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OSCILLATORY CRITERIA FOR TWO-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS

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ABSTRACT. Some oscillation criteria are established for two-dimensional systems of first order linear difference equations.

1. Introduction and notation

This paper is devoted to the oscillatory properties of two-dimensional system of linear difference equations

$$\begin{aligned} \Delta u_k &= q_k v_k\,, \\ \Delta v_k &= -\,p_k u_{k+1}, \end{aligned} \tag{1}$$

where

$$\Delta x_k = x_{k+1} - x_k, \quad p_k, q_k \in R \qquad \text{for} \quad k \in \mathbb{N}.$$

System (1) is one of the possible discrete analogies of the linear Hamiltion system of differential equations

$$\begin{aligned} u' &= q(t)v, \\ v' &= -\,p(t)u, \end{aligned}$$

where $p(t)$, $q(t)$ are continuous functions defined on $[t_0, +\infty)$.

By a solution of system (1) we understand a vector sequence $\{(u_k, v_k)\}_{k=1}^{+\infty}$ satisfying system (1) for every natural k. A nontrivial solution $\{(u_k, v_k)\}_{k=1}^{+\infty}$ of system (1) is said to be oscillatory if there exists an infinite set $N_0 \subseteq N$ such that

$$u_k u_{k+1} \le 0 \qquad \text{for} \quad k \in N_0.$$

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It is known (see, e.g., [1]) that if

$$q_k \geq 0 \qquad \text{for} \quad k \in N$$

and system (1) has at least one oscillatory solution, then all its solutions are oscillatory. That is why we can introduce the following definition.

Definition 1.1. System (1) is said to be oscillatory if all its solutions are oscillatory, otherwise (1) is said to be non-oscillatory.

Oscillatory properties of system (1) are known in the case where

$$0 < m \leq q_k \qquad \text{for} \quad k \in \mathbb{N} \qquad \text{and} \qquad \sum_{j=1}^{+\infty} p_j = +\infty$$

hold (see, e.g., [1]) or in the case where following conditions

$$0 < m \leq q_k \qquad \text{for} \quad k \in \mathbb{N} \qquad \text{and} \qquad -\infty = \liminf_{k\to\infty} \sum_{j=1}^{k} p_j < \limsup_{k\to\infty} \sum_{j=1}^{k} p_j,$$

are fulfilled. System (1) is oscillatory in both mentioned cases.

In this paper sufficient conditions guaranteeing that the system (1) is oscillatory are established for cases where the series $\sum_{j=1}^{+\infty} p_j$ converges to a finite number, i.e.,

$$\left| \sum_{j=1}^{+\infty} p_j \right| < +\infty \tag{2}$$

and

$$0 < m \leq q_k \leq M < +\infty \qquad \text{for} \quad k \in N, \tag{3}$$

where m, M are real positive constants.

In what follows we will suppose that there exists a finite limit

$$c_0 = \lim_{k\to\infty} c_k,$$

where

$$c_k = \frac{1}{\sum\limits_{j=1}^{k-1} q_j} \sum_{j=1}^{k-1} q_j \sum_{i=1}^{j-1} p_i \qquad \text{for} \quad k \in N.$$

Let us introduce following notations

$$Q_k = \left(c_0 - \sum_{j=1}^{k-1} p_j \right) \sum_{j=1}^{k-1} q_j = \sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} p_j \qquad \text{for} \quad k \in \mathbb{N},$$

$$H_k = \frac{1}{\sum\limits_{j=1}^{k-1} q_j} \sum_{j=1}^{k} p_j \left(\sum_{i=1}^{j} q_i \right)^2 \qquad \text{for} \quad k \in \mathbb{N},$$

$$Q_* = \liminf_{k\to\infty} Q_k, \qquad Q^* = \limsup_{k\to\infty} Q_k,$$

$$H_* = \liminf_{k\to\infty} H_k, \qquad H^* = \limsup_{k\to\infty} H_k.$$

2. Main results

Below formulated theorems generalize and make more complete previous well--known criteria of analogous types. Presented results can be understood as a difference analogy of oscillatory theorems for ordinary differential equations which can be found in [3]–[5].

