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ON LOCAL PROBLEM OF ENTROPY FOR FUNCTIONS FROM ZAHORSKI CLASSES

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ABSTRACT. We investigate some properties of functions belonging to the Zahorski classes. In particular, we answer the question: Is it possible to find close to a function from ith Zahorski class ($i = 1, 2, 3, 4, 5$) a function belonging to exactly the same Zahorski class and having a special (local) properties connected with the entropy?

1. Introduction

Studies on entropies of various functions led to distinguishing the interesting property of functions, which can be described as follows: *There exists a fixed point of a function such that topological entropy is focused on each neighbourhood of this point.* In the case of functions mapping the unit interval into itself, the investigations related to this issue are closely connected with the considerations characteristic for real analysis ([4]–[6]).

The Zahorski classes are classical basis for a lot of considerations connected with real analysis. Therefore, the combination of local aspects of entropy with the Zahorski classes seems to be fully justified.

Obviously, there exist (even continuous) functions, which have no strong entropy point. So, it is of interest to know whether arbitrarily close to a function belonging to established Zahorski class one can find another function belonging to the same Zahorski class and having this property. The aim of this paper is to answer the above question.

In the theory of discrete dynamical systems, the topics connected with topological entropy have been considered mainly for continuous funcions. The starting point for investigations related to these topics for Darboux-like functions (also with a *big set* of discontinuity points) were papers [3] and [9].

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Combining these directions of investigations and taking into account the well-known properties of functions belonging to the suitable Zahorski classes, we will focus our attention on functions having only one discontinuity point, which is simultaneously a strong entropy point. Continuing the investigations contained, among others, in papers [5]–[7] we will consider Γ-approximation of functions. We shall say that a function $f\colon [0,1] \to [0,1]$ is Γ-approximated by functions belonging to a class $\mathcal{K}$ of functions from $[0,1]$ into itself if, for each open set $U \subset [0,1] \times [0,1]$ containing the graph of f, there exists $g \in \mathcal{K}$ such that the graph of g is a subset of U.

Now, we introduce the notation used throughout the paper and briefly recall the concepts associated with Zahorski classes and strong entropy points.

Let $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the set of positive integers, rational numbers and real numbers, respectively. We will use the letter λ to denote the Lebesgue measure on $\mathbb{R}$. The symbol $\mathfrak{DB}_1$ stands for the family of all Darboux Baire one functions from $[0,1]$ to $[0,1]$. The distance between a point x and a set A will be denoted by $\operatorname{dist}(x,A)$. We write $h(f)$ for the topological entropy[1] of a function $f\colon [0,1] \to [0,1]$. The symbol $\operatorname{Fix}(f)$ stands for the set of all fixed points of f and $\Gamma(f)$—for a graph of f. If A is a nonempty subset of $[0,1]$ and $x \in [0,1]$, then

$$A+x = \{z+x : z \in A\}, \quad A-x = \{z-x : z \in A\} \quad \text{and} \quad -A = \{-z : z \in A\}.$$

Zygmunt Zahorski in [10] introduced a hierarchy of classes of functions connected with special classes of subsets of $\mathbb{R}$. In this paper, we will consider the classes of functions associated with subsets of $[0,1]$. The class $\mathfrak{M}_0$ consists of the empty set and all nonempty sets $E \subset [0,1]$ of type F_σ such that every point of E is a bilateral accumulation point of E. Obviously, if 0 (1) belongs to E, it is only a right (left) accumulation point of E. The family of all nonempty sets $E \subset [0,1]$ of type F_σ such that every point of E is a bilateral condensation point of E complemented with the empty set, constitutes the class $\mathfrak{M}_1$. Similarly as previously, the point 0 and 1 need to be a one-sided condensation point of E. A set $E \subset [0,1]$ belongs to the class $\mathfrak{M}_2$ if it is empty or if it is a nonempty set of type F_σ such that for each $x \in E \setminus \{0,1\}$ and any $\varepsilon > 0$ the sets $(x, x+\varepsilon) \cap E$ and $(x-\varepsilon, x) \cap E$ have a positive measure. Moreover, if $0 \in E$ ($1 \in E$), then $\lambda\big((0,\varepsilon) \cap E\big) > 0$ ($\lambda\big((1-\varepsilon, 1) \cap E\big) > 0$) for any $\varepsilon > 0$. The class $\mathfrak{M}_3$ consists of all nonempty sets $E \subset [0,1]$ of type F_σ such that there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of closed sets such that $E = \bigcup_{n\in\mathbb{N}} K_n$ and a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of numbers such that $0 \le \eta_n < 1$ ($n \in \mathbb{N}$), and for each $n \in \mathbb{N}$, each $x \in K_n$ and

[1]The notion of entropy is used in the definition of strong entropy point, but to obtain all the results presented in this paper, it suffices to use Lemma 2.8, which is an obvious consequence of the results contained in [7]. For these reasons, we do not recall a formal definition (it can be found, for example, in [3]).

