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AN OPTIMAL DESIGN WITH RESPECT TO A VARIABLE THICKNESS OF A VISCOELASTIC BEAM IN A DYNAMIC BOUNDARY CONTACT

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ABSTRACT. We deal with the optimal control problem governed by a hyperbolic variational inequality describing the perpendicular vibrations of a beam clamped on the left end with a rigid obstacle at the right end. A variable thickness of a beam plays the role of a control parameter.

1. Introduction

The dynamic contact problems are not frequently solved in the framework of variational inequalities. The inner dynamic obstacle problem for a viscoelastic plate with moderately large deflections has been solved in [2]. We deal here with an optimal design problem for a viscoelastic cantilever beam in a dynamic contact on one part of the boundary. A variable thickness of a beam plays the role of a control variable. A similar problem has been solved in [3] for the stationary elastic case. In contrast to it there is no uniqueness result in the dynamic case and hence the minimum will depend both on the thickness as the control and the deflection as the state variable. In order to achieve *a priori estimates* of solutions in a minimizing sequence of a cost functional we assume the bounded admissible set of solutions. Solving the state hyperbolic variational inequality we apply the method of penalization in a similar way as in the case of Mindlin–Timoshenko model considered in [1].

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2. Solving of the state problem

2.1. Setting of the state problem

We consider a short memory viscoelastic beam of the length $L > 0$. Its variable thickness is expressed by a positive function $x \mapsto e(x)$, $x \in [0, L]$, the constants $d_i > 0$, $i = 0, 1$, involve the material and geometrical characteristics. For simplicity we assume $\rho = 1$ the density of the material. The beam is clamped on the left end and free on its right end. Moreover the right end is unilaterally supported. If $f\colon (0, T] \times (0, L) \mapsto \mathbb{R}$ is a perpendicular load acting on the beam, $u_0\colon (0, L) \mapsto \mathbb{R}$, $v_0\colon (0, L) \mapsto \mathbb{R}$ the initial displacement and velocity respectively, then its vertical displacement $u\colon (0, T] \times (0, L) \mapsto \mathbb{R}$ solves the following hyperbolic initial-boundary value problem with an unknown contact force $g\colon (0, T] \mapsto \mathbb{R}$ and the complementary conditions in the point L.

$$e(x)u_{tt} + \big[e^3(x)(d_1 u_{txx} + d_0 u_{xx})\big]_{xx} = f(t, x) \text{ in } (0, L) \times (0, T], \tag{1}$$

$$u(t, 0) = u_x(t, 0) = 0,\ t \in (0, T], \tag{2}$$

$$u_{xx}(t, L) = 0,\ t \in (0, T], \tag{3}$$

$$u(t, L) \geq 0,\ \big[e^3(x)(d_1 u_{txx} + d_0 u_{xx})\big]_x(t, L) = g(t) \geq 0,\ u(t, L)g(t) = 0, \tag{4}$$

$$u(0, x) = u_0(x), \tag{5}$$

$$u_t(0, x) = v_0(x),\ x \in (0, L). \tag{6}$$

In order to solve the problem (1)–(6) we formulate its weak solution as a solution of a hyperbolic variational inequality.

We set $I = (0, T)$, $Q = I \times (0, L)$ and introduce the following spaces:

$$L_2(0, L) = \left\{ y : (0, L) \mapsto \mathbb{R};\ \int_0^L y^2\, dx < \infty \right\},$$

$$H^k(0, L) = \big\{ y \in L_2(0, L)\colon y^{(k)} \in L_2(0, L) \big\}, \quad k > 0,$$

$$V = \big\{ y \in H^2(0, L)\colon y(0) = y'(0) = 0 \big\}.$$

The spaces $L_2(0, L)$ and V are the Hilbert spaces with the inner products and the norms

$$(y, z) = \int_0^L y(x)z(x)\, dx, \quad |y|_0 = (y, y)^{1/2}, \quad y, z \in L_2(0, L),$$

$$\big((y,z)\big) = \int_0^L y''(x)z''(x)\,dx, \qquad \|y\| = \big((y,y)\big)^{1/2}, \quad y,z \in V.$$