Theorem 2.1. Let

$$0 \le Q_* \le \frac{1}{4} \quad \text{and} \quad H^* > \frac{1}{2}\Big(1 + \sqrt{1 - 4Q_*}\Big). \tag{4}$$

Then system (1) is oscillatory.

Theorem 2.2. Let

$$0 \le H_* \le \frac{1}{4} \quad \text{and} \quad Q^* > \frac{1}{2}\Big(1 + \sqrt{1 - 4H_*}\Big). \tag{5}$$

Then system (1) is oscillatory.

3. Auxiliary proposition

We establish some properties of solutions of equation (1) in this section. These properties are used to prove main results in what follows.

Lemma 3.1. Let $\big\{(u_k, v_k)\big\}_{k=k_0}^{+\infty}$ be a nonoscillatory solution of system (1). Then

$$\sum^{\infty} R_j < +\infty, \tag{6}$$

where

$$w_j = \frac{v_j}{u_j} \quad \text{and} \quad R_j = \frac{w_j^2 q_j}{1 + w_j q_j}. \tag{7}$$

P r o o f. We suppose on the contrary that

$$\sum^{\infty} R_j = +\infty\,. \tag{8}$$

Let us put $w_k = \frac{v_k}{u_k}$ for $k \geq k_0$, then we can rewrite system (1) as follows

$$\Delta w_k + p_k + R_k = 0 \qquad \text{for} \quad k \geq k_0, \tag{9}$$

where R_k is defined by (7). If we summarize (9) from k_0 to $k-1$, we get

$$-w_k + w_{k_0} = \sum_{j=k_0}^{k-1} p_j + \sum_{j=k_0}^{k-1} R_j \qquad \text{for} \quad k > k_0.$$

The multiplication of the last equality by q_k and the summarization from k_0 to $k-1$ results in

$$\sum_{j=k_0}^{k-1} -w_j q_j = \sum_{j=k_0}^{k-1} q_j \sum_{i=k_0}^{j-1} p_i + \sum_{j=k_0}^{k-1} q_j \sum_{i=k_0}^{j-1} R_i - w_{k_0} \sum_{j=k_0}^{k-1} q_j \quad \text{for } \ k > k_0. \tag{10}$$

Hence,

$$\frac{\sum\limits_{j=k_0}^{k-1} -w_j q_j}{\sum\limits_{j=k_0}^{k-1} q_j} = \frac{\sum\limits_{j=k_0}^{k-1} q_j \sum\limits_{i=k_0}^{j-1} p_i}{\sum\limits_{j=k_0}^{k-1} q_j} + \frac{\sum\limits_{j=k_0}^{k-1} q_j \sum\limits_{i=k_0}^{j-1} R_i}{\sum\limits_{j=k_0}^{k-1} q_j} + w_{k_0} \qquad \text{for } \ k > k_0. \tag{11}$$

Consequently, in view of the assumptions (2) and (8), we get from (11)

$$\limsup_{k\to\infty} \frac{\sum\limits_{j=k_0}^{k-1} -w_j q_j}{\sum\limits_{j=k_0}^{k-1} q_j} = +\infty. \tag{12}$$

On the other hand, it is clear that

$$R_k = \frac{w_k^2 q_k}{w_k q_k + 1} = \frac{w_k^2}{\frac{1}{q_k} + w_k} \geq 0 \qquad \text{for} \quad k \geq k_0. \tag{13}$$

Put

$$A_k = \begin{cases} \frac{w_k^2 q_k}{R_k} & \text{for} \quad k w_k \neq 0, \\ 0 & \text{for} \quad k w_k = 0. \end{cases}$$

