# Modeling a Distribution of Mortgage Credit Losses[1]

*Petr GAPKO* – Martin ŠMÍD***

**Abstract**

*In our paper, we focus on the credit risk quantification methodology. We demonstrate that the current regulatory standards for credit risk management are at least not perfect. Generalizing the well-known KMV model, standing behind Basel II, we build a model of a loan portfolio involving a dynamics of the common factor, influencing the borrowers' assets, which we allow to be non-normal. We show how the parameters of our model may be estimated by means of past mortgage delinquency rates. We give statistical evidence that the non-normal model is much more suitable than the one which assumes the normal distribution of risk factors. We point out in what way the assumption that risk factors follow a normal distribution can be dangerous. Especially during volatile periods comparable to the current crisis, the normal-distribution-based methodology can underestimate the impact of changes in tail losses caused by underlying risk factors.*

**Keywords:** *credit risk, mortgage, delinquency rate, generalized hyperbolic distribution, normal distribution*

**JEL Classification:** G21

## Introduction

Minimum standards for credit risk quantification are often prescribed in developed countries with regulated banking. A system of financial regulation has been developed and is maintained by European supervisory institutions (Basel

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Committee on Banking Supervision, Committee of European Banking Supervisors – CEBS) with its standards formalized in the Second Basel Accord (Basel II, see BIS, 2006) which is implemented into European law by the Capital Requirements Directive (CRD) (European Commission, 2006).

For credit risk, Basel II allows only two possible quantification methods – a *Standardized Approach* (STA) and an *Internal Rating Based Approach* (IRB) (for more details on these two methods see BIS, 2006). The main difference between STA and IRB is that while the STA methodology is based on prescribed parameters, under IRB banks are required to use internal measures for both the quality of the deal (measured by the counterparty's *probability of default* – PD) and the quality of the deal's collateral (measured by the deal's *loss given default* – LGD).

The PD is the chance that the counterparty will default (or, in other words, fail to pay back its liabilities) in the upcoming 12 months. A common definition of default is that the debtor is delayed in its payments for more than 90 days (90+ days past due).

The LGD, on the other hand, is the percentage of the size of the defaulted debt which the bank will actually lose given that the default happens – in practice, the potential 100% loss decreases by expected recoveries from the default, i.e., the amount that the creditor expects to be able collect back from the debtor after the debtor defaults; these recoveries are mainly realized from collateral sales and bankruptcy proceedings.

It is possible to say that PD and LGD are two major and common measures of deal quality and basic parameters for credit risk measurement. The PD is usually obtained either from a scoring model, from a Merton-based distance-to-default model (e.g., Moody's KMV, mainly used for commercial loans; Merton, 1973 and 1974) or as a long-term stable average of past 90+ delinquencies.[2] The model, presented later in the paper, provides a connection between the scoring models and those based on past delinquencies. The LGD can generally be understood as a function of collateral value; however, we view LGD as fixed in the present paper for simplicity.

Once PDs and LGDs have been obtained, we are able to calculate the *expected loss*. The expected loss is the first moment of a loss distribution, i.e., a mean measure of the credit risk. The expected loss is a sufficient measure of credit risk on the long-term horizon. However, in the short-term (e.g., the one-year horizon), it is insufficient to be protected only against expected losses because of the stochasticity of the losses. Thus a bank should look into the right tail

[2] Delinquency is often defined as a delay in installment payments, e.g., 90+ delinquencies can be interpreted as a delay in payments for more than 90 days.

or the distribution of the losses and decide which quantile (probability level) of the loss should be covered by holding a sufficient amount of capital.

Banks usually cover a quantile level suggested by a rating agency, which, however, has to be no less than the *regulatory level* 99.9%. This level may seem a bit excessive, as it can be interpreted as meaning that banks should cover a loss which occurs once in a thousand years. The reason for choosing such a conservative value is the usual absence of data for an exact estimation of the quantiles, resulting in a large error of the quantiles estimation.

The quantile is usually calculated by Value-at-risk type models, such as Saunders and Allen (2002), Andersson et al. (2001) or by the IRB approach which assumes that the credit losses are caused by two normally distributed risk factors: credit quality of the debtor and a common risk factor for all debtors, often interpreted as the macroeconomic environment (see Vasicek, 1987).

In this paper, we will introduce a new approach to quantifying credit risk which can be classed with the Value-at-risk models. Our approach is different from the IRB method in the choice of the loss distribution. In the general version of our model, we assume a generally non-normal distribution of the risk factors. Moreover, we model a dynamics of the common factor (modeling a dynamics of the factor being necessary especially with respect to the present financial crisis). In the simpler version of our model, which we later apply to the mortgage data, we keep the *IRB assumption* of the normal individual factor (credit quality of a debtor) while allowing a non-normal common factor; in its general form, however, our approach allows a non-normal individual factor, too, which could be useful to measure the credit risk of many types of banking products, e.g., consumer loans, overdraft facilities, commercial loans with a lot of variance in collateral, exposures to sovereign counterparties and governments, etc.