each $c > 0$ there exists a number $\varepsilon(x,c) > 0$ such that if h and h_1 satisfy conditions $h \cdot h_1 > 0$, $\frac{h}{h_1} < c$, $|h+h_1| < \varepsilon(x,c)$, then

$$\frac{\lambda(E \cap (x+h, x+h+h_1))}{|h_1|} > \eta_n. \tag{1}$$

Obviously, if $x = 0$ ($x = 1$), then we consider h, h_1 greater (smaller) than 0. In addition, we assume that the empty set belongs to the class $\mathfrak{M}_3$. A slight change in the definition of the class $\mathfrak{M}_3$ leads us to the class $\mathfrak{M}_4$. More specifically, in this case, we replace the above condition $0 \leq \eta_n < 1$ ($n \in \mathbb{N}$) with the condition $0 < \eta_n < 1$ ($n \in \mathbb{N}$). We say that $E \subset [0,1]$ belongs to the class $\mathfrak{M}_5$ if it is empty or if it is a nonempty set of type F_σ such that for each $x \in E \setminus \{0,1\}$ we have

$$\lim_{h \to 0^+} \frac{\lambda(E \cap [x-h, x+h])}{2h} = 1, \tag{2}$$

that is every point of E is a density point of E. Moreover, if 0 (1) belongs to E, then $\lim_{h\to 0^+} \frac{\lambda(E\cap[0,h])}{h} = 1$ ($\lim_{h\to 0^+} \frac{\lambda(E\cap[1-h,1])}{h} = 1$).

Using the above hierarchy of sets, we can define some classes of functions. Let $i \in \{0,1,\ldots,5\}$. We say that a function $f : [0,1] \to [0,1]$ belongs to the class $\mathcal{M}_i$, if the sets $E^{f,\alpha} = \{x \in [0,1] \colon f(x) > \alpha\}$ and $E_{f,\alpha} = \{x \in [0,1] \colon f(x) < \alpha\}$ belong to the class $\mathfrak{M}_i$ for any $\alpha \in \mathbb{R}$. Moreover, to shorten some notations, we will use the symbol $\mathcal{M}_6$ to denote the family of all continuous functions $f \colon [0,1] \to [0,1]$.

Now, we will briefly recall the notion of a strong entropy point introduced in [7]. As in the case of Zahorski classes, we will limit our considerations to the functions having $[0,1]$ as the domain and as the range.

Let $f : [0,1] \to [0,1]$. A pair $(\mathcal{F}, J) = B_f$, where $\mathcal{F}$ is a family of pairwise disjoint (nonsingletons) continuums in $[0,1]$ and $J \subset [0,1]$ is a connected set such that $J \subset f(A)$ for any $A \in \mathcal{F}$, is called an f-bundle. Moreover, if we additionally assume that $A \subset J$ for all $A \in \mathcal{F}$ then such an f-bundle is called an f-bundle with dominating fibre.

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. A set $M \subset \bigcup \mathcal{F}$ is (B_f, n, ε)-separated if for each $x, y \in M$, $x \neq y$ there is $0 \leqslant i < n$ such that $f^i(x), f^i(y) \in J$ and $\rho(f^i(x), f^i(y)) > \varepsilon$.

The entropy of an f-bundle B_f is defined in the following way:

$$h(B_f) = \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \left[\frac{1}{n} \log\left(s_n^{B_f}(\varepsilon)\right)\right],$$

where

$$s_n^{B_f}(\varepsilon) = \max\{\mathrm{card}(M) : M \subset [0,1] \text{ is } (B_f, n, \varepsilon)\text{-separated set}\}.$$

A sequence of f-bundles $\{B_f^k\}_{k\in\mathbb{N}}$, where $B_f^k = (\mathcal{F}_k, J_k)$ for $k \in \mathbb{N}$, converges to a point x_0 ($B_f^k \underset{k\to\infty}{\longrightarrow} x_0$), if for any $\varepsilon > 0$ there exists $k_0 \in N$ such that

$\bigcup \mathcal{F}_k \subset B(x_0, \varepsilon)$ and $B\big(f(x_0), \varepsilon\big) \cap J_k \neq \emptyset$ for any $k \geqslant k_0$. Let

$$E_f(x_0) = \left\{ \limsup_{n\to\infty} h(B_f^n) : B_f^n \underset{n\to\infty}{\longrightarrow} x \right\}.$$

We shall say that a point $x_0 \in X$ is a strong entropy point of the function $f\colon [0,1] \to [0,1]$ if $h(f) \in E_f(x_0)$ and $x_0 \in \mathrm{Fix}(f)$. The symbol $\mathfrak{E}_s^D$ stands for the family of all functions $f\colon [0,1] \to [0,1]$ such that f has the only one point of discontinuity and this point is simultaneously a strong entropy point of f.

2. Auxiliary statements

In order to facilitate the reading of this paper in this section, we will present a few statements (some new with not very difficult proofs and some well-known) which are useful in the main parts of this paper.

One can observe that in order to prove that a nonempty set E of type F_σ belongs to the class $\mathfrak{M}_3$ or to the class $\mathfrak{M}_4$, we should find a sequence of sets and a sequence of numbers satisfying condition (1). It is worth adding that to check whether a nonempty set E of type F_σ belongs to the class $\mathfrak{M}_3$, it is sufficient to show the following condition: for each $x \in E$ and each sequence $\{I_n\}_{n\in\mathbb{N}}$ of closed intervals contained in $[0,1]$ converging to x (i.e., $\lim_{n\to\infty} \mathrm{dist}(x, I_n) = 0$) and not containing x such that $\lambda(I_n \cap E) = 0$ for each $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \frac{\lambda(I_n)}{\mathrm{dist}(x,I_n)} = 0$.