Further, we set the Hilbert space

$$\mathcal{V} := H^1(I;V) = \big\{y \in L_2(I;V)\colon \dot{y} \equiv y_t \in L_2(I;V)\big\}$$

with the inner product and the norm

$$(y,z)_{\mathcal{V}} = \int_0^T \big[\big((y,z)\big) + \big((\dot{y},\dot{z})\big)\big]\,dt, \quad \|y\|_{\mathcal{V}} = (y,y)_{\mathcal{V}}^{1/2}.$$

Let X be a Banach space. We denote by X^* the dual Banach space of all linear continuous functionals over X and by $L_p(I;X)$ the Banach space of all functions $y\colon I \mapsto X$ such that

$$\|y(\cdot)\|_X \in L_p(0,T), \quad p \in [1,+\infty].$$

Further we denote by

$$C(\bar{I};X) \quad \text{and} \quad C_w(\bar{I};X)$$

the spaces of continuous, respectively weakly continuous functions

$$y\colon \bar{I} \mapsto X, \quad \bar{I} = [0,T].$$

We introduce the convex cones

$$K = \{w \in V\colon w(L) \ge 0\}, \quad \mathcal{K} = \big\{y \in \mathcal{V}\colon y(t,L) \ge 0 \quad \text{for all} \;\; t \in (0,T]\big\}$$

and assume

$$f \in L_2(Q), \quad u_0 \in K, \quad v_0 \in L_2(\Omega), \quad e \in C^2\big([0,L]\big), \quad 0 < e_1 \le e(x) \le e_2.$$

We utilize the fact that the set $\mathcal{K}$ is a convex cone and formulate a weak solution of the problem (1)–(6) as a solution of a variational inequality with a complementarity condition.

Definition 2.1. Function $u \in \mathcal{K}$ is a weak solution of the problem (1)–(6) if $\ddot{u} \in \big(L_\infty(I;V)\big)^*$, $\dot{u} \in C_w\big(\bar{I}, L_2(0,L)\big)$, the initial condition (5) holds, the condition (6) is satisfied in a weak sense and

$$\langle\langle \ddot{u}, ey\rangle\rangle + \int_Q \big[e^3(x)(d_1\dot{u} + d_0 u)_{xx} y_{xx} - f(t,x)y(t,x)\big]\,dx\,dt = \langle g, y(\cdot,L)\rangle_I, \qquad (7)$$
$$\text{for all} \quad y \in L_\infty(I;V),$$

where $g \in \big(L_\infty(I)\big)^*$ is a functional satisfying

$$\langle g, v(\cdot,L)\rangle_I \ge 0, \quad \text{for all} \quad v \in \mathcal{K}, \qquad (8)$$

$$\langle g, u(\cdot,L)\rangle_I = 0. \qquad (9)$$

We remark that the expression $\langle\langle\cdot,\cdot\rangle\rangle$ means the duality between $\big(L_\infty(I;V)\big)^*$ and $L_\infty(I;V)$ as the extension of the inner product in the space $L_2(Q)$ and the expression $\langle\cdot,\cdot\rangle_I$ means the duality between $\big(L_\infty(I)\big)^*$ and $L_\infty(I)$.

2.2. Penalization

We define for $\varepsilon > 0$ the *penalized problem* in the variational form:

To find $u_\varepsilon \in \mathcal{V}$ such that $\ddot{u}_\varepsilon \in L_2(Q)$ and

$$\begin{aligned}&\int\limits_Q \big[e(x)\ddot{u}_\varepsilon y + e^3(x)(d_1\dot{u}_\varepsilon + d_0u_\varepsilon)_{xx}y_{xx} - fy\big]\,dx\,dt\\ &= \int\limits_I \varepsilon^{-1}u_\varepsilon^-(t,L)y(t,L)\,dt \qquad \text{for all} \quad y \in L_2(I,V),\end{aligned} \tag{10}$$

$$u_\varepsilon(0,x) = u_0(x),\ \dot{u}_\varepsilon(0,x) = v_0(x)\ x \in (0,L),\ \text{with } u_\varepsilon^- = \max\{0,-u_\varepsilon\}. \tag{11}$$

We verify the existence of a solution to the penalized problem and useful *a priori* estimates by the Galerkin method.

