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ASYMPTOTIC ESTIMATE FOR DIFFERENTIAL EQUATION WITH POWER COEFFICIENTS AND POWER DELAYS

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ABSTRACT. The paper analyzes the asymptotic bounds of solutions of differential equation with power coefficients, power delays and a forcing term in the form $\dot{y}(t) = \sum_{j=0}^{m} a_j t^{\alpha_j} y(t^{\lambda_j}) + f(t)$, where a_0 is a negative real, $\lambda_0 = 1$ and $0 < \lambda_i < 1$, $i = 1, \dots, m$. Some additional assumptions on power coefficients and a forcing term $f(t)$ are considered to obtain an asymptotic estimate for solutions of the studied differential equation. The application of the result is illustrated by several examples.

1. Introduction

The paper deals with the asymptotic estimate of solutions of the differential equation with several power delays and a forcing term in the form

$$\dot{y}(t) = \sum_{j=0}^{m} a_j t^{\alpha_j} y(t^{\lambda_j}) + f(t), \qquad t \in I := [t_0, \infty), \quad t_0 \geq 1, \tag{1}$$

where

$a_0 < 0,\ \lambda_0 = 1,\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m < 1,\ a_i, \alpha_i \in \mathbb{R},\ i = 1, \dots, m$ *and* f is a continuous function on I.

The asymptotic behaviour of solutions of delay differential equations is still widely investigated as evidenced by many papers dealing with this topic, see, e.g.,

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Iserles [5], [6], Liu [7] or Makay and Terjéki [8]. To recall some papers closely related to the studied problem we mention Čermák [2], where the equation with power coefficients and a proportional delays have been studied. Paper [1] gives asymptotic bounds of solutions of differential equation with a more general form of coefficients and delays and with a forcing term

$$\dot{y}(t) = \sum_{j=0}^{m} a_j(t) y\big(\tau_j(t)\big) + f(t), \qquad t \in [t_0, \infty),$$

where $a_0(t) < 0$ and $a_i(t)$, $i = 1, \dots, m$ are continuous functions, $\tau_0(t) = t$, $\tau_i(t) < t$, $i = 1, \dots, m$ and some additional assumptions have been considered. The following analysis of the asymptotics of the equation (1) takes advantage of the approach utilized in [1].

The main goal of the paper is to formulate the asymptotic estimate of (1) and discuss the obtained result on several examples. The structure of the paper is the following: Section 2 introduces some auxiliary statements. In Section 3 is derived the asymptotic estimate of delay differential equation (1) and Section 4 illustrates the obtained result on several examples and gives some final remarks.

2. Preliminaries

In this section we introduce some fundamentals connected with the studied problem. First, we recall the notion of solution of (1). Denote

$$t_{-1} := t_0^{\lambda_1} \quad \text{and} \quad I_{-1} := [t_{-1}, \infty).$$

By a solution of (1) we understand a real-valued function $y \in C(I_{-1}) \cap C^1(I)$ such that y satisfies (1) on I.

Analogously, as in papers [1] or [2], we utilize a system of auxiliary functional equations, which can be in our special case of power delays rewritten as

$$\psi(t^{\lambda_i}) = \psi(t) - \log \lambda_i^{-1}, \qquad t \in [1, \infty), \quad i = 1, \dots, m. \tag{2}$$

One can see that the common solution of the system (2) is

$$\psi(t) = \log \log t. \tag{3}$$

The existence of the common solution of (2) enables us to embed the system of delayed arguments $\{t^{\lambda_1}, \dots, t^{\lambda_m}\}$ into an iteration group $[\psi]$. For more information about this topic see, e.g., Neuman [9].

3. Asymptotic estimate of the equation (1)

The aim of this section is to formulate and prove the asymptotic estimate of solutions of the delay differential equation (1).

