\ldots,2^n\}$. We will show that ϕ is continuous at each point $x \in [0,1] \setminus E$. Let $\varepsilon > 0$, $x_0 \in [0,1] \setminus E$ and i_0 be such a positive integer that

$$\sum_{i=i_0+1}^{\infty} \frac{2}{5^i} < \varepsilon. \tag{2}$$

There exists $k_{i_0} \in \{0,1,\ldots,2^{i_0}-1\}$ such that

$$x_0 \in \left(\frac{k_{i_0}}{2^{i_0}}, \frac{k_{i_0}+1}{2^{i_0}}\right).$$

It is easy to see that for all $x \in \left(\frac{k_{i_0}}{2^{i_0}}, \frac{k_{i_0}+1}{2^{i_0}}\right)$, if $x_0 = (0.\,x_1^0x_2^0\ldots)_2$ and $x = (0.\,x_1x_2\ldots)_2$, where $x_i^0, x_i \in \{0,1\}$ for $i \in \mathbb{N}$, then $x_i^0 = x_i$ for $i = 1,\ldots,i_0$. Hence, using (2), we obtain

$$|\phi(x) - \phi(x_0)| = \left|\sum_{i=i_0+1}^{\infty} 2\frac{x_i - x_i^0}{5^i}\right| \leq \sum_{i=i_0+1}^{\infty} \frac{2}{5^i} < \varepsilon.$$

Consequently, ϕ is continuous except for some countable set, so ϕ is Baire one function. □

Lemma 2. *There exists a Cantor-type set C which is not microscopic and*

$$\left(C + \sum_{i=1}^{\infty} \frac{2x_i}{5^i}\right) \cap \left(C + \sum_{i=1}^{\infty} \frac{2y_i}{5^i}\right) = \emptyset$$

for all zero-one sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ such that $\{x_i\}_{i\in\mathbb{N}} \neq \{y_i\}_{i\in\mathbb{N}}$.

P r o o f. Put $I_0 = \left[0, \frac{1}{5}\right]$ and $I_1 = \left[\frac{4}{5}, 1\right]$. Let $k \geq 2$. Assume that we have constructed 2^k closed disjoint intervals $I_{(i_1\ldots i_k)}$ (where $i_l = 0$ or $i_l = 1$ for $l = \{1,2,\ldots,k\}$) of the length $\frac{1}{5^k}$. From the middle of each of these intervals, we remove an open interval of the length $\frac{3}{5}\cdot\frac{1}{5^k}$. We obtain 2^{k+1} closed intervals $I_{(i_1\ldots i_{k+1})}$, where $i_l = 0$ or $i_l = 1$ for $l = \{1,2,\ldots,k+1\}$, each one of the length $\frac{1}{5^{k+1}}$. Put

$$C = \bigcap_{k=1}^{\infty} \bigcup_{i_1\ldots i_k} I_{(i_1\ldots i_k)}.$$

Obviously, C is a closed nowhere dense subset of $[0,1]$.

Moreover, C is not a microscopic set. Indeed, for $\varepsilon < \frac{1}{5}$, there does not exist a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $C \subset \bigcup_{n\in\mathbb{N}} I_n$ and $m(I_n) < \varepsilon^n$ for $n \in \mathbb{N}$.

Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be two arbitrary zero-one sequences such that $\{x_i\}_{i\in\mathbb{N}} \neq \{y_i\}_{i\in\mathbb{N}}$. Put $k = \min\{i \in \mathbb{N} : x_i \neq y_i\}$. Without loss of generality, we can assume that $y_k = 1$ and $x_k = 0$. Then,

$$\sum_{i=1}^{\infty} \frac{2y_i}{5^i} - \sum_{i=1}^{\infty} \frac{2x_i}{5^i} = \frac{2}{5^k} + \alpha, \tag{3}$$

where

$$| \alpha | \leq \sum_{i=k+1}^{\infty} \frac{2 \mid y_i - x_i \mid}{5^i} \leq 2 \sum_{i=k+1}^{\infty} \frac{1}{5^i} = \frac{1}{2 \times 5^k}. \tag{4}$$

Clearly,

$$C \subset \bigcup_{i_1 \dots i_k} I_{(i_1 \dots i_k)} \quad \text{and} \quad m(I_{(i_1 \dots i_k)}) = 5^{-k}$$

for each zero-one sequence $(i_1 \dots i_k)$. Observe that if $(i_1 \dots i_k)$ and $(j_1 \dots j_k)$ are two arbitrary zero-one sequences such that $(i_1 \dots i_k) \neq (j_1 \dots j_k)$ then,

$$\operatorname{dist}(I_{(i_1 \dots i_k)}), I_{(j_1 \dots j_k)}) \geq 3 \times 5^{-k}.$$