Theorem 2.2. *There exists a unique solution $u \equiv u_\varepsilon$ of the problem* (10), (11) *satisfying the estimate*

$$\begin{aligned}&\|\dot{u}_\varepsilon\|^2_{C(\bar{I},L_2(0,L))} + \|\dot{u}_\varepsilon\|^2_{L_2(I,V)}\\ &\le\ C(d_0,d_1,e_1,e_2,u_0,v_0,f),\ C(d_0,d_1,e_1,e_2,u_0,v_0,f)\\ &=\ \left(\frac{2}{e_1} + \frac{1}{d_1e_1^3}\right)\left(e_2|v_0|_0^2 + d_0e_2^3\|u_0\|^2 + \frac{2}{e_1}\|f\|^2_{L_1(I,L_2(0,L))}\right).\end{aligned} \tag{12}$$

Proof. Let us denote a basis of V by $\{w_i \in V;\ i \in \mathbb{N}\}$. We construct the Galerkin approximation u_m of a solution in the form

$$\begin{aligned}&u_m(t) = \sum_{i=1}^m \alpha_i(t)w_i, \qquad \alpha_i(t) \in \mathbb{R}, \quad i = 1,\dots,m, \quad m \in \mathbb{N},\\ &\int\limits_0^L \big(e(x)\ddot{u}_m(t)w_i + e^3(x)(d_1\dot{u}_m + d_0u_m)_{xx}w_{ixx}\big)\,dx\\ &= \int\limits_0^L f(t)w_i\,dx + \varepsilon^{-1}u_m^-(t,L)w_i(L), \qquad i = 1,\dots,m,\end{aligned} \tag{13}$$

$$u_m(0) = u_{0m},\ \dot{u}_m(0) = v_{0m},\ u_{0m} \to u_0 \text{ in } V \ \text{ and } \ v_{0m} \to v_0 \text{ in } L_2(0,L). \tag{14}$$

The solution u_m is defined on a certain interval $I_m = (0,t_m),\ t_m < T$ after applying the theorem on local existence and the uniqueness of a solution $\{\alpha_1,\dots,\alpha_m\}$ of the 2nd-order system of ordinary differential equations. It can be extended to the whole interval $[0,T]$ as a consequence of *a priori* estimates that we prove next.

After multiplying the equation (13) by $\dot{\alpha}_i(t)$, summing up with respect to i and integrating we obtain the estimate

$$\begin{aligned} \|\dot{u}_m\|^2_{C(\bar{I}_m;L_2(0,L))} + \|\dot{u}_m\|^2_{L_2(I_m;V)} + \|u_m\|^2_{C(\bar{I}_m;V)} \\ + \varepsilon^{-1}\|u_m^-(\cdot,L)\|^2_{C(\bar{I}_m)} \le c_1, \\ c_1 \equiv c_1(d_0,d_1,e_1,e_2,u_0,v_0,f). \end{aligned} \tag{15}$$

As the right-hand side of this estimate does not depend on t_m a solution can be prolonged to the whole interval I with the *a priori* estimate

$$\begin{aligned} \|\dot{u}_m\|^2_{C(\bar{I};L_2(0,L))} + \|\dot{u}_m\|^2_{L_2(I;V)} \\ + \|u_m\|^2_{C(\bar{I};V)} + \varepsilon^{-1}\|u_m^-(\cdot,L)\|^2_{C(\bar{I})} \le c_2. \end{aligned} \tag{16}$$

Moreover, after multiplying (13) with $\ddot{\alpha}$ summing up and integrating we have the estimate

$$\|\ddot{u}_m\|^2_{L_2(Q)} \le c_\varepsilon, \quad m \in \mathbb{N}. \tag{17}$$