Obviously, $A_k \geq 0$ for $k \geq k_0$ and

$$1 \geq A_k - w_k q_k. \tag{14}$$

The summarization of latter inequalities from k_0 to $k-1$ and (3) yield

$$\frac{k-k_0}{\sum\limits_{j=k_0}^{k-1} q_j} \geq \frac{\sum\limits_{j=k_0}^{k-1} A_j}{\sum\limits_{j=k_0}^{k-1} q_j} - \frac{\sum\limits_{j=k_0}^{k-1} w_j q_j}{\sum\limits_{j=k_0}^{k-1} q_j} \qquad \text{for} \quad k > k_0. \tag{15}$$

Further, the condition (3) implies

$$\frac{k-k_0}{\sum\limits_{j=k_0}^{k-1} q_j} \leq \frac{k-k_0}{\sum\limits_{j=k_0}^{k-1} m} = \frac{1}{m}.$$

Hence, by virtue of (15), we get

$$\frac{1}{m} \geq \frac{\sum\limits_{j=k_0}^{k-1} A_j}{\sum\limits_{j=k_0}^{k-1} q_j} - \frac{\sum\limits_{j=k_0}^{k-1} w_j q_j}{\sum\limits_{j=k_0}^{k-1} q_j} \qquad \text{for} \quad k, \quad k > k_0.$$

In view of $A_j \geq 0$ for $j \geq k_0$ the last inequality yields

$$\limsup_{k\to\infty} \frac{\sum\limits_{j=k_0}^{k-1} -w_j q_j}{\sum\limits_{j=k_0}^{k-1} q_j} < +\infty,$$

which contradicts (12). □

Lemma 3.2. Let $0 \leq Q_* \leq \frac{1}{4}$ and $\big\{(u_k, v_k)\big\}_{k=k_0}^{+\infty}$ be a nonoscillatory solution of system (1). Then

$$\liminf_{k\to\infty} \frac{v_k}{u_k} \sum_{j=1}^{k-1} q_j \geq \frac{1}{2}\Big(1 - \sqrt{1-4Q_*}\Big). \tag{16}$$

Proof. Let us put $w_k = \frac{v_k}{u_k}$ for $k \geq k_0$. The sum of (9) from k to l results in

$$w_k - w_{l+1} = \sum_{j=k}^{l} p_j + \sum_{j=k}^{l} R_j \qquad \text{for} \quad k \geq k_0. \tag{17}$$

According to Lemma 3.1, it is clear that $\lim_{j\to\infty} w_j = 0$. Therefore we get from (17)

$$w_k = \sum_{j=k}^{\infty} p_j + \sum_{j=k}^{\infty} R_j \qquad \text{for} \quad k \geq k_0. \tag{18}$$

Let us denote

$$A = \liminf_{k\to\infty} w_k \sum_{j=1}^{k-1} q_j. \tag{19}$$

Obviously, if $A = +\infty$, then (16) holds.

Let us assume that $A < +\infty$. Now if $Q_* = 0$, then, in view of (13) and (18), the inequality (16) holds.

Let $Q_* > 0$. Obviously, for every $\varepsilon \in]0, Q_*[$ there exists $k^{[\varepsilon]} > k_0$ such that

$$Q_k > Q_* - \varepsilon \qquad \text{for} \quad k \geq k^{[\varepsilon]}. \tag{20}$$

Hence, by virtue of (18), we get

$$w_k \sum_{j=1}^{k-1} q_j > Q_* - \varepsilon \qquad \text{for} \quad k \geq k^{[\varepsilon]}.$$

and $A \geq Q_*$. Now we can choose $k_0^{[\varepsilon]} \geq k^{[\varepsilon]}$ such that the inequalities

$$w_k \sum_{j=1}^{k-1} q_j \geq A - \varepsilon, \quad |w_k q_k| \leq \varepsilon \qquad \text{for} \quad k, \quad k \geq k_0^{[\varepsilon]} \tag{21}$$

are fulfilled. Further, we can rewrite equality (18) as follows

$$w_k \sum_{j=1}^{k-1} q_j = \sum_{j=k}^{\infty} p_j \sum_{j=1}^{k-1} q_j + \sum_{j=k}^{\infty} R_j \sum_{j=1}^{k-1} q_j \qquad \text{for} \quad k, \quad k \geq k_0. \tag{22}$$

