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ON THE OSCILLATION OF THIRD-ORDER QUASI-LINEAR DELAY DIFFERENTIAL EQUATIONS

Tongxing Li — Chenghui Zhang — Blanka Baculíková — — Jozef Džurina

ABSTRACT. The aim of this work is to study asymptotic properties of the third-order quasi-linear delay differential equation

$$\left[a(t)\left(x''(t)\right)^{\alpha}\right]' + q(t)x^{\alpha}\left(\tau(t)\right) = 0, \tag{E}$$

where $\alpha > 0$, $\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)}\mathrm{d}t < \infty$ and $\tau(t) \le t$. We establish a new condition which guarantees that every solution of (E) is either oscillatory or converges to zero. These results improve some known results in the literature. An example is given to illustrate the main results.

1. Introduction

We are concerned with the oscillation and asymptotic behavior of the third-order differential equation

$$\left[a(t)\left(x''(t)\right)^{\alpha}\right]' + q(t)x^{\alpha}\left(\tau(t)\right) = 0, \tag{E}$$

where $\alpha > 0$ is the quotient of odd positive integers, $q(t), \tau(t) \in C\left([t_0, \infty)\right)$ and

$$a(t) \in C^1\left([t_0, \infty)\right),\ a'(t) \ge 0,\ a(t) > 0,\ \int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)}\mathrm{d}t < \infty,\ q(t) \ge 0, q(t)$$

is not identically zero on any ray of the form $[t_*, \infty)$ for any $t_* \ge t_0$, $\tau(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a solution of Eq. (E) we mean a function $x(t) \in C^2\left([T_x, \infty)\right)$, $T_x \ge t_0$, which has the property $a(t)\left(x''(t)\right)^{\alpha} \in C^1\left([T_x, \infty)\right)$ and satisfies (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\left\{|x(t)| : t \ge T\right\} > 0$ for all $T \ge T_x$. We assume that (E) possesses such a solution. A solution of (E)

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is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise, it is said to be nonoscillatory. Equation (E) itself is said to be almost oscillatory if all its solutions are oscillatory or convergent to zero asymptotically.

Recently, great attention has been devoted to the oscillation of differential equations; see, e.g., the papers [1]–[17]. Especially, differential equations of the form (E) and its special cases have been the subject of intensive research. Hartman and Wintner [4], Hanan [5] and Erbe [6] studied a particular case of (E), namely, the third-order differential equation

$$x'''(t) + q(t)x(t) = 0.$$

Baculíková et al. [11] considered the oscillation of third-order differential equation

$$\left[b(t)\left(\left[a(t)x'(t)\right]'\right)^{\alpha}\right]' + q(t)x^{\alpha}(t) = 0.$$

Baculíková and Džurina [12], [13], [14], Grace et al. [15], Saker and Džurina [17] examined the oscillation behavior of (E) under the cases when

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \mathrm{d}t = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \mathrm{d}t < \infty.$$

However, those results cannot be applied when

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \mathrm{d}t < \infty \quad \text{and} \quad \tau(t) = t.$$

We utilize a new method to complement this gap.

Remark 1. All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

Remark 2. Without loss of generality we can deal only with the positive solutions of (E).

2. Main results

In this section, we obtain a new oscillatory criterion for (E).

Lemma 1 ([14, Lemma 3]). *Assume that $u(t) > 0$, $u'(t) \geq 0$, $u''(t) \leq 0$ on (t_0, ∞). Then for each $l \in (0,1)$ there exists a $T_l \geq t_0$ such that*

$$\frac{u(\tau(t))}{u(t)} \geq l \frac{\tau(t)}{t}, \qquad t \geq T_l.$$

Lemma 2 ([14, Lemma 4]). *Assume that $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leq 0$ on (T_l, ∞). Then*

$$\frac{z(t)}{z'(t)} \geq \frac{t - T_l}{2}, \qquad t \geq T_l.$$

Theorem 1. *Assume that*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{a(u)}\int_{u}^{\infty}q(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u\,\mathrm{d}v=\infty \tag{2.1}$$

and

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left[l^\alpha sq(s)\left(\frac{\tau(s)-T_l}{2}\frac{\tau(s)}{s}\right)^\alpha-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{a(s)}{s^\alpha}\right]\mathrm{d}s=\infty \tag{2.2}$$

for some $l\in(0,1)$ and for sufficiently large $T_l\geq t_0$, $t_2\geq T_l$. Furthermore, assume that

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left[k^\alpha q(s)\tau^\alpha(s)\delta^\alpha(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\delta(s)a^{1/\alpha}(s)}\right]\mathrm{d}s=\infty \tag{2.3}$$

holds for some $k\in(0,1)$ and for sufficiently large $t_3\geq t_0$, where

$$\delta(t):=\int_{t}^{\infty}\frac{1}{a^{1/\alpha}(s)}\,\mathrm{d}s.$$

Then (E) *is almost oscillatory.*

Proof. Assume that x is a positive solution of (E). Then there exist three possible cases:

(1) $x(t)>0,\ x'(t)<0,\ x''(t)>0,\ x'''(t)\leq 0,$

(2) $x(t)>0,\ x'(t)>0,\ x''(t)>0,\ x'''(t)\leq 0,$

(3) $x(t)>0,\ x'(t)>0,\ x''(t)<0,\ \big(a(t)(x''(t))^\alpha\big)'\leq 0$

for $t\geq t_1$, t_1 large enough.