As we said previously, we apply our model to the US nationwide mortgage portfolio assuming the normal distribution of the individual factor and a generalized hyperbolic distribution of the common factor. We compare our results to the IRB approach, showing that the assumption of a normal common factor is inappropriate.

There are several other extensions of Vasicek's model; however, they mainly focus on the randomness of LGD. The simplest (and the most natural) enhancement of the Vasicek model incorporating LGD is the one proposed in Frye (2000), which assumes that LGD is a second risk indicator driving credit losses. An extension of the Frye model can be found in Pykhtin (2003), who supposes that the risk factor driving LGD depends on one systemic and two idiosyncratic factors. Another extension of the Vasicek model can be found in Witzany (2011) where LGD is assumed to be driven by a specific factor different from the one driving defaults and by two systemic factors, one common to the defaults and the other specific to LGD. None of these models allows a non-normal distribution for the PD systemic factor.

The paper is organized as follows. After the introduction we describe the usual credit risk quantification methods and Basel II – embedded requirements in detail. Then we derive our method of measuring credit risk, based on the class of generalized hyperbolic distributions and Value-at-risk methodology. In the last part, we focus on the data description and verification of our approach's ability to capture the credit risk more accurately than the Basel II IRB. Further, we demonstrate that the class of distributions we use better fits the empirical data than several distributions that are, alongside the IRB's standard normal distribution, commonly used for credit risk quantification. At the end we summarize our findings and offer recommendations for further research.

## 1. Credit Risk Measurement Methodology

The Basel II document is organized into three separate pillars. The first pillar requires banks to quantify credit risk, operational risk, and market risk by a method approved by the supervisor.[3] For credit risk there are two possible quantification methods: the method STA and the method IRB.[4] Both methods are based on quantification of risk-weighted assets for each individual exposure.

The STA method uses measures defined by the supervisor, i.e., each deal is assigned a risk-weight based on its characteristics. Risk-weighted assets are obtained by multiplying the assigned risk-weight by the amount that is exposed to default. The IRB approach, on the other hand, is more advanced than STA. It is based on a Vasicek-Merton credit risk model (Vasicek, 1987) and calculation of its risk-weighted assets is more complicated than in the STA case. First of all, PD and LGD are used to define the riskiness of each deal. These measures are then used to calculate risk-weighted assets based on the assumption of normal distribution for the asset value. In both cases, the largest permitted loss that could occur at the 99.9% level of probability[5] is stated as 8% of the risk-weighted assets (for more details on calculations of risk-weighted assets see BIS, 2006). The loss itself is defined as the amount that is really lost when a default occurs. The default is defined as a delay in payments for more than 90 days (90+ delinquency).

[3] A supervisor is a regulator of a certain country's financial market; for the Czech Republic, the supervisor is the Czech National Bank.

[4] As defined in the provision of the Czech National Bank No. 123/2007 Sb.

[5] The 99.9% level of probability is defined by the Basel II document and is assumed to be a far-enough tail for calculating losses that do not occur with a high probability. Note that a 99.9% loss at the one-year horizon means that the loss occurs once in 1 000 years on average. Because the human race lacks such a long dataset, 99.9% was chosen based on rating agencies' assessments.

### 1.1. Expected and Unexpected Loss for an Individual Exposure

The expected and unexpected losses are the two basic measures of credit risk. The expected loss is the mean loss, i.e., the expectation of the loss distribution, whereas the unexpected loss is the random difference between the expected and the actual loss. First, let us focus on expected and unexpected loss quantification for a single exposure, e.g., one particular loan based on PD and LGD. As there is no PD or LGD feature in the STA method, and because regulatory institutions are interested only in unexpected losses, under STA it is impossible to calculate the expected loss, and even the unexpected loss calculation is highly simplified and based on benchmarks only. On the other hand, the advantage of this method is its simplicity. The IRB approach uses PDs and LGDs and thus is more accurate than the STA but relatively difficult to maintain.

A bank using the IRB method has to develop its own scoring and rating models to estimate PDs and LGDs. These parameters are then used to define each separate exposure.[6] The average loss that could occur in the following 12 months is calculated as follows:

$$EL = PD \;.\; E(LGD) \;.\; EAD \tag{1}$$

where

EAD – the exposure-at-default,[7]

EL – the abbreviation for *Expected Loss*.

The EAD is usually regarded as a random variable as it is a function of a *Credit Conversion Factor* – CCF;[8] however, for mortgage portfolios, CCF is prescribed by the regulator as a fixed value. For our calculations we assume that if a default is observed, it happens on a 100% drawn credit line, so we don't treat EAD as a variable but as a constant.

EL can be regarded as an "ordinary" loss that would occur each year and thus is something that banks incorporate into their loan-pricing models so it has to be covered by ordinary banking fees and/or interest payments. However, banks also have to protect themselves against the randomness of the loss, which they do by holding capital to cover the maximum loss that could occur at the regulatory

[6] Exposure is the usual expression for the balance on a separate account that is currently exposed to default. We will adopt this expression and use it in the rest of our paper.