Moreover, we have

Lemma 2.1 ([2]). *If $E \in \mathfrak{M}_4$ and $x \in E \setminus \{0,1\}$, then*

$$\underline{d}(E,x) = \liminf_{h\to 0^+} \frac{\lambda(E \cap [x-h, x+h])}{2h} > 0.$$

Lemma 2.2. *There exist decreasing sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ of positive numbers such that $b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$, $\lim_{n\to\infty} b_n = 0$ and the set $\bigcup_{n\in\mathbb{N}}(-a_n, -b_n) \cup \{0\} \cup \bigcup_{n\in\mathbb{N}}(a_n, b_n)$ belongs to the class $\mathfrak{M}_2$ and does not belong to the class $\mathfrak{M}_3$.*

The above lemma is possibly known. However, we are not able to give a reference. For the proof, it is sufficient to put $a_n = \frac{2^{n+2}-1}{2^{2n+2}}$ and $b_n = \frac{1}{2^n}$ for $n \in \mathbb{N}$.

The following technical lemma will also be useful in the proof of the main theorem of this paper.

Lemma 2.3. *Let $L_n^k = \left[\frac{1}{2^k} + \frac{n}{2^{2k}}, \frac{1}{2^k} + \frac{n+1}{2^{2k}}\right]$ for $k \in \mathbb{N}\setminus\{1\}$ and $n \in \{0, 1, \dots, 2^k - 1\}$. Then, for any $c > 0$ and any $k_0 \in \mathbb{N}$, there exists $k_1 \in \mathbb{N}$ such that $k_1 \geq k_0$ and for any $h, h_1 \in (0, \infty)$ such that $\frac{h}{h_1} < c$ and $h + h_1 < \frac{1}{2^{k_1}}$ there are two points*

α and β ($\alpha < \beta$) in the interval $(h, h+h_1)$ being endpoints of some intervals L_n^k for some $k > k_1$ and some $n \in \{0, 1, \ldots, 2^k - 1\}$ such that $\frac{\beta-\alpha}{h_1} > \frac{1}{2}$.

P r o o f. Let $c > 0$ and $k_0 \in \mathbb{N}$. Obviously, one can find $k_1 \in \mathbb{N}$ such that $k_1 \geq k_0$ and $\frac{1}{2^{k_1}} < \frac{1}{4(c+1)}$. Note further that if $k > k_1$, $n \in \{0, 1, \ldots, 2^k - 1\}$ and J is an interval such that $J \subset L_n^k$ then

$$\frac{\lambda(J)}{\operatorname{dist}(0, J)} \leq \frac{\lambda(L_n^k)}{\operatorname{dist}(0, L_n^k)} \leq \frac{1}{2^k} < \frac{1}{4(c+1)}. \tag{3}$$

Let $h, h_1 \in (0, \infty)$ be such that $\frac{h}{h_1} < c$ and $h+h_1 < \frac{1}{2^{k_1}}$. Suppose, contrary to our claim, that no endpoint of no interval L_n^k (for $k > k_1$ and $n \in \{0, 1, \ldots, 2^k - 1\}$) belongs to $(h, h+h_1)$. Thus, there is $k_* > k_1$ and $n_* \in \{0, 1, \ldots, 2^{k_*}-1\}$ such that $[h, h+h_1] \subset L_{n_*}^{k_*}$. Clearly, $h+h_1 < \frac{1}{2^{k_1}}$ and $\frac{\operatorname{dist}(0,[h,h+h_1])}{\lambda([h,h+h_1])} = \frac{h}{h_1} < c$, which contradicts (3). Therefore, there is at least one point belonging to $(h, h+h_1)$ and being an endpoint of some interval L_n^k for some $k > k_1$ and some $n \in \{0, 1, \ldots, 2^k - 1\}$.

Now, we will prove that there are at least two different points having required properties. Conversely, suppose that there is only one such point. Let us denote it by w. Obviously, there are $k', k'' \in \{k_1+1, k_1+2, \ldots\}$, $n' \in \{0, 1, \ldots, 2^{k'} - 1\}$ and $n'' \in \{0, 1, \ldots, 2^{k''} - 1\}$ such that $[h, w] \subset L_{n'}^{k'}$ and $[w, h+h_1] \subset L_{n''}^{k''}$. Condition (3) implies that

$$\lambda([h, w]) < \frac{h}{4(c+1)} \quad \text{and} \quad \lambda([w, h+h_1]) < \frac{w}{4(c+1)},$$

so, in consequence, we obtain that

$$h_1 = \lambda([h, h+h_1]) = \lambda([h, w]) + \lambda([w, h+h_1]) < \frac{2h+h_1}{4(c+1)},$$

so, $2h > 4h_1c + 3h_1$. Hence, $\frac{h}{h_1} > 2c$, which is impossible.

