Attending Meetings: The Use of Mixed Strategies[1]

Wilfried PAUWELS – Daniel DUJAVA***

Abstract

We consider a game in which each of n players is invited to a meeting, and has to decide whether or not to attend the meeting. A quorum has to be attained if the meeting is to have the power of making binding decisions. We consider all possible preferences of the players. These preferences are assumed to be the same for all players. Restricting ourselves to symmetric Nash equilibria, we identify three different classes of preferences. In a first class the game has a unique Nash equilibrium, defined in mixed strategies. In a second class the game has two Nash equilibria, defined in pure strategies. In a final class of preferences the game has a Nash equilibrium in pure strategies, and possibly also in mixed strategies. If there is a mixed strategy Nash equilibrium, we show that the equilibrium probability of attending the meeting increases when the quorum increases. Furthermore, if the number of players becomes very large, this equilibrium probability tends to the value of the quorum. Finally, we show how the underlying game structure can also be used in other applications.

Keywords: *mixed strategies, attending meetings*

JEL Classification: D72, D81

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* Wilfried PAUWELS, University of Antwerp, Faculty of Applied Economics, Department of Economics, City Campus, Prinsstraat 13, 2000 Antwerp, Belgium; e-mail: wilfried.pauwels@uantwerpen.be

** Daniel DUJAVA, University of Economics in Bratislava, Faculty of National Economy, Department of Economic Theory, Dolnozemská cesta 1/b, 852 35 Bratislava 5, Slovak Republic; e-mail: daniel.dujava@euba.sk

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Introduction

Suppose you are invited to a meeting the agenda of which does not particularly interest you. Your decision to attend or not to attend the meeting may then be rather difficult. On the one hand, you may not want to lose your time preparing and attending a meeting in which you are not interested. But on the other hand, by not attending the meeting you may disappoint the chairman of the meeting. She may show you her disapproval, or she may even give you a warning. Moreover, if a meeting can only make binding decisions if a quorum is attained, you may feel guilty when, because of your absence, this quorum was not attained. The upshot of all these considerations could be that you decide to follow a mixed strategy, assigning positive probabilities to the actions „attending“ and „not attending“ the meeting.

In this paper we formalize the above situation as a game, and we study the use of mixed strategies in this game. In particular, we want to know which types of preferences give rise to an equilibrium mixed strategy. Furthermore, we want to analyze how the equilibrium mixed strategy depends on the exact value of the quorum and on the number of members of the meeting.

The main results of the paper are as follows. First, we identify three classes of preferences, each class having its own type of equilibrium. For preferences belonging to a first class, the game has a unique Nash equilibrium, defined in mixed strategies. For preferences belonging to a second class, there are two Nash equilibria, defined in pure strategies. Finally, in a third class of preferences, there always exists a Nash equilibrium in pure strategies, and possibly also a Nash equilibrium in mixed strategies. Secondly, we show that, in all classes of preferences with a mixed strategy Nash equilibrium, an increase of the quorum always leads to an increase in the equilibrium probability of attending the meeting. Furthermore, if the number of players becomes very large, then – for all preferences of the first class – the equilibrium probability of attending the meeting converges to the quorum, expressed as a fraction of the number of players. Finally, we show that there are interesting game situations, different from our „meeting game“, that can be analyzed using the underlying game structure of our model. One example refers to the well-known game in which players witness a crime and can report it to the police (see, e.g., Osborne, 2004). Other examples are the public good games analyzed by Palfrey and Rosenthal (1983).

The paper is structured as follows. In a first section we present the model. In a second section we derive and analyze equilibrium mixed strategies. A final section concludes. In an Appendix we prove some more technical results, used in section 2.

The model

We consider the following normal form game. The players of the game are the members of a board who are invited to a meeting. We denote the set of players by $N = \{1,\ 2,\ ...,\ n\}$. The quorum of the meeting is denoted by αn, meaning that at least αn members have to attend if the meeting is to have the power to make binding decisions. α is a fraction which can be written as $\alpha = x/n$, with $x = 1,\ 2,\ ...,\ n$. Clearly, $1/n \leq \alpha \leq 1$.

Each player in the game has two possible actions: she can attend or not attend the meeting. These two actions are denoted by *A* and *NA*. A mixed strategy is defined by a probability *p*, $p \in [0,\ 1]$ assigned to the action *A*. The probability that the action *NA* is chosen is $1 - p$. We define a strictly mixed strategy as a mixed strategy *p* with $p \in (0,\ 1)$. Note that the chairman of a meeting is not a player of the game. As she *has* to attend the meeting, she cannot choose between the actions *A* and *NA*.

An individual player $i \in N$, taking the actions chosen by the other players as given, can find herself in three different regimes. Regime I applies when the number of players, different from player i, having decided to attend the meeting is smaller than $\alpha n - 1$. In this regime the meeting will never attain its quorum, even when player i decides to attend. Clearly, in the extreme case when $\alpha = 1/n$, regime I cannot occur. In this case, even if all the other $n - 1$ players decide not to attend, the quorum is attained if the player we are considering decides to attend. Regime II applies when exactly $\alpha n - 1$ of the players, different from player *i*, have decided to attend. In this case player *i* is decisive, meaning that the quorum is attained if and only if player *i* decides to attend. Finally, in regime III, at least αn players, different from player *i*, have decided to attend the meeting. In this regime the quorum is attained, even when player *i* decides not to attend. In the extreme case when $\alpha = 1$, regime III cannot occur. In this case, even if all the other $n - 1$ players decide to attend, the quorum is not attained if the player we are considering decides not to attend.

We assume all the players have the same preferences, so that the game is symmetric. A player's payoffs, given in terms of Bernoulli utility numbers,[2] depend on the action that player has chosen, and on the actions chosen by all the other players. These latter actions are given by the regime that applies. The notation used to indicate the payoffs of an individual player is given in the following scheme. When a player decides to attend the meeting, her payoffs in each of the three regimes are denoted by a_I, a_{II} and a_{III}. When she decides not to attend the

[2] We are using the terminology introduced by Mas-Colell, Whinston and Green (1995), *p*. 184.

meeting, these payoffs are denoted by n_I, n_{II} and n_{III}. As we noted before, in the extreme cases when $\alpha = 1/n$ or $\alpha = 1$ regime I or regime III does not occur.

	Regime I	Regime II	Regime III
A	a_I	a_{II}	a_{III}
NA	n_I	n_{II}	n_{III}

At this point of the analysis, we do not impose any restrictions on these pay--offs. In fact, each of these utility numbers can be given by any real number. In the next section we will derive restrictions on these payoffs which are necessary and/or sufficient for the existence and/or uniqueness of an equilibrium mixed strategy.

