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ON A SOLUTION OF MONOTONE TYPE PROBLEMS WITH UNCERTAIN INPUTS

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ABSTRACT. The paper deals with a nonlinear weak monotone type problem and its solution with respect to uncertain coefficients in the equation. The so-called worst scenario method is adopted. The formulation of suitable conditions and a proof of the existence of a solution of the worst scenario problem is presented.

1. Introduction

Mathematical modelling of the physical phenomena usually exhibits a sort of error due to neglecting or simplifying some factors in a model, working with uncertain input parameters, approximate numerical solution, etc. The paper deals with the second mentioned issue, i.e., discusses how to proceed with problems, where an exact description of the input data is not available—we talk about the problems with uncertain input data. By input data we mean coefficients in the equations/variational inequalities, right-hand sides, functions from initial/boundary conditions, etc.

We shall investigate a nonlinear elliptic monotone type boundary value problem in dimension one with the equation

$$-\big(a(u')\big)' = f.$$

Contrary to the usual approach, the coefficient a is assumed to be uncertain, however ranging in some bounds and still satisfying certain hypothesis on continuity and monotonicity.

The motivation for studying such a problem is the following. Imagine a situation when we model some physical phenomena set in a highly heterogeneous medium like some composite material or perforated/porous body. If a structure

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is too fine we can expect problems from the numerical point of view—to catch the structure we would need very fine meshes and thus the resulting system of equations can exceed the computer capabilities. To overcome this problem, the natural idea is to consider a homogeneous medium (i.e., described by constant parameters in the spatial variable) instead of heterogeneous one having the same properties from the macroscopic point of view. It remains to answer the question, how to determine such effective (homogeneous) properties from the knowledge of the microstructure (we note that it is not just an average of particular components of the structure). To this end, the mathematical method of homogenization was developed, for introduction see, e.g., [1]. It is quite easy and powerful tool, however its practical use is restricted to the case of periodic structures. In the real world we encounter rather almost periodic and sometimes even completely stochastic structures. In these cases the evaluation of the effective parameters by the homogenization method fails, although we assume that certain bounds can be guessed. In other words, we arrive at the problem with uncertain input data mentioned above.

To solve such a problem, we adopt a deterministic method called the worst scenario method introduced by Hlaváček, see [3]. The main idea consists in defining a functional over a suitable set of admissible data serving as a criterion that evaluates some state/physical quantity from certain point of view. Then the optimization (here we should say antioptimization) of the functional yields the worst state. In other words, the strategy of the method is to be on a safe side, since the knowledge of the "critical" data can help to adjust a technological process properly.

The presented results are based on the author's recent research [5], [2]. The paper is intended rather as an introduction to the topic for readers that are not familiar with the topic than a rigorous mathematical study (we restrict ourselves to the one-dimensional problem only). However, formulation of conditions under which the corresponding worst scenario problem has a solution and its proof seems to be new. The text is structured as follows. After some preliminaries, the model problem is introduced in Section 2. Here we recall some known results and therefore the proofs are omitted here. Section 3 is devoted to the worst scenario method containing the main result on solvability of the corresponding worst scenario problem. Some concluding remarks in Section 4 close the paper.

2. Model problem

Let us start with some preliminaries. Throughout the paper, $I = (b_\ell, b_r)$ is an open and bounded interval in $\mathbb{R}$. For a subset $\Omega \subset \mathbb{R}$ we denote by $|\Omega|$ its Lebesgue measure. We employ the Lebesgue space $L^2(I)$ of integrable functions on I equipped with the standard norm $\|u\|_{L^2(I)} = \int_I u^2 \,\mathrm{d}x$ and the Sobolev space $H_0^1(I)$ of all functions $u \in L^2(I)$ with the integrable (distributive) derivative such that $u(b_\ell) = u(b_r) = 0$. The norm is taken as

$$\|u\|_{H_0^1(I)} = \big(\|u\|^2_{L^2(I)} + \|u'\|^2_{L^2(I)}\big)^{1/2}.$$

The Friedrichs' inequality

$$\|u\|_{L^2(I)} \le C\|u'\|_{L^2(I)}$$

(see, e.g., [4]) makes this norm equivalent with the seminorm, i.e., we have

$$\|u'\|_{L^2(I)} \le \|u\|_{H_0^1(I)} \le c\|u'\|_{L^2(I)}.$$

Both spaces are Hilbert spaces. Further, the space of continuous functions on the real line, $C(\mathbb{R})$, equipped with the norm $\|u\|_{C(\mathbb{R})} = \sup_{x\in\mathbb{R}} |u(x)|$ is used (the convergence in this norm corresponds to the uniform convergence).