If case (1) holds, similar to the proof of [14, Lemma 2], we can obtain that $\lim_{t\to\infty}x(t)=0$ due to condition (2.1).

Suppose that case (2) holds. We define the function w by

$$w(t)=t\frac{a(t)(x''(t))^\alpha}{(x'(t))^\alpha},\qquad t\geq t_1. \tag{2.4}$$

Then $w(t)>0$. From (2.4), we obtain

$$\begin{aligned} w'(t) &= \frac{a(t)(x''(t))^\alpha}{(x'(t))^\alpha}+\frac{t(a(t)(x''(t))^\alpha)'}{(x'(t))^\alpha}-\frac{\alpha ta(t)(x''(t))^{\alpha+1}}{(x'(t))^{\alpha+1}}\\ &= -tq(t)\left(\frac{x(\tau(t))}{x'(t)}\right)^\alpha+\frac{w(t)}{t}-\alpha\frac{w^{(\alpha+1)/\alpha}(t)}{(ta(t))^{1/\alpha}}. \end{aligned} \tag{2.5}$$

From [14, Lemma 3, Lemma 4], we see that

$$\frac{x'(\tau(t))}{x'(t)} \geq \frac{l\tau(t)}{t} \quad \text{and} \quad \frac{x(t)}{x'(t)} \geq \frac{(t-T_l)}{2} \quad \text{for} \quad t \geq T_l \geq t_1, \ \text{respectively.}$$

Hence by (2.5), there exists a $t_2 \geq T_l$ such that

$$w'(t) \leq -l^\alpha t q(t) \left(\frac{\tau(t)-T_l}{2} \frac{\tau(t)}{t} \right)^\alpha + \frac{w(t)}{t} - \alpha \frac{w^{(\alpha+1)/\alpha}(t)}{(ta(t))^{1/\alpha}}, \qquad t \geq t_2. \quad (2.6)$$

Using the inequality

$$Bv - Av^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \qquad A > 0,$$

with

$$v = w(t), \ A := \alpha / \big(ta(t)\big)^{1/\alpha} \quad \text{and} \quad B := 1/t,$$

we have

$$w'(t) \leq -l^\alpha t q(t) \left(\frac{\tau(t)-T_l}{2} \frac{\tau(t)}{t} \right)^\alpha + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{a(t)}{t^\alpha}.$$

Integrating the last inequality from t_2 to t, we obtain

$$\int_{t_2}^{t} \left[l^\alpha s q(s) \left(\frac{\tau(s)-T_l}{2} \frac{\tau(s)}{s} \right)^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{a(s)}{s^\alpha} \right] \mathrm{d}s \leq w(t_2),$$

which contradicts (2.2).

Assume that case (3) holds. We define the function u by

$$u(t) = -\frac{a(t)(-x''(t))^\alpha}{(x'(t))^\alpha}, \qquad t \geq t_1. \quad (2.7)$$

Then $u(t) < 0$. Noting that $a(t)\big(-x''(t)\big)^\alpha$ is nondecreasing, we get

$$a^{1/\alpha}(s) x''(s) \leq a^{1/\alpha}(t) x''(t), \qquad s \geq t \geq t_1.$$

Dividing the above inequality by $a^{1/\alpha}(s)$, and integrating it from t to l, we obtain

$$x'(l) \leq x'(t) + a^{1/\alpha}(t) x''(t) \int_t^l \frac{\mathrm{d}s}{a^{1/\alpha}(s)}.$$

Letting $l \to \infty$, we have

$$0 \leq x'(t) + a^{1/\alpha}(t) x''(t) \delta(t).$$

That is

$$-\delta(t) \frac{a^{1/\alpha}(t) x''(t)}{x'(t)} \leq 1.$$

Hence by (2.7), we get

$$-\delta^\alpha(t) u(t) \leq 1. \quad (2.8)$$

Differentiating (2.7), we obtain

$$u'(t) = \frac{\left(-a(t)(-x''(t))^{\alpha}\right)'(x'(t))^{\alpha} + \alpha a(t)(-x''(t))^{\alpha}(x'(t))^{\alpha-1}x''(t)}{(x'(t))^{2\alpha}}.$$

Thus

$$u'(t) = \frac{\left(-a(t)(-x''(t))^{\alpha}\right)'}{(x'(t))^{\alpha}} - \alpha\frac{(-u(t))^{(\alpha+1)/\alpha}}{a^{1/\alpha}(t)}. \tag{2.9}$$