Theorem 1. *Let y be a solution of (1), where $-a_0 t^{\alpha_0} \geq K(\log^{-\eta} t)$, $0 < \sum_{i=1}^{m} |a_i t^{\alpha_i}| \leq -M a_0 t^{\alpha_0}$ for all $t \in I$ and suitable real constants $K > 0$, $M > 0$, $\eta < 1$. Let $f(t) \in C(I)$ be such that $f(t) = O(\log^{\nu} t)$ as $t \to \infty$ for a suitable real ν. Then*

$$y(t) = O(\log^{\gamma} t) \quad \text{as } t \to \infty, \quad \gamma > \max\left\{\eta + \nu, \frac{\log M}{\log \lambda_1^{-1}}, \dots, \frac{\log M}{\log \lambda_m^{-1}}\right\}. \tag{4}$$

P r o o f. First we transform the equation (1). Utilizing the substitution (arising from the common solution of the auxiliary system (2))

$$s = \psi(t) = \log\log(t), \qquad z(s) = \exp\big\{-\gamma \log\log(t)\big\} y(t) \tag{5}$$

we obtain

$$\begin{aligned} z'(s) = {} & \Big[a_0\big(h(s)\big)^{\alpha_0} h'(s) - \gamma\Big] z(s) \\ & + \sum_{i=1}^{m} a_i \big(h(s)\big)^{\alpha_i} \exp\big\{-\gamma \log \lambda_i^{-1}\big\} h'(s) z\big(\mu_i(s)\big) \\ & + f\big(h(s)\big) \exp\{-\gamma s\} h'(s), \end{aligned}$$

where "$'$" means the differentiation with respect to s, $h(s) = \psi^{-1}(s) = \exp\exp(s)$ and $\mu_i(s) = s - \log \lambda_i^{-1}$ on $\psi(I)$, $i = 1, \dots, m$. We reorganize the equation into the form

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}s}\left[\exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} z(s)\right] \\ & = \sum_{i=1}^{m} a_i \big(h(s)\big)^{\alpha_i} \exp\big\{-\gamma \log \lambda_i^{-1}\big\} h'(s) \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} z\big(\mu_i(s)\big) \\ & \quad + \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} f\big(h(s)\big) \exp\{-\gamma s\} h'(s), \end{aligned} \tag{6}$$

where $s_0 \in \psi(I)$ is such that

$$\gamma > a_0\big(h(s)\big)^{\alpha_0} h'(s) \quad \text{for all } s \geq s_0.$$

Denote

$$s_k := s_0 + k\lambda_m^{-1}, \quad J_k := [s_{k-1}, s_k], \quad k = 1, 2, \dots$$

Let $s^* \in J_{k+1}$. The integration of (6) over $[s_k, s^*]$ gives

$$\begin{aligned}
&\exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0} \mathrm{d}u\right\} z(s)\Bigg|_{s_k}^{s^*} \\
&= \sum_{i=1}^{m} \int_{s_k}^{s^*} a_i\big(h(s)\big)^{\alpha_i} \exp\{-\gamma \log \lambda_i^{-1}\} h'(s) \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0} \mathrm{d}u\right\} z\big(\mu_i(s)\big)\,\mathrm{d}s \\
&\quad + \int_{s_k}^{s^*} \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0} \mathrm{d}u\right\} f\big(h(s)\big) \exp\{-\gamma s\} h'(s)\,\mathrm{d}s\,.
\end{aligned}$$

We express the value of z in the instant $s^* \in J_{k+1}$ as

$$\begin{aligned}
&z(s^*) \\
&= \exp\left\{\gamma(s_k - s^*) + \int_{h(s_k)}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} z(s_k) \\
&\quad + \exp\left\{\int_{s_0}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u - \gamma s^*\right\} \\
&\quad \times \sum_{i=1}^{m} \int_{s_k}^{s^*} a_i\big(h(s)\big)^{\alpha_i} \exp\{-\gamma \log \lambda_i^{-1}\} h'(s) \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} z\big(\mu_i(s)\big)\,\mathrm{d}s \\
&\quad + \exp\left\{\int_{s_0}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u - \gamma s^*\right\} \\
&\quad \times \int_{s_k}^{s^*} \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} f\big(h(s)\big) \exp\{-\gamma s\} h'(s)\,\mathrm{d}s.
\end{aligned}$$