Equilibrium Mixed Strategies

We now study symmetric mixed strategy Nash equilibria of the game defined in the previous section. We will first determine the types of preferences leading to mixed strategy Nash equilibria. We then examine how the parameters α and n affect the precise value of the equilibrium mixed strategy.

Consider an individual player $i \in N$, and assume that all the other players choose the same mixed strategy given by p. The number of these players actually attending the meeting is then a random variable with a binomial distribution. If we assume that $1/n < \alpha < 1$, so that the regimes I, II and III occur with positive probabilities, these probabilities are given by

$$L(p) = \sum_{k=0}^{\alpha n-2} \binom{n-1}{k} p^k (1-p)^{n-1-k} \tag{1}$$

$$M(p) = \binom{n-1}{\alpha n-1} p^{\alpha n-1} (1-p)^{(n-1)-(\alpha n-1)} \qquad 2)$$

$$R(p) = 1 - L(p) - M(p) = \sum_{k=\alpha n}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-l} \tag{3}$$

Clearly, if $\alpha = 1/n$, regime I cannot occur, and the above probabilities simplify to $L(p) = 0$, $M(p) = (1-p)^{n-1}$ and $R(p) = 1-(1-p)^{n-1}$. Similarly, if $\alpha = 1$, regime III cannot occur, and the above probabilities simplify to $L(p) = 1 - p^{n-1}$, $M(p) = p^{n-1}$ and $R(p) = 0$. We will analyze these two extreme cases later on. For the moment we assume that $1/n < \alpha < 1$, so that the probabilities (1), (2) and (3) apply.

The expected payoff of a player choosing action *A*, when all the other players choose a mixed strategy *p*, is given by

$$a_I L(p) + a_{II} M(p) + a_{III} R(p)$$

If a player chooses action *NA*, her expected payoff is

$$n_I L(p) + n_{II} M(p) + n_{III} R(p)$$

If we want a player to choose a mixed strategy, we require that these two expected payoffs be equal:

$$a_I L(p) + a_{II} M(p) + a_{III} R(p) = n_I L(p) + n_{II} M(p) + n_{III} R(p)$$

This is an application of the indifference principle (see, e.g., Maschler, Solan and Zamir, 2013, p. 160). To simplify the notation, we define the numbers $d_I = a_I - n_I$, $d_{II} = a_{II} - n_{II}$ and $d_{III} = a_{III} - n_{III}$, giving the difference in payoffs obtained from the actions *A* and *NA*, in each of the three regimes. We now define a function $\varphi(p)$ as

$$\varphi(p) = d_I L(p) + d_{II} M(p) + d_{III} R(p) \tag{4}$$

We want to find the values $p^* \in (0, 1)$ such that

$$\varphi(p^*) = 0 \tag{5}$$

Such a value of *p** is a symmetric, strictly mixed strategy Nash equilibrium. Moreover, if at the value $p^* \in (0, 1)$ the function φ cuts the horizontal *p*-axis from above, the mixed strategy Nash equilibrium *p** is locally stable. The motivation for this stability property is as follows. If, at a mixed strategy *p*, close to a value *p** where $\varphi(p^*) = 0$, $\varphi(p)$ is negative, the expected payoff of choosing *NA* exceeds the expected payoff of choosing *A*, and the decision maker may be assumed to decrease the value of *p*. Similarly, if $\varphi(p)$ is positive, the decision maker may be assumed to increase the value of *p*. If then the function φ cuts the horizontal *p*-axis at *p** from above, these adjustments will ultimately lead the decision maker to the value *p** where $\varphi(p^*) = 0$.

Clearly, the game can also have Nash equilibria in pure strategies, at values $p = 0$ or $p = 1$. If $p = 0$, all the other players have chosen the action *NA*. If then $\varphi(0) < 0$, the best reply by an individual player is also to choose *NA*. Hence, the combination of actions (*NA*, *NA*, ..., *NA*) is a locally stable pure strategy Nash equilibrium. Similarly, if $\varphi(1) > 0$, the combination (*A*, *A*, ..., *A*) is also a locally stable pure strategy Nash equilibrium.

We now identify the conditions guaranteeing that a value $p^* \in (0,\ 1)$ with property (5) exists, and that this value is unique and stable. As we analyze the case when α satisfies $1/n < \alpha < 1$, the probabilities (1), (2) and (3) apply. As in this case $L(0)=1$, $M(0)=0$, $R(0)=0$, $L(1)=0$, $M(1)=0$ and $R(1)=1$, we have

$$\begin{aligned} \varphi(0) &= d_I \\ \varphi(1) &= d_{III} \end{aligned} \tag{6}$$

Hence, the function φ starts at the value $\varphi(0)=d_I$ and ends at the value $\varphi(1)=d_{III}$.

We now determine the general behaviour of the function φ over the interval $(0,\ 1)$. Is this function monotonically increasing or decreasing over this interval, or does it attain critical points in this interval? And what can we say about the number of these critical points, and about the nature of these critical points (maxima or minima)? In the Appendix, section 1, we prove that the general behaviour of the function φ is fully determined by the relative sizes of the differences d_I, d_{II} and d_{III}. In particular, for each of the following six sequences of inequalities, the function φ shows the indicated behaviour:

$d_I < d_{II} < d_{III}$: φ is monotonically increasing over $(0,\ 1)$

$d_{III} < d_{II} < d_{II}$: φ is monotonically decreasing over $(0,\ 1)$

$d_I < d_{III} < d_{II}$ or $d_{III} < d_I < d_{II}$: φ has a unique critical point $p_e \in (0,\ 1)$, which is a maximum

$d_{II} < d_I < d_{III}$ or $d_{II} < d_{III} < d_I$: φ has a unique critical point $p_e \in (0,\ 1)$, which is a minimum

Moreover, the value p_e at which the function φ has a critical point is given by

$$p_e = \frac{(d_I - d_{III}) - (d_{II} - d_{III})}{(d_I - d_{III}) - \dfrac{n-1}{\alpha n - 1}(d_{II} - d_{III})} \tag{7}$$

For each of the above six sequences of inequalities we can fix the signs of the differences d_I, d_{II} and d_{III}. For example, for the first sequence $d_I < d_{II} < d_{III}$ we can consider the possibilities $0 < d_I < d_{II} < d_{III}$, $d_I < 0 < d_{II} < d_{III}$, $d_I < d_{II} < 0 < d_{III}$ and $d_I < d_{II} < d_{III} < 0$. We call such a signed sequence of inequalities a *preference profile*. For each such preference profile we can now easily determine the nature of the Nash equilibrium (equilibria), if there exists one. We start with the following simple cases. If all the differences d_I, d_{II} and d_{III} are negative, then clearly the combination of actions (*NA*, *NA*, ..., *NA*) is the unique pure strategy Nash equilibrium. Similarly, when all the differences d_I, d_{II} and d_{III} are positive, the combination (A, A, ..., A) is the unique pure strategy Nash equilibrium. More generally, as already noted, whenever $\varphi(0) = d_I < 0$, the combination of actions (NA, NA, ... NA) is a pure strategy Nash equilibrium, and whenever $\varphi(1) = d_{III} > 0$, the combination (A, A, ..., A) is also a pure strategy Nash equilibrium.