Let $\mathcal{E}$ be a set of all endpoints of intervals L_n^k (for $k > k_1$ and $n \in \{0, 1, \ldots \ldots, 2^k - 1\}$) belonging to the interval $(h, h+h_1)$. Put $\alpha = \min \mathcal{E}$ and $\beta = \max \mathcal{E}$. Clearly, $h < \alpha < \beta < h+h_1$. Moreover, one can find $k', k'' \in \{k_1+1, k_1+2, \ldots\}$, $n' \in \{0, 1, \ldots, 2^{k'} - 1\}$ and $n'' \in \{0, 1, \ldots, 2^{k''} - 1\}$ such that $[h, \alpha] \subset L_{n'}^{k'}$ and $[\beta, h+h_1] \subset L_{n''}^{k''}$. Condition (3) gives that $\lambda\big([h, \alpha] \cup [\beta, h+h_1]\big) < \frac{h+h_1}{2(c+1)} < \frac{h_1}{2}$. Thus,

$$\frac{\beta - \alpha}{h_1} = \frac{\lambda([h, h+h_1] \setminus ([h, \alpha] \cup [\beta, h+h_1]))}{h_1} > \frac{1}{2}.$$

$\square$

Now, we will focus our considerations on the functions from Zahorski classes. Let us recall the known relations between these classes:

$$\mathfrak{DB}_1 = \mathcal{M}_0 = \mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \mathcal{M}_4 \supset \mathcal{M}_5 \supset \mathcal{M}_6.$$

It is well-known that all the above inclusions are proper ([2], [10]). In [10], one can also find the property saying that the class $\mathcal{M}_5$ coincides with the family of all approximately continuous functions. Moreover, it is known that the family of all functions $f\colon [0,1] \to [0,1]$ which are almost continuous (i.e., Γ-approximated by continuous functions) and Baire class one is equal to the family $\mathfrak{DB}_1$ (see [1]). This gives

Proposition 2.4. *Let $i \in \{0,1,\dots,6\}$. If $f \in \mathcal{M}_i$, then f is almost continuous.*

The above proposition and result contained in [8] permit us to write:

Proposition 2.5. *If $f \in \mathcal{M}_i$ $(i \in \{0,1,\dots,6\})$, then* $\mathrm{Fix}(f) \neq \emptyset$.

From Proposition 2.4 we immediately obtain that each function from $\mathcal{M}_i$ $(i \in \{0,1,\dots,6\})$ is Γ-approximated by continuous functions $\xi\colon [0,1] \to [0,1]$. Furthermore, one can prove the following theorem, useful in various considerations.

Theorem 2.6. *If $f \in \mathcal{M}_i$ $\big(i \in \{0,1,\dots,6\}\big)$, then f is Γ-approximated by continuous functions $\xi\colon [0,1] \to [0,1]$ such that* $\mathrm{Fix}(\xi) \cap (0,1) \neq \emptyset$.

P r o o f. Let $i \in \{0,1,\dots,6\}$, $f \in \mathcal{M}_i$ and U be an open set containing the graph of f. Proposition 2.4 implies that there exists a continuous function $\zeta : [0,1] \to [0,1]$ whose graph is contained in U. If $\mathrm{Fix}(\zeta) \cap (0,1) \neq \emptyset$, then we put $\xi = \zeta$. Otherwise, there is no loss of generality in assuming that $1 \in \mathrm{Fix}(\zeta)$. Obviously, one can find $\delta > 0$ such that $[1-\delta,1] \times [1-\delta,1] \subset U$. Moreover, there is $w \in (1-\delta,1)$ such that $\zeta(w) \in [1-\delta,1]$. Fix $v \in (w,1)$. The function ξ defined in the following way:

$$\xi(x) = \begin{cases} v & \text{if } x = v, \\ \zeta(x) & \text{if } x \in [0,w], \\ \text{linear} & \text{in } [w,v],[v,1] \end{cases}$$

is the required function. □

Moreover, we have

Theorem 2.7 ([2])**.** *Let E be any subset of $[0,1]$ such that*

(1) *E is an F_σ set,*

(2) *E is a bilaterally dense set.*

Then, there exists a function f belonging to the class $\mathfrak{DB}_1$ such that

$$f(x) = 0 \quad \text{for } x \in [0,1] \setminus E \quad \text{and} \quad f(x) \in (0,1] \quad \text{for } x \in E.$$

We will finish this section with the lemma related to strong entropy points. Taking into account Lemma 3.1, 3.6 and 3.9 [7], we can immediately show

Lemma 2.8. *Let $x_0 \in [0,1]$, $f\colon [0,1] \to [0,1]$ be an arbitrary function and for any $k \in \mathbb{N}$ let $B_f^k = (\mathcal{F}_k, J_k)$ be an f-bundle with dominating fibre such that $\mathcal{F}_k$ is infinite. If $x_0 \in \mathrm{Fix}(f)$ and $B_f^k \underset{k\to\infty}{\longrightarrow} x_0$, then x_0 is a strong entropy point of f.*

3. Main result

Suppose that we consider a suitable property of functions. The natural question arises whether there is a possibility of approximation a function without this property by functions having the considered property.

In our case, one can ask if it is possible to Γ-approximate functions from ith Zahorski class ($i = 1,2,3,4$) by using functions from the same Zahorski class having a strong entropy point and not belonging to $i+1$ Zahorski class. Notice that the last requirement excludes continuous functions from the set of Γ-approximating functions (which leads to considerations essentially different from those regarding entropy of continuous functions). On the other hand, in order to be *close to the continuity*, we can extend the question requiring Γ-approximating functions to have a strong entropy point, which is simultaneously the only one discontinuity point of this function.