Denote

$$M_k := \max\bigl\{|z(s)|, \quad s \in \cup_{p=1}^{k} J_p\bigr\}, \quad k = 1, 2, \ldots$$

Then the value of $z(s^*)$ can be estimated as

$$\begin{aligned}
&|z(s^*)| \\
&\le M_k \exp\left\{\gamma(s_k - s^*) + \int_{h(s_k)}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} \\
&\quad + M_k \exp\left\{\int_{s_0}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u - \gamma s^*\right\} \\
&\quad \times \int_{s_k}^{s^*} \sum_{i=1}^{m} \bigl|a_i\bigl(h(s)\bigr)^{\alpha_i}\bigr| \exp\bigl\{-\gamma \log \lambda_i^{-1}\bigr\} h'(s) \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} \mathrm{d}s \\
&\quad + \exp\left\{\int_{s_0}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u - \gamma s^*\right\} \\
&\quad \times \int_{s_k}^{s^*} \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} \bigl|f\bigl(h(s)\bigr)\bigr| \exp\{-\gamma s\} h'(s)\,\mathrm{d}s. \qquad (7)
\end{aligned}$$

Further, considering the relations

$$\begin{aligned}
&\sum_{i=1}^{m} \bigl|a_i\bigl(h(s)\bigr)^{\alpha_i}\bigr| \exp\bigl\{-\gamma \log \lambda_i^{-1}\bigr\} \\
&\le M \exp\bigl\{-\gamma \log \lambda_i^{-1}\bigr\} \Bigl(-a_0\bigl(h(s)\bigr)^{\alpha_0}\Bigr) \\
&\le -a_0\bigl(h(s)\bigr)^{\alpha_0}
\end{aligned}$$

and

$$\bigl|f\bigl(h(s)\bigr)\bigr| \exp\{-\gamma s\} \le K_1 \exp\bigl\{(\vartheta - \gamma)s\bigr\}, \qquad K_1 > 0$$

the estimate (7) can be rewritten as

$$
\begin{aligned}
&|z(s^*)| \\
&\le M_k \exp\left\{\gamma(s_k - s^*) + \int\limits_{h(s_k)}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} \\
&\quad + M_k \exp\left\{\int\limits_{s_0}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u - \gamma s^*\right\} \\
&\quad \times \int\limits_{s_k}^{s^*} \Big[-a_0\big(h(s)\big)^{\alpha_0}\Big] h'(s) \exp\left\{\gamma s - \int\limits_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\}\mathrm{d}s \\
&\quad + K_1 \exp\left\{\int\limits_{s_0}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u - \gamma s^*\right\} \\
&\quad \times \int\limits_{s_k}^{s^*} \exp\left\{\gamma s - \int\limits_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} h'(s) \exp\big\{(\vartheta - \gamma)s\big\}\,\mathrm{d}s.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|z(s^*)| \\
&\le M_k \exp\left\{\gamma(s_k - s^*) + \int\limits_{h(s_k)}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} \\
&\quad + \Big(M_k + K_2 \exp\big\{(\eta + \vartheta - \gamma)s_k\big\}\Big) \exp\left\{\int\limits_{s_0}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u - \gamma s^*\right\} \\
&\quad \times \int\limits_{s_k}^{s^*} \Big[-a_0\big(h(s)\big)^{\alpha_0}\Big] h'(s) \exp\left\{\gamma s - \int\limits_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\}\mathrm{d}s,
\end{aligned}
$$

where $K_2 = K_1/K$.

Since a_0 is negative, the integral on the right-hand side can be estimated as

$$\int_{s_k}^{s^*} \Big[-a_0\big(h(s)\big)^{\alpha_0}\Big] h'(s) \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\} \mathrm{d}s$$
$$\leq \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\}\Bigg|_{s_k}^{s^*} (1 + K_3 \mathrm{e}^{-\omega s_k}), \qquad K_3 > 0, \quad \omega = 1 - \eta > 0.$$