We now group all the other possible preference profiles into three main classes. A *first class of preference profiles* concerns preferences for which there always exists a stable, strictly mixed strategy Nash equilibrium which, in addition, is the unique Nash equilibrium of the game. These are the preference profiles

$$d_{II} < d_{III} < 0 < d_I \tag{8}$$

$$d_{III} < d_{II} < 0 < d_I \tag{9}$$

$$d_{III} < 0 < d_{II} < d_I \tag{10}$$

$$d_{III} < 0 < d_I < d_{II} \tag{11}$$

The common characteristic of all these profiles is that $d_{III} < 0 < d_I$. Given these inequalities, the preference profiles in (8) – (11) consider all possible locations of d_{II}, relative to d_{III} and d_I. In these preference profiles players basically dislike meetings. In particular, in regime III, when the attainment of the quorum is guaranteed, players do not want to attend the meeting: d_{III} is always negative. At the same time, in regime I, d_I is always positive, meaning that players prefer to attend the meeting even when that choice does not allow the quorum to be attained. This last characteristic may look strange for players who dislike meetings.

However, three possible interpretations are as follows. First, a meeting which does not attain its quorum can be very humiliating and embarrassing for the chairman, and players may want to alleviate the chairman's discomfort by attending the meeting. This would then also require d_{II} to be positive. Secondly, players who dislike meetings still may want to attend a meeting in regime I only to witness and to enjoy (with malicious pleasure) the chairman's discomfort in that regime. This would be consistent with a negative value of d_{II}. Finally, the inequalities $d_{III} < 0 < d_I$ may also characterize players who, for social reasons, like to be seen in the meeting, pretending they are concerned, but who are in fact unwilling to take any responsibility of the outcome. This may be a possible interpretation of the profiles (8) and (10).

Figure 1a

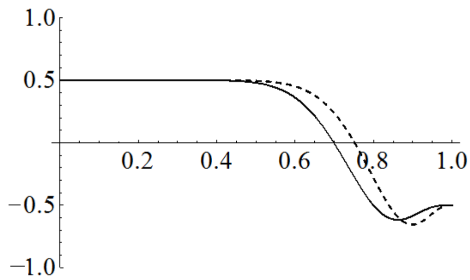


Figure 1b

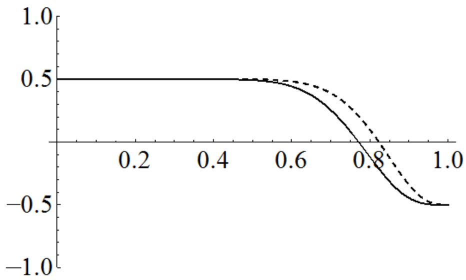


Figure 1c

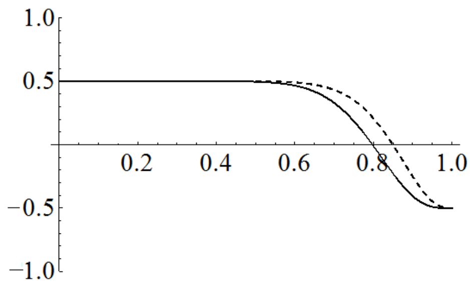
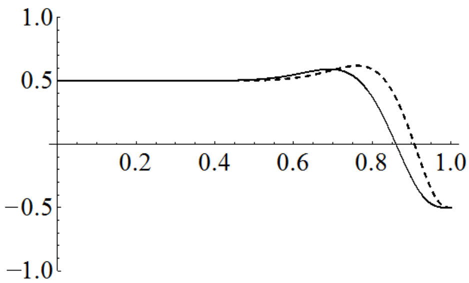


Figure 1d



Source: Own calculation.

The exact behaviour of the function φ in the four preference profiles (8) – (11) is as follows. As in all these profiles $d_{III} < 0 < d_I$, the function φ starts at a positive value d_I, and ends at a negative value d_{III}. In the case of (8) it attains a minimum in the interval $(0, 1)$. This is illustrated in Figure 1.a. In the cases (9) and (10) the function φ monotonically decreases from d_I to d_{III}. Figure 1.b illustrates (9), and Figure 1.c illustrates (10). Finally, in case (11), illustrated in Figure 1.d, the function φ attains a maximum in the interval $(0, 1)$. In all these cases there exists a unique, stable, strictly mixed strategy Nash equilibrium.[3]

A *second class of preference profiles* concerns preferences for which all stable Nash equilibria are pure strategy equilibria. These are profiles of the type

$$d_{II} < d_I < 0 < d_{III} \tag{12}$$

$$d_I < d_{II} < 0 < d_{III} \tag{13}$$

$$d_I < 0 < d_{II} < d_{III} \tag{14}$$

$$d_I < 0 < d_{III} < d_{II} \tag{15}$$

In all these preference profiles the inequalities $d_I < 0 < d_{III}$ hold. Given these inequalities, the preference profiles in (12) – (15) differ only in the exact location of d_{II}. In these profiles all possible locations of d_{II}, relative to d_I and d_{III}, are allowed. In all the profiles $d_I < 0 < d_{III}$ players can be said to be „serious“: in regime I they do not want to lose their time by attending the meeting, while in regime III, when real decisions are being taken, they want to be present. The sign of d_{II} reveals whether a player prefers a meeting which attains the quorum and where she is present, or a meeting which does not attain the quorum and where she is not present. In cases (14) and (15) the player is „concerned“ and prefers the first alternative. She would have guilt feelings if she would not attend. No such guilt feelings exist in cases (12) and (13).