The inequalities (20)–(22) yield

$$w_k \sum_{j=1}^{k-1} q_j \geq Q_* - \varepsilon + \frac{(A-\varepsilon)^2}{1+\varepsilon} \qquad \text{for} \quad k, \quad k \geq k_0^{[\varepsilon]},$$

hence

$$A \geq Q_* - \varepsilon + \frac{(A-\varepsilon)^2}{1+\varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary we can rewrite the last inequality in the form

$$A^2 - A + Q_* \leq 0,$$

i.e.,

$$A \geq \frac{1}{2}\Big(1 - \sqrt{1 - 4Q_*}\Big).$$

□

Lemma 3.3. Let $0 \leq H_* \leq \frac{1}{4}$ and $\big\{(u_k, v_k)\big\}_{k=k_0}^{+\infty}$ be a nonoscillatory solution of system (1). Then

$$\limsup_{k\to\infty} \frac{v_k}{u_k} \sum_{j=1}^{k-1} q_j \leq \frac{1}{2}\Big(1 + \sqrt{1 - 4H_*}\Big). \tag{23}$$

Proof. Let us put $w_k = \frac{v_k}{u_k}$ for $k \geq k_0$. We get from (9)

$$\Delta w_k \left(\sum_{j=1}^{k} q_j \right)^2 = -p_k \left(\sum_{j=1}^{k} q_j \right)^2 - R_k \left(\sum_{j=1}^{k} q_j \right)^2 \qquad \text{for} \quad k \geq k_0.$$

The summation of the last equalities from n to $k-1$, where $n \geq k_0$, implies

$$\sum_{j=n}^{k-1} \Delta w_j \left(\sum_{i=1}^{j} q_i \right)^2 = -\sum_{j=n}^{k-1} p_j \left(\sum_{i=1}^{j} q_i \right)^2 - \sum_{j=n}^{k-1} R_j \left(\sum_{i=1}^{j} q_i \right)^2 \quad \text{for} \quad k > k_0. \tag{24}$$

Obviously,

$$\sum_{j=n}^{k-1} \Delta w_j \left(\sum_{i=1}^{j} q_i \right)^2 \tag{25}$$

$$= \left(\sum_{j=1}^{k-1} q_j \right)^2 w_k - \left(\sum_{j=1}^{n-1} q_j \right)^2 w_n - \sum_{j=n}^{k-1} w_j q_j \left(2 \sum_{i=1}^{j-1} q_i + q_j \right), \qquad \text{for} \quad k > k_0.$$

By virtue of (25), the equality (24) yields

$$\left(\sum_{j=1}^{k-1} q_j \right)^2 w_k = -\sum_{j=n}^{k-1} p_j \left(\sum_{i=1}^{j} q_i \right)^2 + \left(\sum_{j=1}^{n-1} q_j \right)^2 w_n$$
$$+ \sum_{j=n}^{k-1} \left[w_j q_j \left(2 \sum_{i=1}^{j-1} q_i + q_j \right) - R_j \left(\sum_{i=1}^{j} q_i \right)^2 \right] \quad \text{for} \quad k > k_0.$$

Hence

$$w_k \left(\sum_{j=1}^{k-1} q_j \right) = -H_k + \frac{1}{\sum\limits_{j=1}^{k-1} q_j} \sum_{j=n}^{k-1} D_J + P_{k,n} \qquad \text{for} \quad k > k_0, \tag{26}$$

where

$$D_j = w_j q_j \left(2 \sum_{i=1}^{j-1} q_i + q_j \right) - R_j \left(\sum_{i=1}^{j} q_i \right)^2 \tag{27}$$

and

$$P_{k,n} = \frac{1}{\sum\limits_{j=1}^{k-1} q_j} \left(\sum_{j=1}^{n-1} q_j \right)^2 w_n + \frac{1}{\sum\limits_{j=1}^{k-1} q_j} \sum_{j=1}^{n-1} p_j \left(\sum_{i=1}^{j} q_i \right)^2.$$