We consider a class of nonlinear Dirichlet boundary value problems in the form

$$\begin{aligned} -\big(a(u')\big)' &= f \quad \text{in } I, \qquad (1)\\ u(b_\ell) \;=\; u(b_r) &= 0, \end{aligned}$$

where the function a is assumed to be uncertain from the set of admissible data U_{ad} defined as

$$U_{ad} := \big\{g \in S(r,\alpha,\beta,\gamma) : a_{\min}(\xi) \le g(\xi) \le a_{\max}(\xi)\big\},$$

where $a_{\min}$, $a_{\max}$ are given functions from the set $S(r,\alpha,\beta,\gamma)$ of all functions $g\colon \mathbb{R} \to \mathbb{R}$ satisfying the properties: For a fixed $r,\alpha,\beta,\gamma > 0$ ($\gamma \le \beta$)

$$\begin{aligned} |g(\xi_1) - g(\xi_2)| &\le \beta|\xi_1 - \xi_2|, && \text{for all } \ \xi_1,\xi_2 \in [-r,r], && (2)\\ \alpha|\xi_1 - \xi_2|^2 &\le \big(g(\xi_1) - g(\xi_2)\big)(\xi_1 - \xi_2), && \text{for all } \ \xi_2,\xi_2 \in [-r,r] && (3)\\ g(\xi) &= g(r) + \gamma(\xi - r), && \xi > r, &&\\ g(\xi) &= g(-r) + \gamma(\xi + r), && \xi < r. && (4) \end{aligned}$$

Then the weak formulation of the problem (1) reads:

Problem 1. Let $a \in U_{ad}$. Find $u \equiv u(a) \in H_0^1(I)$ such that

$$\int_I a(u')v' \,\mathrm{d}x = \int_I fv \,\mathrm{d}x, \qquad \text{for all} \quad v \in H_0^1(I).$$

The solvability of the problem is based on the following abstract theorem. Let $A\colon V \to V'$, where V is a Hilbert space and V' its dual, be an operator such that:

(1) $\|A(u_1) - A(u_2)\|_{V'} \le \beta\|u_1 - u_2\|_V$, for all $u_1, u_2 \in V$ (Lipschitz continuity),

(2) $\alpha\|u_1 - u_2\|^2_V \le \langle A(u_1) - A(u_2), u_1 - u_2\rangle$ (strong monotonicity),

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V and V'. Then the operator equation $A(u) = b$ has a unique solution for each $b \in V'$.

These two properties are fulfilled under the conditions (2)–(4). More precisely, directly from the construction of the set $S(r,\alpha,\beta,\gamma)$, it can be seen that each function from U_{ad} satisfies

$$|a(\xi_1) - a(\xi_2)| \le \beta|\xi_1 - \xi_2|, \qquad \text{for all} \quad \xi_1, \xi_2 \in \mathbb{R}, \tag{5}$$

$$\alpha|\xi_1 - \xi_2|^2 \le \big(a(\xi_1) - a(\xi_2)\big)(\xi_1 - \xi_2), \quad \text{for all} \quad \xi_1, \xi_2 \in \mathbb{R}. \tag{6}$$

While the Lipschitz continuity of the function $a \in U_{ad}$ (5) ensures also the Lipschitz continuity of the operator A, the condition (6) yields the strong monotonicity of the operator A. To summarize above considerations, we can state:

Theorem 1. *Let $a \in U_{ad}$ and $f \in L^2(I)$. Then there exists a unique solution of Problem 1.*

We note that the existence of the solution can be obtained also under much weaker assumptions, however the introduced conditions are needed in the following section. Details on theory of monotone operators can be found, e.g., in [6].

3. Worst scenario method

In this section we solve the worst scenario problem related to Problem 1. As mentioned in the introduction, the basic idea consists in a suitable choice of a criterial functional that can be generally dependent on both the input (admissible) data as well as the solution of the model problem. This functional is chosen with respect to the aim of interest/expert decision. Although we have quite a lot of freedom in the definition of the criterion, certain continuity conditions must be fulfilled, for details see [3]. In our case we introduce

Definition 1. The functional

$$\Phi\colon U_{ad} \times H_0^1(I) \to \mathbb{R}$$

is called criterial if the following convergence is satisfied: taking $a_n, a \in U_{ad}$, $v_n, v \in H_0^1(I)$ such that $a_n \to a$ in $C(\mathbb{R})$, $v_n \rightharpoonup v$ in $H_0^1(I)$ as $n \to \infty$, then

$$\Phi(a_n, v_n) \to \Phi(a, v).$$

One of the easiest examples is the functional

$$\Phi\big(a, u(a)\big) = |\tilde{I}|^{-1} \int_{\tilde{I}} u(a)\,\mathrm{d}x,$$

where $\tilde{I}$ is a suitable subinterval of I and $u(a)$ is the solution of the Problem 1.