[7] Exposure-at-default is a Basel II expression for the amount that is (at the moment of the calculation) exposed to default.

[8] CCF is a measure of what amount of the loan (or a credit line) amount is on average withdrawn in the case of a default. It is measured in percentage of the overall financed amount and is important mainly for off-balance sheet items (e.g., credit lines, credit commitments, undrawn part of the loan, etc.).

probability level at minimum. To capture the variability in credit losses and to calculate the needed quantile of the loss distribution, we clearly need to know the shape of the loss distribution.

On the deal level, the distribution of the loss can be easily determined: Default is a binary variable occurring with a probability equal to PD. If the LGD is positive, the loss occurs with the same probability as the default, and the distribution of the loss can readily be determined from the distribution of the LGD.[9]

### 1.2. Expected and Unexpected Loss for a Portfolio

On the portfolio level (constructed from a certain number of individual deals), the expected loss can be calculated easily: since the loss of the portfolio as a whole is the sum of the losses of individual deals, its expected loss equals, by the additive property of expectations, to the sum of expressions (1) for all the individual deals. If, in addition, the PDs of the individual deals are identical and the LGDs are equally distributed then the expected loss comes out as PD times the sum of the individual EAD's. However, the calculation of the unexpected loss on the portfolio level is not so straightforward for the reason that deals may be correlated with each other within a complicated correlation structure that is usually unknown.

There are two ways of constructing a model for calculating unexpected loss. If the correlation structure among the individual deals is known, we can calculate the variance of the unexpected loss from the variances of individual deals and the correlation matrix. This approach is often referred to as a *bottom-up* one. Often, however, the correlation matrix of the individual deals is not known and thus a different approach has to be chosen to determine the unexpected loss of the loan portfolio. The second approach is widely known as a *top-down* approach and the main idea is to estimate the loss distribution based on historical data or assume a distribution structure and determine the standard deviation or directly find the difference between the chosen quantile and the mean value.[10]

In the present paper, we assume a rather simple dependence structure of the individual deals, similar to the one from Vasicek (2002): in particular, we assume that the default happens if a *factor variable* of a deal, summing an individual and a common part, falls below a certain threshold.

[9] Please note that the LGD variable may take on negative values in some cases. This is, for example, a situation when a loan's collateral covers the loan value and a bank collects some additional cash on penalty fees and interest.

[10] Remember that the loss mean value equals the expected loss of a deal.

## 2. Our Approach

### 2.1. The Distribution of Loan Portfolio Value

The IRB to modeling the loan portfolio value is based on the famous paper by Vasicek (2002) assuming that the value $A_{i,1}$ or the $i$-th's borrower's assets at time one can be represented as

$$\log A_{i,1} = \log A_{i,0} + \eta + \gamma X_i \tag{2}$$

where

$A_{i,0}$ – the borrower's wealth at time zero,

$\eta$ and $\gamma$ – constants,

$X_i$ – a (unit normal) random variable, which may be further decomposed as

$$X_i = Y + Z_i$$

where

$Y$ – a factor common to all the borrowers,

$Z_i$ – a private factor, specific to each borrower.

It is assumed that, at time $t$ – 1, all the borrowers, having the same initial wealth $A_{i,0}$ = $A_0$, take mortgages of the same size. Assuming the number of the borrowers to be very large and applying the Law of Large Number to the conditional distribution of the wealth given the common factor $Y$, the famous Vasicek distribution of the percentage loss of the portfolio holder (i.e., the percentage of those borrowers whose wealth $A_{i,1}$ at time 1 is not sufficient to repay that mortgage) is obtained (see Vasicek, 2002 for details).

#### 2.1.1. The Generalization

We generalize the model in two ways: we assume a dynamics of the common factor $Y$ over discrete times $t$ = 1, 2… and we allow non-normal distributions of both the common and private factors. For each time, $t \in \mathbb{N}$, we assume, similarly to the original model, that

$$\log A_{i,t} = \log A_{i,t-1} + Y_t + U_{i,t} \tag{3}$$

where

$A_{i,t}$ – the wealth of the $i$-th borrower at time,

$t \in \mathbb{N}$, $U_{i,t}$ – a random variable specific to the borrower,

$Y_t$ – the common factor following a general (adapted) stochastic process with a deterministic initial value $Y_0$.

For simplicity, we assume that the duration of the debt is exactly one period and that different borrowers take mortgages in each period, the initial wealth of each borrower equating to the (cummulative) common factor plus a zero mean borrower-specific *disturbance*, i.e.,

$$\log A_{i,t-1} = \Sigma_{j=1}^{t-1} Y_j + V_{i,t}$$

for all $i \le n$ where $V_{i,t}$ is a centered random variable specific to the borower – such an assumption makes sense, for instance, if $Y_t$ stands for log-returns of a stock index which and the borrower owns a portfolio with the same composition as the index plus some additional assets.