In all four cases of (12) – (15) there exists an unstable mixed strategy Nash equilibrium, where the function φ cuts the p-axis from below. At the same time, the combinations (A, A, ..., A) and (NA, NA, ..., NA) are always stable pure strategy Nash equilibria. In the cases (13) and (14) the function φ monotonically increases from d_I to d_{III}. In case (15) the function φ attains a maximum at the value $p_e \in (0,\ 1)$, while in case (12) it attains a minimum at $p_e \in (0,\ 1)$.

Finally, *a third class of preference profiles* concerns preferences where there is always a stable pure strategy Nash equilibrium. Moreover, it is also possible that there exists a stable mixed strategy Nash equilibrium. These are the preference profiles of the type

$$d_I < d_{III} < 0 < d_{II} \tag{16}$$

$$d_{III} < d_I < 0 < d_{II} \tag{17}$$

$$d_{II} < 0 < d_I < d_{III} \tag{18}$$

$$d_{II} < 0 < d_{III} < d_I \tag{19}$$

In all these preference profiles d_I and d_{III} always have the same sign. In the cases (16) and (17) this sign is negative, meaning that players strongly dislike meetings. At the same time, they have strong guilt feelings in regime II: if the quorum is not attained and if they are decisive, they would feel very guilty if they would not attend. In these two preference profiles the function φ attains a maximum at the value $p_e \in (0,\ 1)$. If the value d_{II} is sufficiently large, and if the probability that regime II applies is also sufficiently large, the maximum

[3] The values of (d_I,d_{II},d_{III}) in each of the Figures 1a, 1b, 1c and 1d are given by (0.5, –2, –0.5), (0.5, –0.25, –0.5), (0.5, 0.25, –0.5) and (0.5, 2, –0.5). On the solid curves α = 16/20, and on the dashed curves α = 17/20. In all the figures n = 20.

value $\varphi(p_e)$ will be positive, so that there exist two values of p at which $\varphi(p)=0$. These are two mixed strategy Nash equilibria, but only the higher value of p is a stable Nash equilibrium. Moreover, the combination of actions (*NA*, *NA*, ..., *NA*) is also a pure strategy Nash equilibrium. These cases are illustrated in Figures 2a and 2b.[4]

Figure 2a

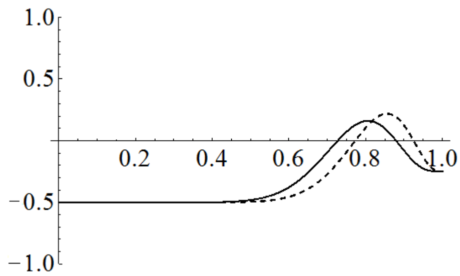
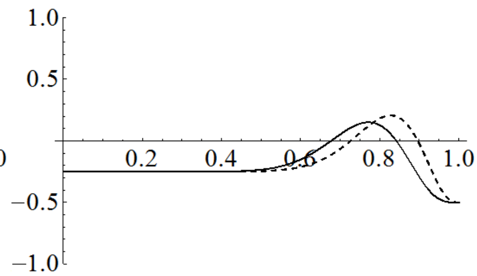


Figure 2b



Source: Own calculation.

In the cases (18) and (19) both d_I and d_{III} are positive, and d_{II} is negative. These preferences look rather eccentric: players prefer to attend in regimes I and III, but in regime II they prefer to boycott the meeting. For this reason we will not discuss these cases any further in the context of our meeting game. The mathematical properties of these cases are, however, easy to derive. The function φ attains a minimum at the value $p_e \in (0,\ 1)$. If the value $\varphi(p_e)<0$, there exist two mixed strategy Nash equilibria, but only the lower value of p is a stable Nash equilibrium. Moreover, the combination of actions (*A*, *A*,..., *A*) is also a pure strategy Nash equilibrium.

We still have to consider the possible extreme values of *α*, viz. $\alpha = 1/n$ and $\alpha = 1$. In case $\alpha = 1/n$ regime I cannot occur, and $L(p)=0$, $M(p)=(1-p)^{n-1}$ and $R(p)=1-(1-p)^{n-1}$. It follows that

$$\varphi(p)=[d_{II}-d_{III}](1-p)^{n-1}+d_{III} \tag{20}$$

We now have $\varphi(0)=d_{II}$ and $\varphi(1)=d_{III}$. This game has a unique, stable, strictly mixed strategy Nash equilibrium if and only if $d_{III}<0<d_{II}$. It is given by

[4] The values of (d_I, d_{II}, d_{III}) in each of the Figures 2a and 2b are given by (–0.5, 2, –0.25) and (–0.25, 2,–0.5). On the solid curves α = 16/20, and on the dashed curves α = 17/20. In both cases n = 20.

$$p^* = 1 - \left[\frac{-d_{III}}{d_{II} - d_{III}} \right]^{\frac{1}{n-1}} \tag{21}$$

In the context of the game in which players have to decide whether or not to attend a meeting, the case where $\alpha = 1/n$ looks very special. However, there is another well known game with exactly this structure. In this game *n* players observe a crime, and each player wants the police to be informed, but prefers that someone else make the phone call (see Osborne, 2004, pp. 131 – 134), and the many references given there. Each player has two actions *C* = Call, and *NC* = Not Call. Payoffs can be described by the following scheme:

	Regime II	**Regime III**
C	υ – c	υ – c
NC	0	υ

υ is the utility each player enjoys when the police is informed, and *c* is the cost of making a phone call. It is natural to assume that $\upsilon > c > 0$. Clearly, in this example the condition $d_{III} < 0 < d_{II}$ is satisfied, and

$$p^* = 1 - \left(\frac{c}{\upsilon} \right)^{\frac{1}{n-1}} \tag{22}$$

is the unique, stable, strictly mixed strategy Nash equilibrium.

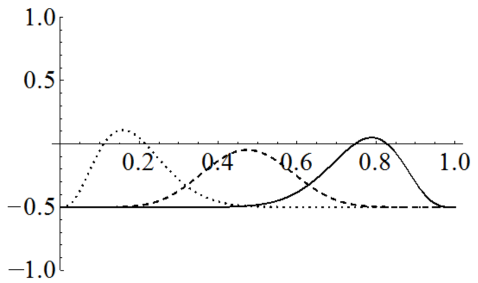
Finally, we consider the other extreme case $\alpha = 1$, in which all players have to attend in order to attain the quorum. Here $L(p) = 1 - p^{n-1}$, $M(p) = p^{n-1}$ and $R(p) = 0$, so that

$$\varphi(p) = (d_{II} - d_I)\, p^{n-1} + d_I \tag{23}$$

We now have $\varphi(0) = d_I$ and $\varphi(1) = d_{II}$. There now exists a stable, strictly mixed strategy Nash equilibrium if and only if $d_I > 0 > d_{II}$. In the context of our game such preferences are unreasonable. In the more reasonable case where $d_{II} > 0 > d_I$, the mixed strategy Nash equilibrium is unstable. The only stable Nash equilibria in this case are the pure strategy combinations (*A*, *A*, ..., *A*) and (*NA*, *NA*, ..., *NA*).