This choice represents the average value of the solution (e.g., the temperature) over $\tilde{I}$. This choice is motivated by the question "what data give the highest/the most danger solution (temperature) in some crucial place of the material?" Similarly, the solution can be replaced by its gradient or the generalized gradient as it is often the aim of interest.

Now, the set of admissible data and the criterial functional with respect to the model problem are given, so that we can introduce the worst scenario problem:

Problem 2. Find $\widehat{a} \in U_{ad}$ such that

$$\Phi\big(\widehat{a}, u(\widehat{a})\big) \geq \Phi\big(a, u(a)\big), \qquad \text{for all} \quad a \in U_{ad},$$

where Φ is a criterial functional and $u(a)$ is the solution of the Problem 1.

Before we prove the existence theorem of the worst scenario problem, we present the following two lemmas.

Lemma 1. *The set U_{ad} is compact in $C(\mathbb{R})$. In other words, every sequence $\{a_n\} \subset U_{ad}$ contains an uniformly convergent subsequence converging to an element $a \in U_{ad}$.*

Proof. Let U_{ad}^r be the set of all functions from U_{ad} restricted on the interval $[-r, r]$. The Lipschitz continuity condition (2) implies the uniform boundedness and equicontinuity of the functions from U_{ad}^r. By the Arzelà-Ascoli theorem, for each sequence $\{a_n\} \subset U_{ad}^r$ there exists a uniformly converging subsequence. The set U_{ad}^r is closed, thus the limit belongs to U_{ad}^r. Since the set U_{ad} is the continuous extension of U_{ad}^r with the lines of the same slope that differ by a constant only, the uniform convergence on the whole real line follows immediately. $\square$

Lemma 2. *Let $a_n, a \in U_{ad}$ be such that $a_n \to a$ in $C(\mathbb{R})$ as $n \to \infty$. Then $u(a_n) \to u(a)$ in H_0^1, where $u(a_n)$ and $u(a)$ are the solutions of the Problem 1 with the coefficient a_n and a, respectively.*

Proof. For lucidity, let us denote $u_n \equiv u(a_n)$ and $u \equiv u(a)$. First, let us prove that the sequence $\{u_n\}$ is bounded in $H_0^1(I)$. By the strong monotonicity condition (6) and the inequality $|a_n(0)| \leq c$, $c = \max\big\{|a_{\min}(0)|, |a_{\max}(0)|\big\}$, we have

$$\begin{aligned} \alpha \|u_n'\|_{L^2(I)}^2 &\leq \left| \int_I \big(a_n(u_n') - a_n(0)\big) u_n' \,\mathrm{d}x \right| \\ &= \left| \int_I f u_n \,\mathrm{d}x - \int_I a_n(0) u_n' \,\mathrm{d}x \right| \\ &\leq \big(C\|f\|_{L^2(I)} + c|I|^{1/2}\big) \|u_n'\|_{L^2(I)}\,, \end{aligned}$$

where C is the constant from the Friedrichs' inequality.

This and the equivalence of the norm and seminorm imply that $\|u_n\|_{H_0^1(I)} \le M$, where the constant M depends on $\alpha, c, C, \|f\|$ and $|I|$ only.

Using again (6) and the definition of the solutions u_n and u we can write

$$\begin{aligned}
\alpha\|u_n' - u'\|^2_{L^2(I)} &\le \int_I \big(a_n(u_n') - a_n(u')\big)(u_n' - u')\,\mathrm{d}x \\
&= \int_I f(u_n - u)\,\mathrm{d}x - \int_I a_n(u')(u_n' - u')\,\mathrm{d}x \\
&\le \Bigg| \int_I f(u_n - u)\,\mathrm{d}x - \int_I a(u')(u_n' - u')\,\mathrm{d}x \\
&\quad + \int_I \big(a(u') - a_n(u')\big)(u_n' - u')\,\mathrm{d}x \Bigg| \\
&\le \Bigg| \int_I f(u_n - u)\,\mathrm{d}x - \int_I f(u_n - u)\,\mathrm{d}x \Bigg| \\
&\quad + \sup_{\xi\in\mathbb{R}} |a(\xi) - a_n(\xi)| \int_I |u_n' - u'|\,\mathrm{d}x \\
&\le |I| \cdot \sup_{\xi\in\mathbb{R}} |a(\xi) - a_n(\xi)| \cdot \|u_n' - u'\|_{L^2(I)}.
\end{aligned}$$

Since $\|u_n' - u'\|_{L^2(I)}$ is bounded, the right-hand side converges to zero due to the uniform convergence of $a_n \rightrightarrows a$ and the proof is complete. □

Now we are in a position to formulate the main result.