Suppose $U_{i,t}$ and $V_{i,t}$ to have the same distribution with zero mean and with a strictly increasing cummulative distribution function for each $i \le n$, $t \in \mathbb{N}$, where $n$ is the number of borrowers and that all $(U_{i,t},\ V_{i,t})_{i \le n, t \in \mathbb{N}}$ are mutually independent and independent of $(Y_t)_{t \in \mathbb{N}}$. Note that we do not require increments of $Y_t$ to be centered (which may be regarded as compensation for the term $\eta$ present in (1) but missing in (2)).

#### 2.1.2. Percentage Loss (Delinquency Rate) in the Generalized Model

Denote $\overline{Y}_t = (Y_\tau)_{\tau \le t}$ the history of the common factor up to time $t$ Analogously to the original model, the conditional probability of the bankruptcy of the $i$-th borrower at time $t$ given $\overline{Y}_t$ equals to

$$\mathbb{P}\left(A_{i,t} < B_{i,t} | \overline{Y}_t\right) = \mathbb{P}\left(Z < log B_{i,t} - \Sigma_{j=1}^{t} Y_j | \overline{Y}_t\right) = \Psi\left(b - \Sigma_{j=1}^{t} Y_j\right)$$

where $Z = U_{1,1} + V_{1,1}$, $\Psi$ – the cummulative distribution function of $Z$, $B_{i,t}$ are the borrower's debts (installments) which we assume to be the same for all the borrowers and all times, i.e., $\log B_{i,t} = b, t \in \mathbb{N},\ i \le n,$ for some $b$.

The primary topic of our interest is the percentage loss (delinquency rate) $L_t$ of the entire portfolio of the loans at time $t$. After taking the same steps as Vasicek (1991) (with conditional non-normal c.d.f.s instead of the unconditional normal ones), we get, for a very large portfolio, that

$$L_t \doteq \Psi\left(b - \Sigma_{j=1}^{t} Y_j\right),\ t \in \mathbb{N}$$

further implying that

$$Y_t \doteq \Psi^{-1}\left(L_{t-1}\right) - \Psi^{-1}(L_t) \tag{4}$$

and

$$L_t \doteq \Psi\left(\Psi^{-1}\left(L_{t-1}\right) - Y_t\right) \tag{5}$$

the latter formula roughly determines the dynamics of the process of the losses (delinquency rates), and the former one allows us to do statistical inference on the common factor based on the time series of the percentage losses (delinquency rates). To see that the Merton-Vasicek model is a special version of the generalized model, see the Appendix.

In the particular version of our general model we work with later, we assume both $U_{i,t}$ and $V_{i,t}$ to be normally distributed and the common factor to be an ARCH process

$$Y_t = \sigma_t \varsigma_t,\ \ \sigma_t = c + Y_{t-1}^2$$

where

$\varsigma_1, \varsigma_2 \dots$ – i.i.d. (possibly non-normal) variables,

$c$ – a constant.

Since the equation (3) may be rescaled by the inverse standard deviation of $Z$ without loss of generality, we may assume that $\Psi$ is the standard normal distribution function. As was already mentioned, we assume the distribution of $\varsigma_1$ to be generalized hyperbolic and we use the ML estimation to get its parameters – see the Appendix for details. In addition to estimation of the parameters, we compare our choice of the distribution to several other distribution classes.

### 2.2. The Class of Generalized Hyperbolic Distributions

Our model is based on the class of generalized hyperbolic distributions, first introduced in Barndorff-Nielsen, Blæsild and Jensen (1985). The advantage of this distribution class is that it is general enough to describe fat-tailed data. It has been shown (Eberlein, 2001; Eberlein and Prause, 2002; Eberlein and von Hammerstein, 2004) that the class of generalized hyperbolic distributions is better able to capture the variability in financial data than a normal distribution, which is used by the IRB approach. Generalized hyperbolic distributions have been used in an asset (and option) pricing formula (Rejman, Weron and Weron, 1997; Eberlein, 2001; Chorro, Guegan and Ielpo, 2008), for the Value-at-risk calculation of market risk (Eberlein and Prause, 2002; Eberlein and Keller, 1995; Hu and Kercheval, 2008) and in the Merton-based distance-to-default model to estimate PDs in the banking portfolio of commercial customers (e.g., Oezkan, 2002). We will show that the class of generalized hyperbolic distributions can be used for an approximation of a loss distribution for the retail banking portfolio with a focus on the mortgage book.