Let us now return to the case where $1/n < \alpha < 1$. In each of the profiles (8) – (11), and possibly also in the profiles (16) and (17), the game has a stable, strictly mixed strategy Nash equilibrium. We now examine how in each of these six profiles the equilibrium mixed strategy changes if the quorum α increases. In the case of preference profiles (16) and (17) we first note the following rather unexpected property. As we noted before, and as is clear from Figures 2a and 2b, a mixed strategy Nash equilibrium only exists if the maximal value $\varphi(p_e)$ exceeds zero. If this is not the case, no mixed strategy Nash equilibrium exists. In the Appendix, section 2, we prove that, in case the preference profiles (16) and (17) satisfy $d_I = d_{III} < 0 < d_{II}$, this maximal value $\varphi(p_e)$ is decreasing in α for $0 < \alpha < 1/2$, and increasing in α for $(1/2) < \alpha < 1$. It then may happen that this maximal value exceeds zero for low and for high values of α, and that it does not exceed zero for intermediate values of α. Such a case is illustrated in Figure 3.[5] Here the value of α increases from 4/20 (dotted line), to 10/20 (dashed line), and to 16/20 (solid line). For α equal to 4/20 or 16/20 there exists a mixed strategy Nash equilibrium, while for α equal to 10/20 there does not exist such an equilibrium.

Figure 3



Source: Own calculation.

Consider now each of the preference profiles (8) – (11), (16) and (17), and assume that the value of the quorum α increases. Assume also that for the profiles (16) and (17) there exists a mixed strategy Nash equilibrium for all the values of α being considered. In the Appendix, section 3, we prove that an the increase of the value of α will always increase the value of the Nash equilibrium mixed strategy p^*. This is illustrated on Figures 1a – 1d and 2a – 2b. The dashed curves in these figures correspond to the higher values of α and give rise to higher values of p^*, compared with the solid curves.

Finally, for the preference profiles (8) – (11), (16) and (17), we consider the effect of an increase of the number of players n on the mixed strategy Nash equilibrium. For the preference profiles (16) and (17) the probability that a player is decisive becomes smaller as n increases. The condition $\varphi(p_e) > 0$ can then no longer be satisfied. For the preference profiles (8) – (11) we prove in the Appendix,

[5] In this Figure $d_I = d_{III} = -0.5$, and $d_{II} = 2$, and $n = 20$. The value of α is 4/20 on the dotted line, 10/20 on the dashed line, and 16/20 on the solid line.

section 4, that the equilibrium mixed strategy tends to the quorum α when n tends to infinity. This is a remarkable and interesting result. Moreover, we also show in the Appendix that in equilibrium the probability that the quorum is attained equals $\frac{d_{III}}{d_{III}-d_I}$. Note that for the preference profiles of class one $0<\frac{d_{III}}{d_{III}-d_I}<1$.

We conclude this section with the following important remark. In this paper we always referred to the game in which players have to decide whether or not to attend a meeting. This is an interesting game in itself, and it allows us to consider a wide variety of possible preferences of the players. However, the underlying game structure can also be used to analyze different strategic situations. A situation which is related to attending a meeting occurs in case players are invited to a seminar. Not all seminars promise to be interesting, and other time intensive tasks may be difficult to postpone. In the case of a seminar there is no formal quorum, but a good seminar often requires the presence of a minimal number of participants. Moreover, if there are only a small number of participants, this could be rather embarrassing for the colleague who invited the speaker.

As already noted, a very different strategic situation occurs in the game in which players witness a crime and can report it to the police. This is a simple special case of the game we studied. Another interesting example is given in Palfrey and Rosenthal (1983). These authors study a game in which players can contribute to the production of a public good. This public good will be provided if and only if a sufficient number of players decide to make a contribution. Let there be n players, and assume that at least αn players have to make a contribution. Let a player's maximum willingness to pay for the public good be given by υ, and denote a player's contribution by c. Assume that $\upsilon > c > 0$. Also assume there is no refund of a player's contribution in case the public good is not provided. A player's payoffs can then be described by the following scheme.

	Regime I	**Regime II**	**Regime III**
C	$-c$	υ – c	υ – c
NC	0	0	υ

Here C and NC stand for the two possible actions „contribute“ and „not contribute“. If a player decides to contribute, she makes a loss of $-c$ in regime I, and a gain of υ – c in regimes II and III. If a player does not contribute, nothing happens in regimes I and II, and she realizes a gain of v in regime III. In this game $d_I=-c$, $d_{II}=\upsilon-c$ and $d_{III}=-c$. This is a preference profile on the borderline between (16) and (17). Hence, if υ is sufficiently large there exists a mixed strategy Nash equilibrium. Moreover, the combination (*NC*, *NC*, ..., *NC*) is always a Nash equilibrium in pure strategies. Finally, as in this game $d_I = d_{III} < 0 < d_{II}$ holds, the property illustrated in Figure 3 holds. Hence, it is possible that there exist mixed strategy Nash equilibria for small and for large values of α, but not for intermediate values of α. This result certainly deserves further investigation.

Concluding Remarks

The main results of this paper can be summarized as follows. We considered a game in which players decide whether or not to attend a meeting. If all the players of the game prefer not to attend a meeting when they know that the quorum cannot be attained, while they prefer to attend when they know that the quorum will be attained, there exist two stable pure strategy Nash equilibria: one in which all players attend, and one in which no player attends. For two types of preferences there exists a stable mixed strategy Nash equilibrium. First, players may hate meetings so much that they do not want to attend, even when they are sure the quorum will be attained. At the same time, when the quorum is not attained, they may want to attend the meeting. Their motivation could be that by attending the meeting they alleviate the resulting humiliation of the chairman. Alternatively, they may maliciously want to witness the chairman's humiliation. For these preferences there exists a strictly mixed strategy Nash equilibrium which is the unique stable Nash equilibrium of the game. Secondly, assume that players do not want to attend the meeting, both in case the quorum will be attained and in case it will not be attained. It then may happen that players, when they are decisive, want to attend the meeting because they would have strong guilt feelings if, because of their absence, the quorum would not be attained. In this case there always exists a pure strategy Nash equilibrium in which no player decides to attend. If the guilt feelings of the players are sufficiently strong, there also exists a stable mixed strategy Nash equilibrium.