Simultaneously, using (3) and (4), we obtain

$$\frac{1}{5^k} < \frac{3}{2} \times \frac{1}{5^k} \leq \sum_{i=1}^{\infty} \frac{2y_i}{5^i} - \sum_{i=1}^{\infty} \frac{2x_i}{5^i} \leq \frac{5}{2} \times \frac{1}{5^k} < \frac{3}{5^k},$$

so,

$$I_{(i_1 \dots i_k)} \cap \left(I_{(j_1 \dots j_k)} + \sum_{i=1}^{\infty} \frac{2y_i}{5^i} - \sum_{i=1}^{\infty} \frac{2x_i}{5^i} \right) = \emptyset$$

for arbitrary two zero-one sequences $(i_1 \dots i_k), (j_1 \dots j_k)$ such that $(i_1 \dots i_k) \neq (j_1 \dots j_k)$. Consequently,

$$C \cap \left(C + \sum_{i=1}^{\infty} \frac{2y_i}{5^i} - \sum_{i=1}^{\infty} \frac{2x_i}{5^i} \right) = \emptyset,$$

and finally,

$$\left(C + \sum_{i=1}^{\infty} \frac{2x_i}{5^i} \right) \cap \left(C + \sum_{i=1}^{\infty} \frac{2y_i}{5^i} \right) = \emptyset.$$

□

Theorem 3. *If $\mathcal{J}$ is a σ-ideal which contains all singletons, then the pair $(\mathcal{M}, \mathcal{J})$ does not have the Fubini Property (FP).*

Proof. We will prove that there exists a Borel set $A \subset \mathbb{R}^2$ such that each vertical section A_x contains at most one point and horizontal sections A^y are not microscopic for $y \in [0,1]$. Let C be as in Lemma 2. Let us consider the family $\mathcal{F}$ of all zero-one sequences used in the definition of the function ϕ.

Put

$$A = \bigcup_{\{y_i\}\in\mathcal{F}} \left(\left(C + \sum_{i=1}^{\infty} \frac{2y_i}{5^i} \right) \times \left\{ \sum_{i=1}^{\infty} \frac{y_i}{2^i} \right\} \right).$$

Clearly,

$$A = \bigcup_{y\in[0,1]} \Big(\big(C + \phi(y)\big) \times \{y\} \Big) = \left\{ (x,y) \in \left[0, \frac{3}{2}\right] \times [0,1] \colon x \in C + \phi(y) \right\},$$

where ϕ is the function defined in (1). Moreover, putting

$$g(x,y) = x - \phi(y),$$

for $(x,y) \in \left[0, \frac{3}{2}\right] \times [0,1]$, we obtain Baire class one function such that

$$A = g^{-1}(C).$$

Consequently, the set A is of type G_δ, so also a Borel set. From Lemma 2, it follows that each vertical section A_x contains at most one point. Simultaneously, $A^y = C + \phi(y) \notin \mathcal{M}$ for each $y \in [0,1]$. $\square$

Corollary 4. *The pair $(\mathcal{M}, \mathcal{M})$ does not satisfy (FP).*

In [7], it is announced that the pair $(\mathcal{N}, \mathcal{M})$ does not satisfy (FP), either.

REFERENCES

[1] APPELL, J.—D'ANIELLO, E.—VÄTH, M.: *Some remarks on small sets,* Ricerche Mat. **50** (2001), 255–274.

[2] BALCERZAK, M.: *Can ideals without ccc be interesting?,* Topology Appl. **55** (1994), 251–260.

[3] KURATOWSKI, K.: *Topology, Vol. 1,* in: Monogr. Mat., PWN, Warszawa, 1966.

[4] HORBACZEWSKA, G.—KARASIŃSKA, A.—WAGNER-BOJAKOWSKA, E.: *Properties of the σ-ideal of microscopic sets,* in: Traditional and Present-Day Topics in Real Analysis, Chapter 20, Łódź University Press, Łódź, 2013, pp. 325–344.

[5] KARASIŃSKA, A.—WAGNER-BOJAKOWSKA, E.: *Microscopic and strongly microscopic sets on the plane. Fubini Theorem and Fubini property,* Demonstratio Math. **47** (2014), 582–595.

[6] OXTOBY, J. C.: *Measure and Category.* Springer-Verlag, Berlin, 1980.

[7] PASZKIEWICZ, A.: *On methods of investigation of ideals contained in the σ-ideal of nullsets* (in preparation).

[8] RECŁAW, I.—ZAKRZEWSKI, P.: *Fubini properties of ideals,* Real Anal. Exchange **25** (1999/2000), 565–578.

Received November 18, 2015

University of Łódź
Faculty of Mathematics and
Computer Science
ul. Banacha 22
PL–90-238 Łódź
POLAND

E-mail: adampasz@math.uni.lodz.pl
wagner@math.uni.lodz.pl