The class of generalized hyperbolic distributions is a special, quite young class of distributions. It is defined by the following Lebesgue density:

$$gh\ \ x;\lambda,\alpha,\beta,\delta,\mu\ \ =$$

$$= a\ \ \lambda,\alpha,\beta,\delta\ \ (\delta^2+(x-\mu)^2)^{\frac{\lambda-0,5}{2}} \times K_{\lambda-0,5}(\alpha\sqrt{((\delta^2+(x-\mu)^2))})\exp(\beta\ \ x-\mu \qquad (6)$$

where

$$\lambda,\alpha,\beta,\delta \; = \frac{(\alpha^2-\beta^2)^{0,5\lambda}}{\sqrt{2\pi}.\ \alpha^{\ \lambda-0,5}\ \delta^{\lambda} K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})}$$

and $K_\lambda$ is a Bessel function of the third kind (or a modified Bessel function – for more details on Bessel functions see Abramowitz, 1968). The GH distribution class is a mean-variance mixture of the normal and generalized inverse Gaussian (GIG) distributions. Both the normal and GIG distributions are thus subclasses of generalized hyperbolic distributions. Here $\mu$ and $\delta$ are scale and location parameters, respectively. Parameter $\beta$ is the skewness parameter, and the transformed parameter $\bar{\alpha}=\alpha\delta$ determines the kurtosis. The last parameter, $\lambda$, determines the distribution subclass. There are several alternative parameterizations described in the literature using transformed parameters to obtain scale- and location-invariant parameters. This is a useful feature that will help us with the allocation of economic capital to individual exposures. For the moment-generating function and for more details on the class of generalized hyperbolic distributions, see the Appendix.

Because the class of generalized hyperbolic distributions has historically been used for different purposes in economics as well as in physics, one can find several alternative parameterizations in the literature. In order to avoid any confusion, we list the most common parameterizations. These are:

$$\zeta=\delta\sqrt{\alpha^2-\beta^2},\ \rho=\frac{\beta}{\alpha}$$

$$\xi=(1+\zeta)^{-0,5},\ \ \chi=\xi\rho$$

$$\bar{\alpha}=\alpha\delta,\ \bar{\beta}=\beta\delta$$

The main reason for using alternative parameterizations is to obtain a location- and scale-invariant shape of the moment-generating function (see the Appendix).

## 3. Data and Results

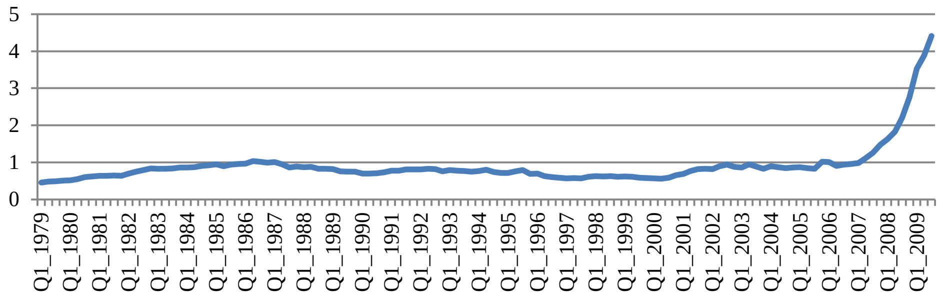
### 3.1. Data Description

To verify whether or not our model based on the class of generalized hyperbolic distributions is able to better describe the behavior of mortgage losses, we used data from the US mortgage market, namely the a dataset consisting of quarterly observations of 90+ delinquency rates on mortgage loans collected by the US Department of Housing and Urban Development and the Mortgage Bankers

Association.[11] This data series is the best substitute for losses that banks faced from their mortgage portfolios, relaxing the LGD variability (i.e., assuming that LGD = 100%). The dataset begins with the first quarter of 1979 and ends with the third quarter of 2009. The development of the US mortgage 90+ delinquency rate is illustrated in Figure 1. We observe an unprecedentedly huge increase in the 90+ delinquency rate beginning with the second quarter of 2007.

Figure 1

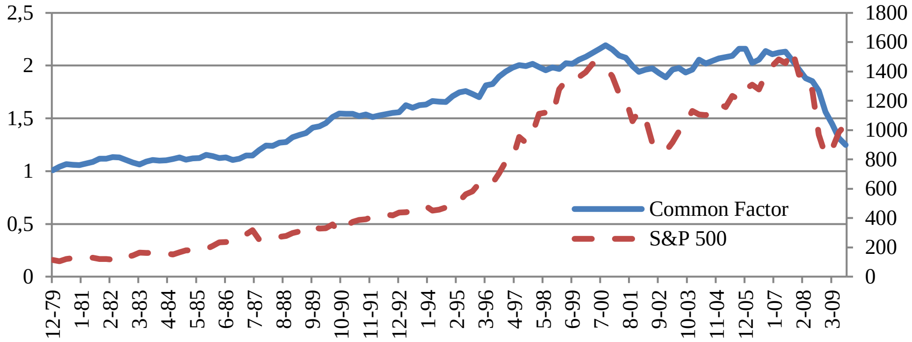
**Development of US 90+ Delinquency Rate**




*Source*: US Department of Housing and Urban Development.

Figure 2

**Comparison of the Development of the Common Factor and Lagged S&P 500 Returns**




*Source*: Own calculations (Common factor), finance.yahoo.com (S&P 500).