We have shown that, in all the cases where there exists a strictly mixed strategy Nash equilibrium, the equilibrium probability of attending the meeting increases if the quorum increases. Furthermore, if the number of players becomes very large, the equilibrium mixed strategy tends to the value of the quorum.

Various extensions of the model may be worth examining. For example, it may be interesting the consider games where players have heterogeneous preferences. Different groups of players may have different preferences. It may also be possible to study different applications of the same underlying game structure.

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A p p e n d i x

1. General Behaviour of the Function φ

We first derive how the behaviour of the function φ over the interval $[0,\ 1]$ depends on the relative values of d_I, d_{II} and d_{III}. To this end we first prove the following lemma.

Lemma 1**:** *For all values of* $p \in (0,\ 1)$, *and for all integers x,* $1 \le x \le n$, *the following equality holds*

$$\sum_{k=0}^{x-1} \binom{n-1}{k} \left(\frac{1-p}{p}\right)^{x-k} (k - np + p) = (1-p)\, x \binom{n-1}{x} \tag{24}$$

Proof

The proof is by induction on *x*:

(1) We first show that (24) holds for $x = 1$. Starting with the LHS of (24) for $x = 1$ we have

$$\binom{n-1}{0}\left(\frac{1-p}{p}\right)(0 - np + p) = \left(\frac{1-p}{p}\right) p(1-n) = (p-1)1\binom{n-1}{1}$$

The last term is exactly the RHS of (24) for $x = 1$.

(2) We now show that, if (24) holds for any integer *x*, $1 \le x \le n-1$, it must also hold for $x+1$. The LHS of (24) for $x+1$ is given by

$$\sum_{k=0}^{x} \binom{n-1}{k} \left(\frac{1-p}{p}\right)^{x+1-k} (k - np + p) =$$
$$= \frac{1-p}{p}\left[\sum_{k=0}^{x-1} \binom{n-1}{k} \left(\frac{1-p}{p}\right)^{x-k} (k - np + p) + \binom{n-1}{x}(x - np + p)\right]$$

As (24) holds for *x*, this can be written as

$$\frac{1-p}{p}\left[(p-1)x\binom{n-1}{x}+\binom{n-1}{x}(x-np+p)\right]=$$
$$=(p-1)(x+1)\binom{n-1}{x+1}$$

This last expression is the RHS of (24) for $x+1$. Q.E.D.

For the following theorem it is useful to change the notation for the function φ, defined in (4), as follows. The function φ can be written as

$$\varphi(p)=d_I L(p)+d_{II} M(p)+d_{III}\left(1-L(p)-M(p)\right)$$

or as

$$\varphi(p)=b_1 L(p)+b_2 M(p)+b_3 \tag{25}$$

where

$$\begin{aligned} b_1 &= d_I - d_{III} \\ b_2 &= d_{II} - d_{III} \\ b_3 &= d_{III} \end{aligned} \tag{26}$$

We now prove the following theorem.

Theorem 1: *The function φ has at most one critical point in the interval $(0,\ 1)$. It is given by*

$$p_e=\frac{b_1-b_2}{b_1-\dfrac{n-1}{\alpha n-1}b_2} \tag{27}$$

provided

$$b_1\neq\frac{n-1}{\alpha n-1}b_2 \tag{28}$$

and

$$0<p_e=\frac{b_1-b_2}{b_1-\dfrac{n-1}{\alpha n-1}b_2}<1 \tag{29}$$

Moreover, for $p_e\in(0,\ 1)$,

$$\varphi''(p_e)<0\Leftrightarrow b_1<\frac{n-1}{\alpha n-1}b_2 \text{ and } \varphi''(p_e)>0\Leftrightarrow b_1>\frac{n-1}{\alpha n-1}b_2 \tag{30}$$

Proof

Let us introduce the notation

$$x^c = \alpha n - 1 \tag{31}$$

Using (25), the first order derivative of $\varphi(p)$ can be written as

$$\varphi'(p) = p^{x^c-1}(1-p)^{n-2-x^c}\left[b_1\sum_{k=0}^{x^c-1}\binom{n-1}{k}p^{k-x^c}(1-p)^{x^c-k}(k-np+p) + b_2\binom{n-1}{x^c}\left(x^c-np+p\right)\right]$$

Replacing the sum in the square brackets by the RHS of (24) we obtain

$$\varphi'(p) = \binom{n-1}{x^c}p^{x^c-1}(1-p)^{n-2-x^c}\left[-b_1x^c(1-p) + b_2\left(x^c - np + p\right)\right] \tag{32}$$

The only possible value of $p \in (0,\ 1)$ solving the equation $\varphi'(p) = 0$ is given by (27). Starting from (32), and using $\varphi'(p_e) = 0$, we also find that

$$\varphi''(p_e) = \binom{n-1}{x^c}(p_e)^{x^c-1}(1-p_e)^{n-2-x^c}\ x^c\left(b_1 - \frac{n-1}{x^c}b_2\right)$$

Property (32) then easily follows. Q.E.D.

We now consider all possible combinations of inequalities and signs of b_1, b_2. Using (27) – (29), we can easily determine how the general shape of the function φ depends on the parameters b_1 and b_2. Using (26), each such combination of inequalities and signs of b_1, b_2 gives rise to a corresponding combination of inequalities involving d_I, d_{II} and d_{III}. This leads finally to the following conclusion.

$b_1 < b_2 < 0 \Leftrightarrow d_I < d_{II} < d_{III}$: φ is monotonically increasing over $(0,\ 1)$

$b_1 < 0 < b_2 \Leftrightarrow d_I < d_{III} < d_{II}$ and

$0 < b_1 < b_2 \Leftrightarrow d_{III} < d_I < d_{II}$: φ has a unique critical point at $p_e \in (0,\ 1)$ which is a maximum,

$b_2 < b_1 < 0 \Leftrightarrow d_{II} < d_I < d_{III}$ and

$b_2 < 0 < b_1 \Leftrightarrow d_{II} < d_{III} < d_I$: φ has a unique critical point at $p_e \in (0,\ 1)$, which is a minimum,

$0 < b_2 < b_1 \Leftrightarrow d_{III} < d_{II} < d_I$: φ is monotonically decreasing over $(0,\ 1)$.