Obviously, $D_j \le q_j$ for $j \ge k_0$. Thus, it follows from (26) that

$$\left(\sum_{j=1}^{k-1} q_j\right) w_k \le -H_k + 1 + P_{k,n} \qquad \text{for} \quad k > k_0, \tag{28}$$

Further,

$$\limsup_{k\to\infty} P_{k,n} = \limsup_{k\to\infty} \left(\frac{1}{\sum\limits_{j=1}^{k-1} q_j} \left(\sum_{j=1}^{n-1} q_j\right)^2 w_n + \frac{1}{\sum\limits_{j=1}^{k-1} q_j} \sum_{j=1}^{n-1} p_j \left(\sum_{i=1}^{j} q_i\right)^2 \right) = 0. \tag{29}$$

Hence, on account of (28), we get

$$B \le 1 - H_*,$$

where

$$B = \limsup_{k\to\infty} \frac{v_k}{u_k} \sum_{j=1}^{k-1} q_j. \tag{30}$$

If $H_* = 0$ or $B \le 0$, then inequality (23) holds.

Let now we suppose that $H_* > 0$ and $B > 0$. We can rewrite D_j as follows

$$D_j = q_j \left[w_j \sum_{i=1}^{j-1} q_i \left(2 - w_j \sum_{i=1}^{j-1} q_i\right) + \frac{w_j q_j}{1 + w_j q_j} \left(w_j \sum_{i=1}^{j-1} q_i - 1\right)^2 \right] \text{ for } k,\ j \ge k_0,$$

then (26) transforms into

$$\left(\sum_{j=1}^{k-1} q_j\right) w_k = -H_k + P_{k,n} \tag{31}$$

$$- \frac{1}{\sum\limits_{j=1}^{k-1} q_j} \sum_{j=n}^{k-1} \left[q_j \left(w_j \sum_{i=1}^{j-1} q_i \left(2 - w_j \sum_{i=1}^{j-1} q_i\right) + \frac{w_j q_j}{1 + w_j q_j} \left(w_j \sum_{i=1}^{j-1} q_i - 1\right)^2 \right) \right].$$

Let $0 < \varepsilon < \min\{H_*, 1 - B\}$ be arbitrary. We can choose $k^{[\varepsilon]} > k_0$ such that

$$\sum_{j=1}^{k-1} q_j w_j < B + \varepsilon, \quad H_k > H_* - \varepsilon \quad \text{and} \quad \left| \frac{q_k w_k}{1 + w_k} \right| \le \varepsilon \qquad \text{for} \quad k \ge k^{[\varepsilon]}. \tag{32}$$

Since the function $f(x) = x(2 - x) + \varepsilon(x - 1)^2$ is nondecreasing in $]\varepsilon, 1[$ and $B + \varepsilon \in]\varepsilon, 1[$, it follows from (31) and (32) that

$$w_k \sum_{j=1}^{k-1} q_j \leq -H_* + \varepsilon + \left[(B+\varepsilon)(2-B-\varepsilon) + \varepsilon(B+\varepsilon-1)^2\right] \frac{\sum\limits_{j=k^{[\varepsilon]}}^{k-1} q_j}{\sum\limits_{j=1}^{k-1} q_j} + P_{k,k^{[\varepsilon]}}. \quad (33)$$

Hence, by virtue of (29) and (30), we get

$$B \leq -H_* + \varepsilon + \left[(B-\varepsilon)(2-B-\varepsilon) + \varepsilon(B+\varepsilon-1)^2\right].$$