Starting our analysis, we have computed the values of the common factor $Y$ using the formula (4) with the standard normal $\Psi$. Quite interestingly, its evolution is indeed similar to the one of the US stock market – see Figure 2, displaying

[11] The Mortgage Bankers Association is the largest US society representing the US real estate market, with over 2 400 members (banks, mortgage brokers, mortgage companies, life insurance companies, etc.).

the common factor (left axis), adjusted for inflation, against the S&P 500 stock index. The correlation analysis indicates that the common factor lags behind the index by two quarters (the value of the Pearson correlation coefficient is about 30%).

### 3.2. Results

We considered several distributions for describing the distribution of $\varsigma_1$ (hence of $(L_t)_{t\geq 1}$ after a transform), namely loglogistic, logistic, lognormal, Pearson, inverse Gaussian, normal, lognormal, gamma, extreme value, beta and the class of generalized hyperbolic distributions.

In the set of the distributions compared, we were particularly interested in the goodness-of-fit of the class of generalized hyperbolic distributions and their comparison to other distributions. In particular, after estimating *c* whose estimate is independent of the distribution of $\varsigma_1$, we have, for each compared distribution, fitted its parameters using the maximum likelihood (see the Appendix for the proof that this procedure is correct) and computed the chi-square goodness-of-fit statistics:

$$\chi^2 = \sum_{i=1}^{k} (O_i - E_i)^2 / E_i \tag{8}$$

where

$O_i$ – the observed frequency in the *i*-th bin,
$E_i$ – the frequency implied by the tested distribution,
$k$ – the number of bins.

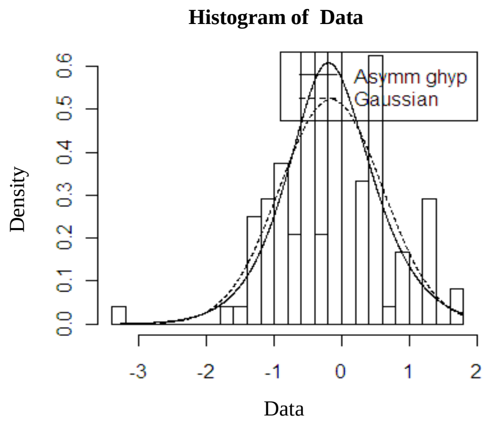
It is well known that the test statistic asymptotically follows the chi-square distribution with $(k – c)$ degrees of freedom, where *c* is the number of estimated parameters. In general, only the generalized hyperbolic distribution from all considered distributions was not rejected to describe the dataset on a 99% level (the statistic value was 22.59 with a p-value 0.0202).

Figure 1 graphically shows the difference between the estimated generalized hyperbolic and normal distributions. From Figure 1 we can see that the GHD is better able to describe both the skewness and the kurtosis of the dataset.

The main result of our estimation is that the class of generalized hyperbolic distributions is the only one suitable to describe the behavior of delinquencies among a wide variety of alternatives. The main reason for this is, in our opinion, the fact that GHD are fat-tailed, which suggests a need for a larger stock of capital to cover a certain percentile delinquency. We demonstrate this in the next Section.

Figure 3

**Compared Histograms: GHD vs. Normal vs. Dataset**




*Source*: Own calculations.

### 3.3. Economic Capital at the One-year Horizon: Implications for the Crisis

The IRB formula, defined in Pillar 1 of the Basel II Accord, assumes that losses follow a distribution that is a mix of two standard normal distributions describing the development of risk factors and their correlation. The mixed distribution is heavy-tailed and the factor determining how heavy the tails are is the correlation between the two risk factors. However, because the common factor is considered to have the standard normal distribution, the final loss distribution's tails may not be heavy enough. If a heavy-tailed distribution is considered for the common factor, the final loss distribution will probably have much heavier tails. Because the regulatory capital requirement is calculated at the 99.9% probability level, this may lead to serious errors in the assessment of capital needs. To show the difference between the regulatory capital requirement (calculated by the IRB method) and the economic capital requirement calculated by our model, we performed the economic capital requirement calculations at the 99.9% probability level as well.

When constructing loss forecasts, we repeatedly used (5) to get

$$L_{t+4} \doteq \Psi(\Psi^{-1}(L_t) - \sum_{1 \le i \le 4} Y_{t+1})$$

(note that our data were quarterly and that a one-year forecast is required). If we wanted to describe the distribution of the forecasted values exactly, we would face complicated integral expressions. We therefore decided to use simulations to obtain annual figures. We were particularly interested in the capital requirement based on average loss and the capital requirement based on last experienced loss.

The average loss was calculated as the mean value from the original dataset of 90+ delinquencies and served as a *through-the-cycle* PD estimate. This value is important for the regulatory-based model (Basel II), as a *through-the-cycle* PD should be used there. The last experienced loss is, on the other hand, important for our model due to the dynamical nature of the model. The Table 1 summarizes our findings. To illustrate how our dynamic model would predict if the normal distribution of the common factor was used, we added this version of the dynamic model as well.