On the basis of these properties we can consider the behaviour of the function φ for all possible preference profiles. This is done in section 2 of the text.

2. The Existence of Mixed Strategy Nash Equilibria for Preference Profiles (12) and (13)

Consider the special profile

$$d_I = d_{III} < 0 < d_{II} \tag{33}$$

This is the borderline case between the first two profiles in (12) and (13). In terms of the notation (26), (33) is equivalent to

$$b_1 = 0\,,\; b_2 > 0\,,\; b_3 < 0$$

From our previous analysis we know that the function $\varphi(p) = b_2 M(p) + b_3$ has a critical point at

$$p_e = \frac{\alpha n - 1}{n-1} = \frac{x-1}{n-1}$$

where the function attains a maximum. This maximum value is given by

$$\varphi(p_e) = b_2 M(p_e) + b_3$$

where

$$M(p_e) = M\left(\frac{x-1}{n-1}\right) = \binom{x-1}{n-1}\left(\frac{x-1}{n-1}\right)^{x-1}\left(\frac{n-x}{n-1}\right)^{n-x}$$

For any given value of n, this maximum value is a function only of x. We denote this maximal value as $\tilde{M}(x) = M\left(\frac{x-1}{n-1}\right)$. We now investigate the behaviour of $\tilde{M}(x)$ as a function of x, x: 1, 2, ..., n. Clearly, the function $\tilde{M}$ is decreasing at the value x if $\tilde{M}(x) > \tilde{M}(x+1)$.

It is easy to see that this inequality is equivalent to

$$\left(\frac{x-1}{x}\right)^{x-1} > \left(\frac{n-1-x}{n-x}\right)^{n-x-1}$$

Using the transformation $y = n - x$, this inequality can be written as

$$\left(\frac{x-1}{x}\right)^{x-1} > \left(\frac{y-1}{x}\right)^{y-1},\; y = n - x \tag{34}$$

Now consider the behaviour of the LHS of (34), which we denote by $f(x)=\left(\frac{x-1}{x}\right)^{x-1}$. For simplicity we assume that *x* is a continuous variable with $x\in[1,\ \infty)$. The following lemma states some interesting properties of the function $f(x)$.

***Lemma 2**: The function* $f(x)=\left(\frac{x-1}{x}\right)^{x-1}$ *has the following properties:* $f(1)=1$, $\lim\limits_{x\to+\infty} f(x)=\frac{1}{e}$ *and* $\forall x\in[1,\ \infty)$, $\frac{df(x)}{dx}<0$.

Proof

Writing $f(x)=\left(\frac{x-1}{x}\right)^{x-1}=\exp\left[\frac{\ln\left(\frac{x-1}{x}\right)}{\frac{1}{x-1}}\right]$ and applying L'Hospital's rule, we obtain

$$f(1)=\lim_{x\to1,x>1} f(x)=\exp\left[\lim_{x\to1,x>1}\frac{\ln\left(\frac{x-1}{x}\right)}{\frac{1}{x-1}}\right]=\exp\left[\lim_{x\to1,x>1}-\frac{x-1}{x}\right]=\exp(0)=1$$

In the same way we find

$$\lim_{x\to+\infty} f(x)=\exp\left[\lim_{x\to+\infty}-\frac{x-1}{x}\right]=\exp(-1)=\frac{1}{e}$$

As *f* is continuously differentiable in the interval $[1,+\infty)$, we find that

$$\frac{df(x)}{dx}=f(x)\left[\ln\left(\frac{x-1}{x}\right)+\frac{1}{x}\right] \tag{35}$$

Define a function $g(x)$ as

$$g(x)=\ln\left(\frac{x-1}{x}\right)+\frac{1}{x}$$

We then have

$$\frac{dg(x)}{dx}=\frac{1}{x^2(x-1)}$$

and

$$\lim_{x\to 1, x>1} g(x) = \ln(0) + 1 = -\infty$$
$$\lim_{x\to +\infty} g(x) = \ln(1) + 0 = 0 \qquad (36)$$

Hence, the function $g(x)$ is negative and increasing for all $x \in [1, +\infty)$. From (35) it then follows $\frac{df(x)}{dx} < 0$ for all $x \in [1, +\infty)$. Q.E.D.

Returning now to (34), it easily follows that

$$\left(\frac{x-1}{x}\right)^{x-1} < \left(\frac{y-1}{y}\right)^{y-1} \Leftrightarrow x < \frac{n}{2}$$
$$\left(\frac{x-1}{x}\right)^{x-1} = \left(\frac{y-1}{y}\right)^{y-1} \Leftrightarrow x = \frac{n}{2}$$
$$\left(\frac{x-1}{x}\right)^{x-1} > \left(\frac{y-1}{y}\right)^{y-1} \Leftrightarrow x > \frac{n}{2}$$

with $y = n - x$.

Hence, the function $\tilde{M}(x)$ is U-shaped over the interval $[1, n]$: it is decreasing for all $1 < x < n/2$, and it is increasing for all $n/2 < x < n$.This property is summarized in the following theorem.

Theorem 2: *For all preference profiles satisfying* $d_I = d_{III} < 0 < d_{II}$, *the function* $\varphi(p)$ *attains a global maximum over the interval* $[0, 1]$ *at the value* $p_e = \frac{n-1}{x-1}$. *For any given value of n, this maximal value* $\varphi(p_e)$ *is decreasing for all* $1 < x < n/2$*, and it is increasing for all* $n/2 < x < n$.

3. The Dependence of the Mixed Strategy Nash Equilibrium on the Size of the Quorum

We consider the effect of an increase of α from $\alpha^L = \frac{x}{n}$ to $\alpha^H = \frac{x+1}{n}$ on the mixed strategy Nash equilibrium for the preference profiles (8) – (11) and (16) – (17). For any value of $p \in [0, 1]$, the two values of α give rise to two corresponding functions $\varphi^L(p)$ and $\varphi^H(p)$, defined by

$$\varphi^L(p) = d_I \sum_{i=0}^{x-2} f(i) + d_{II} f(x-1) + d_{III} \sum_{i=x}^{n-1} f(i)$$
$$\varphi^H(p) = d_I \sum_{i=0}^{x-1} f(i) + d_{II} f(x) + d_{III} \sum_{i=x+1}^{n-1} f(i) \qquad (37)$$

where $f(i)$ is the probability that i players attend the meeting, given any value of $p \in [0,\ 1]$. Using (7), the critical points of these functions are given by

$$p_{eL} = \frac{(d_I - d_{III}) - (d_{II} - d_{III})}{(d_I - d_{III}) - \frac{n-1}{\alpha^L n - 1}(d_{II} - d_{III})} \qquad (38)$$

and

$$p_{eH} = \frac{(d_I - d_{III}) - (d_{II} - d_{III})}{(d_I - d_{III}) - \frac{n-1}{\alpha^H n - 1}(d_{II} - d_{III})} \qquad (39)$$

We use the notation $p_L{}^*$ and $p_H{}^*$ to denote the two Nash equilibria corresponding to α^L and α^H. Therefore $\varphi^L(p_L{}^*) = 0$ and $\varphi^H(p_H{}^*) = 0$.