First, note that in the case of $i = 1$, the last demand is impossible. Indeed, let $f \in \mathcal{M}_1$, $U \subset [0,1]\times[0,1]$ be a nonempty open set containing the graph of f and g be a function from $\mathcal{M}_1 \setminus \mathcal{M}_2$ whose graph belongs to U. Without loss of generality, we can assume that there exist $a \in \mathbb{R}$ and a set $E^{g,a} = \{x : g(x) > a\}$ such that $E^{g,a} \in \mathfrak{M}_1 \setminus \mathfrak{M}_2$. There is no loss of generality in assuming that there are a point $x \in E^{g,a}$ and $h > 0$ such that $\lambda\big((x, x+h) \cap E^{g,a}\big) = 0$. Obviously, $(x, x+h) \cap E^{g,a} \neq \emptyset$ and each point from $(x, x+h) \cap E^{g,a}$ is a discontinuity point of g. The above considerations and the next theorem show that classes of functions being close to each other may have completely different properties.

Theorem 3.1. *Let $i \in \{1,2,\dots,5\}$. Each function from the class $\mathcal{M}_i$ can be Γ-approximated by functions belonging to the class $\mathcal{M}_i \setminus \mathcal{M}_{i+1}$ and having a strong entropy point. Moreover, if $i \neq 1$ then each function from the class $\mathcal{M}_i$ can be Γ-approximated by functions belonging to the class $(\mathcal{M}_i \setminus \mathcal{M}_{i+1}) \cap \mathfrak{E}_s^D$.*

P r o o f. Let $i \in \{1,2,\dots,5\}$ and $f \in \mathcal{M}_i$. To prove the first part of the theorem, it is sufficient to show that for any open set $U \subset [0,1]\times[0,1]$ such that $\Gamma(f) \subset U$ there exists a function $g \in \mathcal{M}_i \setminus \mathcal{M}_{i+1}$ having a strong entropy point and such that $\Gamma(g) \subset U$. To prove the second part of the theorem, we additionally need to show that if $i \neq 1$ then the established strong entropy point is the only one discontinuity point of g. Although the proofs connected with individual Zahorski classes run along similar lines, there are subtle but essential adjustments necessary to fit the argument to each class separately.

Let us first note that if $f \in \mathcal{M}_5$, then it is easy to see that the function constructed in the proof of Theorem 3.10 part (b) [5] has the required properties.

The proof in other cases will be divided into two parts. In the first one, we will construct functions $g_i\colon [0,1] \to [0,1]$ for $i \in \{1,2,3,4\}$. In the second one, we will prove that for $i \in \{1,2,3,4\}$ we have $g_i \in \mathcal{M}_i \setminus \mathcal{M}_{i+1}$ and g_i has a strong entropy point. Moreover, we will show that $g_i \in \mathfrak{E}_s^D$ for $i \in \{2,3,4\}$.

So, let $f \in \mathcal{M}_i$ $(i \in \{1,2,3,4\})$ and $U \subset [0,1] \times [0,1]$ be an open set such that $\Gamma(f) \subset U$. Theorem 2.6 now gives that there exists a continuous function $f^* : [0,1] \to [0,1]$ such that $\Gamma(f^*) \subset U$ and $\mathrm{Fix}(f^*) \cap (0,1) \neq \emptyset$. Fix $x_0 \in \mathrm{Fix}(f^*) \cap (0,1)$. Clearly, one can find $\delta_0 > 0$ such that

$$[x_0 - \delta_0, x_0 + \delta_0] \times [x_0 - \delta_0, x_0 + \delta_0] \subset U.$$

Moreover, there is $\delta_1 \in (0,\delta_0) \cap \mathbb{Q}$ such that

$$f^*([x_0 - \delta_1, x_0 + \delta_1]) \subset (x_0 - \delta_0, x_0 + \delta_0).$$

What is more, one can find $k_0 \in \mathbb{N}$ such that $\frac{1}{2^{k_0-1}} < \frac{\delta_1}{4}$.

Now, assume that $i=1$ and put

$$E = \left[0, x_0 + \frac{\delta_1}{4}\right) \cup \left([x_0 + \frac{\delta_1}{4}, x_0 + \frac{\delta_1}{2}] \cap \mathbb{Q}\right) \cup \left(x_0 + \frac{\delta_1}{2}, 1\right].$$

Theorem 2.7 gives that there exists a function h belonging to $\mathfrak{DB}_1$ such that $h(x) = 0$ for $x \in [0,1] \setminus E$ and $h(x) \in (0,1]$ for $x \in E$. Putting $w(x) = \delta_0 \cdot h(x) + x_0$ for $x \in [0,1]$, we obtain that $w \in \mathfrak{DB}_1$, $w(x) = x_0$ for $x \in [0,1] \setminus E$ and $w(x) \in (x_0, x_0 + \delta_0]$ for $x \in E$.

We define the function g_1 in the following way:

$$\begin{aligned}
g_1(x) &= f^*(x) && \text{for} \quad x \in [0, x_0] \cup [x_0 + \delta_1, 1],\\
g_1(x) &= w(x) && \text{for} \quad x \in \left[x_0 + \frac{\delta_1}{4}, x_0 + \frac{\delta_1}{2}\right],\\
g_1(x) &= x_0 && \text{for} \quad x \in \left[x_0 + \frac{1}{2^{k_0}}, \frac{\delta_1}{4}\right] \cup \left\{x_0 + \frac{1}{2^k}\right\},\\
g_1\left(x_0 + \frac{3}{2^{k+2}}\right) &= x_0 + \delta_0 && \text{for} \quad k \in \{k_0, k_0 + 1, \dots\}
\end{aligned}$$

and g_1 is linear otherwise.