Table 1

**Comparison of Basel II, Dynamic Normal and Dynamic GHD Models Tail Delinquency Rates**

| **Model** | **Basel II IRB (through-the-cycle PD)** | **Our dynamic model with normal distribution** | **Our dynamic model with GHD** |
|---|---|---|---|
| Distribution used for the individual factor | Standard Normal | Standard Normal | Standard Normal |
| Distribution used for the common factor | Standard Normal | Normal | Generalized Hyperbolic |
| 99.9% loss | 10.2851% | 9.5302% | 12.5040% |

*Source*: Own calculations.

The first column in Table 1 relates to the IRB Basel II model, i.e., a model with a standard normal distribution describing the behavior of both risk factors and the correlation between these factors set to the usual value of 15%. The PD used in the IRB formula (see Vasicek, 2002 for details) was obtained from the original dataset as an average default rate through the whole time period. The second column contains results from the dynamic model with a normal individual common factor. The last column is related to our dynamic model with the GHD of the common factor (for estimated parameter values, see the Appendix). The results in Table 1 show that the dynamic model, based on the last experienced loss, predicts higher quantile losses in the case of GHD and slightly lower in the case of a normal distribution, compared to the IRB formula. Thus, heavy tails of the GHD distribution evoke higher quantile losses than the current regulatory IRB formula, which ultimately leads to a higher capital requirement.

## Conclusion

We have introduced a new model for quantification of credit losses. The model is a generalization of the current framework developed by Vasicek and our main contribution lies in two main attributes: first, our model brings dynamics into the original framework and second, our model is generalized in the sense that any probability distribution can be used to describe the behavior of risk factors.

To illustrate that our model is better able to describe past risk factor behavior and thus better predicts the future need for capital, we compared the performance of several distributions common in credit risk quantification. In this sense, we were particularly interested in the performance of the class of Generalized Hyperbolic distributions, which is often used to describe heavy-tail financial data. For this purpose, we used a quarterly dataset of mortgage delinquency rates from the US financial market. Our suggested class of Generalized Hyperbolic distributions showed much better performance.

We have compared our dynamic model with the current risk measurement system required by the regulations. Our results show that the mix of standard normal distributions used in the Basel II regulatory framework underestimates the potential unexpected loss on the one-year horizon. Therefore, introducing the dynamics with a heavy-tailed distribution describing the common factor may lead to a better capturing of tail losses.

Despite the good results of our model, there are still several questions that need to be answered before our model (with the class of generalized hyperbolic distributions as a noise in the process of the common factor) can be used for credit risk assessment. First question points at the use of the 99.9th quantile. As this was chosen by the Basel II framework based on benchmarks from rating agencies, it is not known whether this particular quantile should be required in our dynamic generalized model. Second, more empirical studies have to be performed to prove the goodness-of-fit of the class of generalized hyperbolic distributions. The final suggestion is to add an LGD feature to the calculation to obtain a general credit risk model.

## Appendix

The moment-generating function for the class of generalized hyperbolic distributions is of the form:

$$M(u) = e^{u\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda/2}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+u)^2})}{K_\lambda(\alpha^2-\beta^2)} \tag{1}$$

where $u$ denotes the moment. For the first moment, the formula is simplified to (for details see, e.g., Eberlein, 2001):

$$M'(0) = E(x) = \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}\frac{K_{\lambda+1}(\delta\sqrt{\alpha^2-\beta^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} \tag{2}$$

The second moment is calculated in a (technically) more difficult way:

$$
\begin{aligned}
M''(0) = Var(x) &= \delta^2 \left( \frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\delta\sqrt{\alpha^2-\beta^2}\; K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \right) + \\
&+ \frac{(\beta\delta)^2}{\alpha^2-\beta^2} \left( \frac{K_{\lambda+2}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} - \left( \frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \right)^2 \right)
\end{aligned} \tag{3}
$$

By substituting from equations (2) and (3) into equation (1) we obtain a much simpler expression for the first and second moments of the class of generalized hyperbolic distributions. The following equations express the first and the second moment of the class of generalized hyperbolic distributions in their scale- and location-invariant shape:

$$
M(1) = E(x) = \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}} \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)}
$$

$$
M(2) = Var(x) = \delta^2 \left( \left( \frac{K_{\lambda+1}(\zeta)}{\zeta K_\lambda(\zeta)} \right) + \frac{\beta^2}{\alpha^2-\beta^2} \left( \frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \left( \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} \right)^2 \right) \right)
$$

***On MLE Estimation of the Parameters***

To estimate the parameters of the model, i.e., the constant $c$ and the vector of parameters $\Theta$ of (the distribution of) $\varsigma_1$, we apply the (quasi) ML estimate to the sample $Y_2, Y_3 \ldots$ computed from (4), using the fact that the conditional density of $Y_t$ given $\bar{Y}_{t-1}$ is

$$
f(y; c,\Theta) = \rho_t(c)\varphi(\rho_t y; \Theta) \qquad \rho_t(c) = \left[ Y_{t-1}^2 + c \right]^{-1/2}
$$

where $\varphi(z;\Theta)$ is the p.d.f. of the distribution $\varsigma_1$ dependent on parameters $\Theta$. The (quasi) log-likelihood function is then

$$
L(c,\Theta) = \sum_{i=2}^{T} \log(\rho_t(c)) + \sum_{i=2}^{T} \log(\varphi(\rho_t(c) Y_i; \Theta))
$$

Therefore, if the distribution of $\varsigma_1$ has a free scaling parameter which is part of $\Theta$, we may find its maximum in two steps: first, estimate the value of $c$ by maximizing the left-hand sum, and, second, find the parameter $\Theta$ by a maximization of the right hand sum which is, incidentaly, the likelihood function of the distribution of $\varsigma_1$ so the standard ML procedure may be used to maximize it (the existence of the scaling parameter guarantees independence of the right sum's minimum on *c*).