We define

$$\Delta\varphi(p) = \varphi^H(p) - \varphi^L(p) = (d_{II} - d_{III}) f(x) + (d_I - d_{II}) f(x-1) \qquad (40)$$

Using the binomial density function, we know that

$$f(x) = f(x-1)\frac{p}{1-p}\frac{n-1-(x-1)}{x}$$

This allows us to write

$$\Delta\varphi(p) = f(x-1)\left[(d_{II} - d_{III})\frac{p}{1-p}\frac{n-x}{x} + (d_I - d_{II})\right] \qquad (41)$$

Provided $(d_I - d_{II}) \neq \frac{n-x}{x}(d_{II} - d_{III})$, there is a unique value of p where $\Delta\varphi(p)$ is zero. It is given by

$$\tilde{p} = \frac{(d_I - d_{III}) - (d_{II} - d_{III})}{(d_I - d_{III}) - \frac{1}{\alpha^L}(d_{II} - d_{III})}$$

Hence, at the value $\tilde{p}$, the two functions $\varphi^L(p)$ and $\varphi^H(p)$ cross each other.

We now want to determine the sign of $\Delta\varphi(p) = \varphi^H(p) - \varphi^L(p)$. We know that in the preference profiles (9) and (10) $d_{II} - d_{III} > 0$ and $d_I - d_{II} > 0$. It then follows from (40) that $\varphi^H(p) > \varphi^L(p)$ for all $p \in (0, 1)$. See Figures 1a and 1b. It is easy to show that in the preference profiles (8), (11), (16) and (17), we have

$$0 < p_{eL} < \tilde{p} < p_{eH} < 1 \tag{42}$$

From (41) it follows that, for small values of $p \in (0, 1)$, the sign of $\Delta\varphi(p) = \varphi^H(p) - \varphi^L(p)$ is given by the sign of $d_I - d_{II}$.

All this leads to the following conclusions. For preference profile (8), $0 < \tilde{p} < 1$, and $\varphi^H(p) > \varphi^L(p)$ for all $0 < p < \tilde{p}$, while $\varphi^H(p) < \varphi^L(p)$ for all $p < \tilde{p} < 1$. As $0 < p*_L < p_{eL} < \tilde{p}$, we must have $\varphi^L(p_L*) = 0 < \varphi^H(p_L*)$, so that $p_H* > p_L*$. For preference profiles (9) and (10), we already know that $\varphi^H(p) > \varphi^L(p)$ for all $p \in (0, 1)$. It follows that $p_H* > p_L*$. For preference profiles (11), (16) and (17), $0 < \tilde{p} < 1$, and $\varphi^H(p) < \varphi^L(p)$ for all $0 < p < \tilde{p}$, while $\varphi^H(p) > \varphi^L(p)$ for all $\tilde{p} < p < 1$. As $\tilde{p} < p_{eH} < p_H* < 1$, we must have $\varphi^L(p*_H) < \varphi^H(p*_H) = 0$, so that $p_H* > p_L*$.

All these properties are illustrated in Figures 1 and 2. All these results are summarized in the following theorem.

Theorem 3*: For all preference profiles (8) – (11) an increase of the quorum always increases the equilibrium mixed strategy p*. The same is true for the preference profiles (16) – (17), provided there exists a mixed strategy Nash equilibrium.*

4. The Dependence of the Mixed Strategy Nash Equilibrium on the Number of Players

For large values of *n* the binomial distribution $\binom{n-1}{k} p^k (1-p)^{n-1-k}$ can be approximated by the normal distribution with mean $(n-1)p$ and variance $(n-1)p(1-p)$. It follows that for large values of *n* the following approximations apply $L(p) = C(k \le \alpha n - 2)$, $M(p) = 0$, and $R(p) = 1 - C(k \le \alpha n - 2)$, where *C* is the cumulative distribution function of the normal distribution, and where $L(p)$, $M(p)$ and $R(p)$ are as given in (1), (2), (3), respectively. Define now $\tilde{k} = \frac{k}{\alpha n - 2}$, and write $C\left(k \le \alpha n - 2\right)$ as $\tilde{C}\left(\tilde{k} \le 1\right)$. C is the cumulative distribution function of the normal distribution with mean $\frac{\left(n-1\right)p}{\alpha n - 2}$ and variance $\frac{\left(n-1\right)p\left(1-p\right)}{\left(\alpha n - 2\right)^2}$. Hence, for large values of *n*, the equation $\varphi\left(p\right) = 0$ can be approximated by

$$d_I \tilde{C}\left(\tilde{k} \le 1\right) + d_{III}\left[1 - \tilde{C}\left(\tilde{k} \le 1\right)\right] = 0$$

or

$$\tilde{C}\left(\tilde{k} \le 1\right) = \frac{d_{III}}{d_{III} - d_I} \tag{43}$$

Note that for preference profiles in class one $0 < \frac{d_{III}}{d_{III} - d_I} < 1$.

As *n* tends to infinity, the mean of *k* approaches p/α, and its variance approaches zero. Therefore, the function $\tilde{C}$ exhibits an almost discrete jump from 0 to 1 in the interval $\left(\frac{p}{\alpha} - \varepsilon,\ \frac{p}{\alpha} + \varepsilon\right)$ for any small value of $\varepsilon > 0$. It then follows that $\frac{p^*}{\alpha} = 1$, or that $p^* = \alpha$. From (43) it also follows that the probability that the quorum is attained in equilibrium equals $\frac{d_{III}}{d_{III} - d_I}$. The following theorem summarizes all these results.

Theorem 4: *If the number of players becomes very large, then, for preference profiles of class one, the equilibrium mixed strategy p* tends to the value of the quorum α. Moreover, the probability that the quorum is attained in the equilibrium equals* $\frac{d_{III}}{d_{III} - d_I}$.