Theorem 2. *There exists a solution of the Problem 2.*

P r o o f. Let us denote $J(a) \equiv \Phi\big(a, u(a)\big)$ and let $\{a_n\} \subset U_{ad}$ be a maximizing sequence of the functional J, i.e.,

$$\lim_{n\to\infty} J(a_n) = \sup_{a\in U_{ad}} J(a)\,. \tag{7}$$

Such sequence surely exists, since the values $J(a_n)$ form a nonempty set in $\mathbb{R}$ and thus there exists a supremum. From Lemma 1 we know that there exists an element $\widetilde{a} \in U_{ad}$ such that, up to a subsequence,

$$a_{n'} \rightrightarrows \widetilde{a} \quad \text{on } \mathbb{R}\,.$$

Lemma 2 yields $u(a_{n'}) \to u(\widetilde{a})$ in $W_0^{1,2}(I)$ and as a consequence of the definition of functional Φ

$$J(a_{n'}) \to J(\widetilde{a}) \quad \text{as} \quad n' \to \infty\,. \tag{8}$$

Combining (7) and (8) we get

$$\lim_{n'\to\infty} J(a_{n'}) = J(\widetilde{a}) = \sup_{a\in U_{ad}} J(a)$$

and thus $\widetilde{a}$ is a maximizing element, i.e., $\widetilde{a} = \widehat{a}$ in $\mathbb{R}$, which is the desired result. □

Note that we have the existence of the solution only. Of course, the uniqueness can be obtained if $J(a)$ is strictly concave on U_{ad}, however, we do not know, under which assumptions it is true. We remind that $J(a)$ is constructed via the functional Φ depending generally also on the solution u and we do not have enough information about the behaviour of the solution u with resect to $a \in U_{ad}$.

4. Remarks

The worst scenario method significantly extends the solvability of problems, where some uncertain behaviour in the input has to be taken into account. Compared to stochastic methods it can be sometimes too pessimistic—it searches for critical data even in the case, when the probability of their occurrence is small. On the other hand, the method does not require any probabilistic information on the data distribution and another pros is the relative simplicity and wide applicability of the method. For comprehensive guide on the method we refer [3] and the references therein. This monograph contains also clues to other approaches to problems with uncertainties.

We have adopted the method in the case of nonlinear one-dimensional problem of the monotone type, where suitable conditions on the uncertain data (the coefficient of the equation) were introduced so that we could apply the general abstract scheme of the method presented in [3]. The method's keystone is the compactness of the set of admissible functions U_{ad}. Here we have been successful due to the restriction of the range of uncertainty on the interval of a final length, so that Arzelà-Ascoli theorem could be applied. Since this interval can be arbitrarily large, this limitation is not significant from the technical point of view (the range of possible values of the solution can be usually estimated based on a concrete physical problem). We note that the reformulation of analogous conditions to a higher dimension is a more difficult task and is the subject matter (not in a full generality) of the forthcoming paper [2]. A possible relaxation of these conditions remains an open problem.

We have not discussed the numerical aspects of the problem. It requires a discretization of both the space, where the solution is looked for and the set of admissible data. It means that we reformulate the Problems 1 and 2 on the finite-dimensional subsets $V_h \subset H_0^1(I)$ and $U_{ad}^M \subset U_{ad}$, where h is the discretization parameter, e.g., this one from the finite element method and M means an M-dimensional subset of U_{ad}. The next step would consist in investigations of the existence and uniqueness of approximate solutions u_h, $\widehat{a}_h^M$. Finally, the last step deals with the convergence analysis $\widehat{a}_h^M \to \widehat{a}$, $u_h(\widehat{a}_h^M) \to u(\widehat{a})$, $\Phi\big(\widehat{a}_h^M, u_h(\widehat{a}_h^M)\big) \to \Phi\big(\widehat{a}, u(\widehat{a})\big)$ in relevant function spaces as $h \to 0$ and $M \to \infty$. This as well as some numerical experiments are subject of further research.

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