***The Merton-Vasicek Model as a Special Case of Our Generalized Framework***

In the current section, we show how our generalized model relates to the original one. Let us start with the computation of the loss's distribution. Reall that, in our model, the probability of default equals

$$p_t = \mathbb{P}(A_{i,t} < B_{i,t} \mid \bar{Y}_{t-1}) = \mathbb{P}(\xi_t < b - \Sigma_{j=1}^{t} Y_j \mid \bar{Y}_{t-1}) = \chi_t(b - \Sigma_{j=1}^{t-1} Y_j)$$

where $\xi_t := Y_t + Z_{1,t}$, $\chi_t$ is its conditional c.d.f. given $\bar{Y}_{t-1}$ and $\Phi_t$ is the conditional c.d.f. of $Y_t$. Further, by section III.2,

$$\begin{aligned}
&\mathbb{P}(L_t < \theta \mid \bar{Y}_{t-1}) = \mathbb{P}(\Psi\ \ b - \Sigma_{j=1}^{t-1} Y_j\ \ < \theta \mid \bar{Y}_{t-1}) = \\
&= \mathbb{P}(\Psi\ \ \chi^{-1}(p_t) - Y_t\ \ < \theta) = \mathbb{P}(Y_t > \chi^{-1}(p_t) - \Psi^{-1}(\theta)) = \\
&= 1 - \Phi_t\ \ \chi^{-1}\ \ p_t\ \ - \Psi^{-1}\ \ \theta
\end{aligned}$$

recall that $\Psi$ the c.d.f. of $Z_{1,t}$. Further, denoting $X_{i,t} := Y_t + Z_{i,t}$, we get

$$\mathrm{cov}(X_{i,t}, X_{j,t} \left| \bar{Y}_{t-1} \right.) = \mathrm{var}(X_{i,t}, \left| \bar{Y}_{t-1} \right.) = \mathrm{var}(Y_t \left| \bar{Y}_{t-1} \right.)$$

and, consequently,

$$corr(X_{i,t}, X_{j,t} \left| \bar{Y}_{t-1} \right.) = \frac{\mathrm{var}(Y_t \left| \bar{Y}_{t-1} \right.)}{\mathrm{var}(Y_t \left| \bar{Y}_{t-1} \right.) + \mathrm{var}(Z_t)}$$

Now, if we assume, with Vasicek (2002), that

$$Y_1 : N(0,\ \rho),\ Z_{1,1} : N(0,\ 1-\rho)$$

for a certain *ρ*, then clearly $\xi_1 : N(0,\ 1)$ implying

$$\begin{aligned}
\mathbb{P}(L_1 < \theta) &= 1 - N\left( \frac{N^{-1}(p_1) - \sqrt{1-\rho}\, N^{-1}(\theta)}{\sqrt{\rho}} \right) = \\
&= N\left( \frac{\sqrt{1-\rho}\, N^{-1}(\theta) - N^{-1}(p_1)}{\sqrt{\rho}} \right)
\end{aligned}$$

and

$$corr(X_{i,1},\ X_{j,1}) = \rho$$

i.e., the formulas of Vasicek (2002).

| Estimated parameters of the GHD distribution: | | | | |
|---|---|---|---|---|
| **lambda** | **alpha** | **mu** | **sigma** | **gamma** |
| 0.0995296201 | 0.5711172510 | –0.0005942671 | 0.0234327245 | 0.0081063782 |
| log-likelihood: 292.9479 AIC –575.8958 | | | | |

| *Parameter variance covariance matrix* | | | | | |
|---|---|---|---|---|---|
| | **lambda** | **alpha** | **mu** | **sigma** | **gamma** |
| **lambda** | 1.3666785019 | –0.3367366178 | –6.805791e–04 | –1.974175e-02 | 6.803657e-04 |
| **alpha** | –0.3367366178 | 0.3816490666 | –5.602050e–04 | –2.925449e-02 | 4.998918e-04 |
| **mu** | –0.0006805791 | –0.0005602050 | 9.634706e–06 | 1.405187e-04 | –9.641320e-06 |
| **sigma** | –0.0197417490 | –0.0292544924 | 1.405187e–04 | 1.285614e-02 | –9.736328e-05 |
| **gamma** | 0.0006803657 | 0.0004998918 | –9.641320e–06 | –9.736328e-05 | 1.481876e-05 |

*Source*: Own calculations.

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