Since $\varepsilon > 0$ was chosen arbitrary we have

$$B^2 - B + H_* \leq 0.$$

Obviously, latter inequality implies

$$B \leq \frac{1}{2}\left(1 + \sqrt{1 - 4H_*}\right).$$

□

4. Proofs of main results

P r o o f o f T h e o r e m 2.1. Let us assume on the contrary, that system (1) is nonoscillatory, i.e., there exists a solution $\left\{(u_k, v_k)\right\}_{k=k_0}^{+\infty}$ such that

$$u_k u_{k+1} > 0 \qquad \text{for} \quad k \geq k_0.$$

Now we can rewrite system (1) in the form

$$\Delta w_k + p_k + R_k = 0 \qquad \text{for} \quad k \geq k_0,$$

where

$$w_k = \frac{v_k}{u_k} \quad \text{and} \quad R_k = \frac{w_k^2 q_k}{1 + w_k q_k} \qquad \text{for} \quad k \geq k_0.$$

Let us denote

$$\tilde{A} = \frac{1}{2}\left(1 - \sqrt{1 - 4Q_*}\right).$$

According to Lemma 3.2 there exists $k^{[\varepsilon]} > k_0$ such that

$$w_k \sum_{j=1}^{k-1} q_j > \tilde{A} - \varepsilon \qquad \text{for} \quad k \geq k^{[\varepsilon]} \quad (34)$$

for arbitrary $\varepsilon > 0$. Now we can show, in the similar way as in the proof of Lemma 3.3, that (26) and inequality $D_j \leq q_j$ for $k \geq k^{[\varepsilon]}$ hold. Thus,

$$w_k \sum_{j=1}^{k-1} q_j \leq -H_k + 1 + P_{k,k^{[\varepsilon]}} \qquad \text{for} \quad k \geq k^{[\varepsilon]}.$$

On account of (34), the latter inequality yields

$$H_k \le -\tilde{A} + \varepsilon + 1 + P_{k,k^{[\varepsilon]}} \qquad \text{for} \quad k \ge k^{[\varepsilon]},$$

i.e.,

$$H_k \le -\frac{1}{2}\left(1 - \sqrt{1-4Q_*}\right) + \varepsilon + 1 + P_{k,k^{[\varepsilon]}} \qquad \text{for} \quad k \ge k^{[\varepsilon]}.$$

Hence

$$H^* \le \frac{1}{2}\left(1 + \sqrt{1-4Q_*}\right) + \varepsilon,$$

which contradicts (4), since $\varepsilon > 0$ was chosen arbitrary. $\square$

P r o o f o f T h e o r e m 2.2. Let us assume on the contrary, that system (1) is nonoscillatory, i.e., there exists a solution $\left\{(u_k, v_k)\right\}_{k=k_0}^{+\infty}$ such that

$$u_k u_{k+1} > 0 \qquad \text{for} \quad k \ge k_0.$$

Now we can rewrite system (1) in the form

$$\Delta w_k + p_k + R_k = 0 \qquad \text{for} \quad k \ge k_0,$$

where

$$w_k = \frac{v_k}{u_k} \quad \text{and} \quad R_k = \frac{w_k^2 q_k}{1 + w_k q_k} \qquad \text{for} \quad k \ge k_0.$$

Put

$$\tilde{B} = \frac{1}{2}\left(1 + \sqrt{1-4H_*}\right).$$

According to Lemma 3.3 there exists $k^{[\varepsilon]} > k_0$ such that

$$\tilde{B} + \varepsilon \; > \; w_k \sum_{j=1}^{k-1} q_j \qquad \text{for} \quad k \ge k^{[\varepsilon]} \tag{35}$$

for arbitrary $\varepsilon > 0$. Analogously as in proof of Lemma 3.2 we can prove that the inequality (22) for $k \ge k^{[\varepsilon]}$ holds. Thus, it follows from (35) that

$$\tilde{B} + \varepsilon > \sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} p_j + \sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} R_j \qquad \text{for} \quad k \ge k^{[\varepsilon]}.$$

By virtue of (3) and (7), the latter inequality implies

$$\frac{1}{2}\left(1 + \sqrt{1-4H_*}\right) + \varepsilon > Q_k \qquad \text{for} \quad k \ge k^{[\varepsilon]},$$

which contradicts (5), since $\varepsilon > 0$ was chosen arbitrary. $\square$

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