Now, let $i = 2$ and $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be the same as in Lemma 2.2. Without loss of generality, we can assume that $b_1 < \delta_1$.

Put $g_2(x) = f^*(x)$ for $x \in [0, x_0 - \delta_1] \cup [x_0 + \delta_1, 1] \cup \{x_0\}$, $g_2(x) = x_0 - \delta_0$ for $x \in \bigcup_{n\in\mathbb{N}} \big([x_0 + b_{n+1}, x_0 + a_n] \cup [x_0 - a_n, x_0 - b_{n+1}]\big)$, $g_2(x_0 - b_1) = g(x_0 + b_1) = x_0 - \delta_0$,

$g_2(x_0+\frac{a_n+b_n}{2})=g_2(x_0-\frac{a_n+b_n}{2})=x_0$ for $n\in\mathbb{N}$ and g_2 is linear otherwise.

Now, we turn to a construction of a function g_3. For $k\in\{k_0,k_0+1,\dots\}$ and $n\in\{0,1,\dots,2^k-1\}$, let the set Z_n^k be a closed interval concentric with the interval $\left[\frac{1}{2^k}+\frac{n}{2^{2k}},\frac{1}{2^k}+\frac{n+1}{2^{2k}}\right]$ such that $\lambda(Z_n^k)=\frac{1}{2^{3k}}$.

Set $g_3(x)=f^*(x)$ for $x\in[0,x_0-\delta_1]\cup[x_0+\delta_1,1]\cup\{x_0\}$ and $g_3(x)=x_0-\delta_0$ for $x\in\left[x_0-\frac{1}{2^{k_0-1}},x_0+\frac{1}{2^{k_0-1}}\right]\setminus\bigcup_{k=k_0}^{\infty}\bigcup_{n=0}^{2^k-1}\big(\mathrm{int}(Z_n^k+x_0)\cup\mathrm{int}(-Z_n^k+x_0)\big)$. Furthermore, let the value of the function g_3 in the center of each interval Z_n^k ($k\in\{k_0,k_0+1,\dots\}$, $n\in\{0,1,\dots,2^k-1\}$) be equal to x_0, and g_3 be linear otherwise.

Now, we construct a function g_4. Let l_n^k denote the center of the interval $L_n^k=\left[x_0+\frac{1}{2^k}+\frac{n}{2^{2k}},x_0+\frac{1}{2^k}+\frac{n+1}{2^{2k}}\right]$ for $k\in\{k_0+1,\dots\}$ and $n\in\{0,1,\dots,2^k-1\}$.

Put $g_4(x)=f^*(x)$ for $x\in[0,x_0-\delta_1]\cup[x_0+\delta_1,1]\cup\{x_0\}$, $g_4\big(x_0+\frac{1}{2^{k_0}}\big)=g_4\big(x_0+\frac{1}{2^k}+\frac{n}{2^{2k}}\big)=g_4\big(x_0+\frac{1}{2^k}+\frac{n+1}{2^{2k}}\big)=x_0$ and $g_4\big(l_n^k\big)=x_0+\delta_0$ for $k>k_0$ and $n\in\{0,1,\dots,2^k-1\}$. Moreover, let g_4 be linear otherwise.

Now, we will prove that the constructed functions have the required properties. We check at once that $\Gamma(g_i)\subset U$ for each $i\in\{1,2,3,4\}$. Moreover, it is easy to see that for any $i\in\{1,2,3,4\}$ the point x_0 is a discontinuity point of g_i. What is more, if $i\in\{2,3,4\}$ then g_i has the only one point of discontinuity.

It is clear that $g_1\in\mathfrak{DB}_1$. On the other hand, for $t_0\in\big(x_0+\frac{\delta_1}{4},x_0+\frac{\delta_1}{2}\big)$ such that $g_1(t_0)>x_0$, we have $\lambda\big(E^{g_1,x_0}\cap(t_0,t_0+\frac{\delta_1}{8})\big)=0$, so the set E^{g_1,x_0} does not belong to class $\mathfrak{M}_2$ and, in consequence, we have $g_1\notin\mathcal{M}_2$. Furthermore, it is easy to prove that $g_2\in\mathcal{M}_2$. However, since the set $E^{g_2,x_0-\delta_0}\notin\mathfrak{M}_3$, so $g\notin\mathcal{M}_3$.

We check at once that for any $\alpha\in\mathbb{R}$ the set $E_{g_3,\alpha}$ belongs to $\mathfrak{M}_3$. Similarly, it is easy to see that $E^{g_3,\alpha}\in\mathfrak{M}_3$ for $\alpha\in\mathbb{R}\setminus[x_0-\delta_0,x_0)$. To prove that $g_3\in\mathcal{M}_3$ it suffices to show that $E^{g_3,\alpha}\in\mathfrak{M}_3$ for $\alpha\in[x_0-\delta_0,x_0)$. So, let $\alpha\in[x_0-\delta_0,x_0)$. Clearly, $E^{g_3,\alpha}=\big\{x\in[0,x_0)\cup(x_0,1]:g_3(x)>\alpha\big\}\cup\{x_0\}$. Since the set $\big\{x\in[0,x_0)\cup(x_0,1]:g_3(x)>\alpha\big\}$ is open in the natural topology, it is sufficient to prove that for any $c>0$ there exists a number $\varepsilon(x_0,c)>0$ such that if h and h_1 satisfy conditions $h\cdot h_1>0$, $\frac{h}{h_1}<c$, $|h+h_1|<\varepsilon(x_0,c)$, then