Since

$$x(t) > 0,\; x'(t) > 0,\; x''(t) < 0,$$

we have

$$x(t) \geq ktx'(t) \quad \text{for each} \quad k \in (0,1) \quad \text{and} \quad t \geq T_k \geq t_1.$$

It follows from (E) and (2.9) that there exists a $t_3 \geq T_k$ such that

$$u'(t) \leq -q(t)\big(k\tau(t)\big)^{\alpha} - \alpha\frac{(-u(t))^{(\alpha+1)/\alpha}}{a^{1/\alpha}(t)}, \qquad t \geq t_3. \tag{2.10}$$

Multiplying (2.10) by $\delta^{\alpha}(t)$, and integrating it from t_3 to t, we have

$$\begin{aligned} u(t)\delta^{\alpha}(t) - u(t_3)\delta^{\alpha}(t_3) + \alpha\int_{t_3}^{t} \frac{\delta^{\alpha-1}(s)u(s)}{a^{1/\alpha}(s)}\,\mathrm{d}s & \\ + \alpha\int_{t_3}^{t}\frac{\delta^{\alpha}(s)}{a^{1/\alpha}(s)}\big(-u(s)\big)^{(\alpha+1)/\alpha}\mathrm{d}s & \\ + \int_{t_3}^{t} q(s)\big(k\tau(s)\big)^{\alpha}\delta^{\alpha}(s)\,\mathrm{d}s \;\leq\; 0. & \end{aligned}$$

Using the inequality

$$Av^{(\alpha+1)/\alpha} - Bv \geq -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \qquad A > 0,$$

with

$$v := -u(s),\; A := \alpha\delta^{\alpha}(s)/a^{1/\alpha}(s) \quad \text{and} \quad B := \alpha\delta^{\alpha-1}(s)/a^{1/\alpha}(s),$$

we obtain

$$\int_{t_3}^{t}\left[k^{\alpha}q(s)\tau^{\alpha}(s)\delta^{\alpha}(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\delta(s)a^{1/\alpha}(s)}\right]\mathrm{d}s \leq u(t_3)\delta^{\alpha}(t_3) + 1.$$

Letting $t \to \infty$, we obtain a contradiction to (2.3). This completes the proof. □

Example 1. We consider the third-order differential equation

$$\left[t^{\frac{3}{2}}\left(x''(t)\right)\right]' + \frac{\lambda}{t^{\frac{3}{2}}}x(t) = 0, \qquad \lambda > 0,\ t \geq 1. \tag{2.11}$$

Let

$$\alpha = 1,\ a(t) = t^{\frac{3}{2}},\ q(t) = \frac{\lambda}{t^{\frac{3}{2}}},\ \tau(t) = t.$$

It is easy to verify that conditions (2.1) and (2.2) hold. Moreover, we have

$$\limsup_{t\to\infty} \int_{t_3}^{t} \left[k^\alpha q(s)\big(\tau(s)\big)^\alpha \delta^\alpha(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{\delta(s)a^{1/\alpha}(s)}\right] \mathrm{d}s$$

$$= \left[2k\lambda - \frac{1}{8}\right] \int_{t_3}^{t} \frac{1}{s}\mathrm{d}s = \infty, \qquad \lambda > \frac{1}{16k} \quad \text{for some} \quad k \in (0,1).$$

Hence by Theorem 1, equation (2.11) is almost oscillatory when $\lambda > \frac{1}{16k}$ for some $k \in (0,1)$. However, the results given in [13], [15] cannot be applied to equation (2.11), since

$$\int_{t_0}^{\infty} \left(\frac{1}{a(u)} \int_{t_0}^{u} q(s)\tau(s) \int_{\tau(s)}^{\infty} \frac{1}{a^{1/\alpha}(v)}\,\mathrm{d}v\,\mathrm{d}s\right)^{1/\alpha} \mathrm{d}u < \infty.$$

3. Conclusions

In this paper, we establish a new oscillation theorem for (E). Our result improves and complements results given in the literature. The method can be applied to the following equation

$$\left[a(t)\big(x''(t)\big)^\alpha\right]' + q(t)x^\beta\big(\tau(t)\big) = 0.$$

Further, the method can be extended to the corresponding dynamic equations on time scales, the details are left to the reader.

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Tongxing Li
Chenghui Zhang
School of Control Science and Engineering
Shandong University, Jinan, Shandong 250061
P. R. China
E-mail: litongx2007@163.com
zchui@sdu.edu.cn

Blanka Baculíková
Jozef Džurina
Department of Mathematics
Faculty of Electrical Engineering and Informatics
Technical University of Košice
Letná 9
SK–042-00 Košice
SLOVAKIA
E-mail: blanka.baculikova@tuke.sk
jozef.dzurina@tuke.sk