Then we obtain

$$|z(s^*)|$$
$$\leq M_k \exp\left\{\gamma(s_k - s^*) + \int_{h(s_k)}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\}$$
$$+ \Big(M_k + K_2 \exp\big\{(\eta + \vartheta - \gamma)s_k\big\}\Big) \exp\left\{\int_{s_0}^{h(s^*)} a_0 u^{\alpha_0}\,\mathrm{d}u - \gamma s^*\right\}$$
$$\times \exp\left\{\gamma s - \int_{s_0}^{h(s)} a_0 u^{\alpha_0}\,\mathrm{d}u\right\}\Bigg|_{s_k}^{s^*} \big(1 + K_3 \exp\{-\omega s_k\}\big)$$
$$\leq M_k\Big(1 + M \exp\big\{-\omega s_k\big\}\Big) + K_2 \exp\big\{(\eta + \vartheta - \gamma)s_k\big\}\big(1 + K_3 \exp\{-\omega s_k\}\big)$$
$$\leq M_k^*\big(1 + N \exp\{-\kappa s_k\}\big),$$

where

$$M_k^* = \max\,(M_k, K_2) \quad \text{and} \quad \kappa = \min\,(\omega, \gamma - \eta - \vartheta) > 0 \quad \text{and} \quad N > 0$$

is a constant large enough. Since $s^* \in J_{k+1}$ was arbitrary,

$$M_{k+1}^* \leq M_k^*\big(1 + N \exp\{-\kappa s_k\}\big) \leq M_1^* \prod_{j=1}^{k} \big(1 + N \exp\{-\kappa s_j\}\big).$$

The boundedness of the sequence (M_k^*) as $k \to \infty$ gives the asymptotic estimate (4) with respect to the substitution (5). □

4. Examples and final remarks

First we introduce such a case of differential equation (1) that there are considered constant coefficients and one delayed term in the equation. Furthermore, the forcing term is omitted in this case:

EXAMPLE 1. Consider the equation

$$\dot{y}(t) = a_0 y(t) + a_1 y(t^{\lambda_1}), \qquad t \in [1, \infty), \tag{8}$$

where $a_0 < 0$, a_1 are real constants, $0 < \lambda_1 < 1$. Then Theorem 1 gives the asymptotic estimate

$$y(t) = O(\log^\gamma t), \qquad \text{as} \quad t \to \infty, \quad \gamma = \frac{\log \frac{|a_1|}{-a_0}}{\log \lambda_1^{-1}}.$$

The above asymptotic estimate coincides with the result of Heard [4], which can be reformulated for equation (8) as follows.

THEOREM 2 (Heard, 1975). *Let $a_0 < 0$, $a_1 \neq 0$ be real constants, $0 < \lambda_1 < 1$. Then for every solution of (8) there exists a continuous periodic function g with period $\log \lambda_1^{-1}$ such that*

$$y(t) = (\log t)^\xi g(\log \log t) + O\big(\log^{\xi_r - 1} t\big) \qquad \text{as} \quad t \to \infty,$$

where ξ is a root of $a_0 + a_1 \lambda_1^\xi = 0$ and $\xi_r = \Re(\xi)$.

The next example illustrates the application of the Theorem 1 to the equation (1) with nonconstant coefficients at delayed terms.

EXAMPLE 2. Consider the equation

$$\dot{y}(t) = a_0 y(t) + \frac{1}{t}\Big[a_1 y\big(t^{\lambda_1}\big) + a_2 y\big(t^{\lambda_2}\big)\Big] + f(t), \tag{9}$$

where $a_0 < 0$, a_1, a_2 are real constants, $0 < \lambda_1 < \lambda_2 < 1$ and $f(t) = O(\log^\nu t)$ as $t \to \infty$ for a suitable real ν. Then in accordance with Theorem 1 we obtain the asymptotic estimate of solution of (9) in the form

$$y(t) = O(\log^\gamma t), \quad \text{as} \quad t \to \infty,$$

where

$$\gamma = \max\left\{\nu, \frac{\log \frac{|a_1| + |a_2|}{-a_0}}{\log \lambda_1^{-1}}, \frac{\log \frac{|a_1| + |a_2|}{-a_0}}{\log \lambda_2^{-1}}\right\}.$$

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