$$\frac{\lambda(E^{g_3,\alpha}\cap(x_0+h,x_0+h+h_1))}{|h_1|}>0. \tag{4}$$

Without loss of generality, we can assume that $h,h_1>0$ (the proof for $h,h_1<0$ runs analogously). Let $c>0$. Lemma 2.3 gives that there is $k_1\in\mathbb{N}$ such that $k_1\ge k_0$ and for any $h,h_1\in(0,\infty)$ such that $\frac{h}{h_1}<c$ and $h+h_1<\frac{1}{2^{k_1}}$ there are two points $\overline{h}$ and $\overline{h^*}$ in the interval $(h,h+h_1)$ being endpoints of some intervals L_n^k for some $k>k_1$ and some $n\in\{0,1,\dots,2^k-1\}$. There is no loss of generality in assuming that $\overline{h}<\overline{h^*}$. Put $\overline{h_1}=\overline{h^*}-\overline{h}$. Clearly, there exist $k'\in\{k_1+1,k_1+2,\dots\}$ and $n'\in\{0,1,\dots,2^{k'}-1\}$ such that $L_{n'}^{k'}\subset[\overline{h},\overline{h}+\overline{h_1}]$.

Putting $\varepsilon(x_0, c) = \frac{1}{2^{k_1}}$, we obtain that

$$\begin{aligned}&\lambda\big(E^{g_3,\alpha} \cap (x_0 + h, x_0 + h + h_1)\big)\\ &\geq \lambda\big((E^{g_3,\alpha} - x_0) \cap L^{k'}_{n'}\big)\\ &\geq \lambda\big((E^{g_3,\alpha} - x_0) \cap Z^{k'}_{n'}\big)\\ &= \lambda\big(E^{g_3,\alpha} \cap (Z^{k'}_{n'} + x_0)\big)\\ &= \frac{(x_0 - \alpha) \cdot \lambda(Z^{k'}_{n'})}{\delta_0} > 0\end{aligned}$$

for any $h, h_1 \in (0, \infty)$ such that $\frac{h}{h_1} < c$ and $h + h_1 < \varepsilon(x_0, c)$. It means that condition (4) is fullfiled and, in consequence, we obtain that $g_3 \in \mathcal{M}_3$.

Now, let us consider the set $E^{g_3,x_0-\delta_0}$. Obviously, $x_0 \in E^{g_3,x_0-\delta_0}$. Moreover, for $k > k_0$ we have

$$\begin{aligned}&\lambda\left(E^{g_3,\, x_0-\delta_0} \cap \left[x_0 - \frac{1}{2^k},\ x_0 + \frac{1}{2^k}\right]\right)\\ &\qquad = 2 \cdot \lambda\left(\bigcup_{j=k+1}^{\infty} \bigcup_{n=0}^{2^j-1} Z^j_n\right) = 2 \cdot \sum_{j=k+1}^{\infty} \sum_{n=0}^{2^j-1} \frac{1}{2^{3j}} = \frac{2}{3} \cdot \frac{1}{2^{2k}}.\end{aligned}$$

Hence,

$$\lim_{k\to\infty} \frac{\lambda(E^{g_3,x_0-\delta_0} \cap [x_0 - \frac{1}{2^k},\ x_0 + \frac{1}{2^k}])}{\frac{1}{2^k}} = 0.$$

Thus, $\underline{d}(E^{g_3,x_0-\delta_0}, x_0) = 0$. Lemma 2.1 now shows that $E^{g_3,x_0-\delta_0} \notin \mathfrak{M}_4$, and finally, we obtain that $g_3 \notin \mathcal{M}_4$.

Note further that the set $E_{g_4,x_0+\frac{\delta_0}{2}}$ does not belong to $\mathfrak{M}_5$. Indeed, clearly $x_0 \in E_{g_4,x_0+\frac{\delta_0}{2}}$. Moreover, for $k > k_0$ we have

$$\begin{aligned}&\lambda\left(E_{g_4,x_0+\frac{\delta_0}{2}} \cap \left[x_0, x_0 + \frac{1}{2^k}\right]\right)\\ &= \sum_{j=k+1}^{\infty} \lambda\left(E_{g_4,x_0+\frac{\delta_0}{2}} \cap \left[x_0 + \frac{1}{2^j},\ x_0 + \frac{1}{2^{j-1}}\right]\right)\\ &= \sum_{j=k+1}^{\infty} \sum_{n=0}^{2^j-1} \lambda\left(E_{g_4,x_0+\frac{\delta_0}{2}} \cap L^j_n\right)\\ &= \frac{1}{2^{k+1}}.\end{aligned}$$

Thus,

$$\lim_{k\to\infty}\frac{\lambda(E_{g_4,x_0+\frac{\delta_0}{2}}\cap[x_0,x_0+\frac{1}{2^k}])}{\frac{1}{2^k}}=\frac{1}{2},$$

and we obtain immediately that $E_{g_4,x_0+\frac{\delta_0}{2}}\notin\mathfrak{M}_5$, so $g_4\notin\mathcal{M}_5$.

Moreover, it is easy to see that $E^{g_4,\alpha}\in\mathfrak{M}_4$ for any $\alpha\in\mathbb{R}$ and $E_{g_4,\alpha}\in\mathfrak{M}_4$ for any $\alpha\in\mathbb{R}\setminus(x_0,x_0+\delta_0]$. If $\alpha\in(x_0,x_0+\delta_0]$ then $x_0\in E_{g_4,\alpha}$ and we check at once that the set $E_{g_4,\alpha}\setminus\{x_0\}$ is open in the natural topology. Furthermore, one can find $\beta>0$ such that $(x_0-\beta,x_0)\subset E_{g_4,\alpha}$. It means that

$$\lim_{h\to 0}\frac{\lambda(E_{g_4,\alpha}\cap[x_0-h,x_0])}{h}=1.$$

Thus, to prove that $E_{g_4,\alpha}\in\mathfrak{M}_4$, it suffices to show that there exists $\eta\in(0,1)$ such that for any $c>0$ one can find $\varepsilon(x_0,c)>0$ such that for any $h,h_1\in(0,\infty)$ such that $\frac{h}{h_1}<c$ and $h+h_1<\varepsilon(x_0,c)$ we have

$$\frac{\lambda((E_{g_4,\alpha}-x_0)\cap[h,h+h_1])}{h_1}>\eta.$$

Put $\eta=\frac{\alpha-x_0}{2\cdot\delta_0}$. Clearly, $\eta\in(0,1)$. Let $c>0$. Lemma 2.3 implies that there exists $k_1\in\mathbb{N}$ such that $k_1\geq k_0$ and for any $h,h_1\in(0,\infty)$ such that $\frac{h}{h_1}<c$ and $h+h_1<\frac{1}{2^{k_1}}$ there are two points $\overline{h}$ and $\overline{h^*}$ in the interval $(h,h+h_1)$ being endpoints of some intervals L_n^k for some $k>k_1$ and some $n\in\{0,1,\ldots,2^k-1\}$, such that $\frac{|\overline{h}-\overline{h^*}|}{h_1}>\frac{1}{2}$. Obviously, we can assume that $\overline{h}<\overline{h^*}$ and put $\overline{h_1}=\overline{h^*}-\overline{h}$. Clearly, we have

$$\lambda\big((E_{g_4,\alpha}-x_0)\cap L_n^k\big)=\frac{\alpha-x_0}{\delta_0}\cdot\lambda(L_n^k)$$

for any $L_n^k\subset[\overline{h},\overline{h}+\overline{h_1}]$. Thus,

$$\lambda\big((E_{g_4,\alpha}-x_0)\cap[\overline{h},\overline{h}+\overline{h_1}]\big)=\lambda\left((E_{g_4,\alpha}-x_0)\cap\bigcup_{L_n^k\subset[\overline{h},\overline{h}+\overline{h_1}]}L_n^k\right)>\frac{\alpha-x_0}{\delta_0}\cdot\frac{1}{2}\cdot h_1.$$

Putting $\varepsilon(x_0,c)=\frac{1}{2^{k_1}}$, we obtain that

$$\frac{\lambda((E_{g_4,\alpha}-x_0)\cap[h,h+h_1])}{h_1}\geq\frac{\lambda((E_{g_4,\alpha}-x_0)\cap[\overline{h},\overline{h}+\overline{h_1}])}{h_1}>\eta$$

for any $h,h_1\in(0,\infty)$ such that $\frac{h}{h_1}<c$ and $h+h_1<\varepsilon(x_0,c)$. Finally, we get that $E_{g_4,\alpha}\in\mathfrak{M}_4$. This finishes the proof that $g_4\in\mathcal{M}_4$.

To end the proof, it is sufficient to show that for $i \in \{1,2,3,4\}$ the point x_0 is a strong entropy point of g_i. If $k \in \mathbb{N}$ then set $S_k = \{k_0 + k, k_0 + k + 1, \dots\}$, $J_k^1 = J_k^4 = [x_0, x_0 + \delta_0]$, $J_k^2 = J_k^3 = [x_0 - \delta_0, x_0]$,

$$\mathcal{F}_k^1 = \left\{\left[x_0 + \frac{3}{2^{m+2}}, x_0 + \frac{1}{2^m}\right] : m \in S_k\right\},$$

$$\mathcal{F}_k^2 = \big\{[x_0 - b_m, x_0 - a_m] : m \in S_k\big\},$$

$$\mathcal{F}_k^3 = \{-Z_0^m + x_0 : m \in S_k\} \quad \text{and}$$

$$\mathcal{F}_k^4 = \left\{\left[x_0 + \frac{1}{2^m}, x_0 + \frac{1}{2^m} + \frac{1}{2^{2m}}\right] : m \in S_k\right\}.$$

Putting $B_{g_i}^k = \big(\mathcal{F}_k^i, J_k^i\big)$ for $i \in \{1,2,3,4\}$ and $k \in \mathbb{N}$, we obtain sequences $\big\{B_{g_i}^k\big\}_{k\in\mathbb{N}}$ (for $i \in \{1,2,3,4\}$) of g_i-bundles with dominating fibre such that $B_{g_i}^k \underset{k\to\infty}{\longrightarrow} x_0$. Clearly, $\mathcal{F}_k^i$ is infinite for any $k \in \mathbb{N}$ and $i \in \{1,2,3,4\}$. Lemma 2.8 gives that x_0 is a strong entropy point of g_i for $i \in \{1,2,3,4\}$. $\square$

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