We proceed with the convergence of the Galerkin approximation. Applying the estimates (16), (17) and the compact imbedding theorem we obtain for a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$) a function $u \in \mathcal{V}$ with $\ddot{u} \in L_2(Q)$ and the convergences

$$\begin{aligned} \ddot{u}_m &\rightharpoonup \ddot{u} && \text{in } L_2(Q), \\ \dot{u}_m &\rightharpoonup^* \dot{u} && \text{in } L_\infty\big(I;L_2(0,L)\big), \\ \dot{u}_m(t) &\rightharpoonup \dot{u}(t) && \text{in } L_2(0,L) && \text{for all } t \in \bar{I}, \\ \dot{u}_m &\rightharpoonup \dot{u} && \text{in } L_2(I;V), \\ u_m &\rightharpoonup^* u && \text{in } L_\infty(I;V), \\ u_m(t) &\rightharpoonup u(t) && \text{in } V && \text{for all } t \in \bar{I}. \end{aligned} \tag{18}$$

Let

$$\mu \in \mathbb{N}, \quad z_\mu = \sum_{i=1}^{\mu} \phi_i(t) w_i, \quad \phi_i \in \mathcal{D}(0,T), \quad i = 1,\dots,\mu.$$

The convergence process (18) implies

$$\int_Q \big[e(x)\ddot{u} z_\mu + e^3(x)(d_1\dot{u} + d_0 u)_{xx} z_{\mu xx} - f z_\mu\big]\,dx\,dt = \int_I \varepsilon^{-1} u^-(t,L) z_\mu(t,L)\,dt.$$

Functions $\{z_\mu\}$ form a dense subset of the set $L_2(I;V)$, hence a function $u \equiv u_\varepsilon$ fulfils the identity (10). The initial conditions (11) follow due to (14) and the proof of the existence of a solution is complete. ☐

In order to achieve the *a priori* estimate (12) we put

$$y = \begin{cases} \dot u_\varepsilon & \text{for } t \le s, \\ 0 & \text{for } t > s \end{cases}$$

in (10) with an arbitrary $s \in I$.
After performing the integration we obtain the inequalities

$$\begin{aligned} \frac{1}{2} e_1 |\dot u_\varepsilon|_0^2(s) &\le \frac{1}{2} e_2 |v_0|_0^2 + \frac{1}{2} d_0 e_2^3 \|u_0\|^2 + \int_0^s (f, \dot u_\varepsilon)(t)\, dt \\ &\le \frac{1}{2} e_2 |v_0|_0^2 + \frac{1}{2} d_0 e_2^3 \|u_0\|^2 + \frac{1}{e_1} \|f\|^2_{L_1(I;L_2(0,L))} \\ &\quad + \frac{1}{4} e_1 \|\dot u_\varepsilon\|^2_{L_\infty(I,L_2(0,L))} \qquad \text{for all} \quad s \in I, \end{aligned}$$

$$\frac{1}{4} e_1 \|\dot u_\varepsilon\|^2_{L_\infty(I,L_2(0,L))} \le \frac{1}{2} e_2 |v_0|_0^2 + \frac{1}{2} d_0 e_2^3 \|u_0\|^2 + \frac{1}{e_1} \|f\|^2_{L_1(I;L_2(0,L))},$$

$$\begin{aligned} d_1 e_0^3 \|\dot u_\varepsilon\|^2_{L_2(I,V)} &\le e_2 |v_0|_0^2 + d_0 e_2^3 \|u_0\|^2 + \frac{1}{e_1} \|f\|^2_{L_1(I;L_2(0,L))} \\ &\quad + \frac{1}{4} e_1 \|\dot u_\varepsilon\|^2_{L_\infty(I,L_2(0,L))} \end{aligned}$$

and the estimate (12) follows.

2.3. The limit process to the original state problem

Let us denote by u_ε a solution of the penalized problem (10), (11). The *a priori* estimates and the convergence process derived in the previous section imply the estimate

$$\begin{aligned} &\|\dot u_\varepsilon\|^2_{C(\bar I;L_2(0,L))} + \|\dot u_\varepsilon\|^2_{L_2(I;V)} \\ &\qquad + \|u_\varepsilon\|^2_{C(\bar I;V)} + \varepsilon^{-1} \|u_\varepsilon^-(\cdot, L)\|^2_{C(\bar I)} \le c_2. \end{aligned} \tag{19}$$

Let us set

$$y(x) = x^2, \quad x \in [0, L]$$

in (10). After performing the integration the estimate (19) implies

$$0 \le \int_0^T \varepsilon^{-1} u_\varepsilon^-(t, L)\, dt \le c_3, \tag{20}$$

$$\|\ddot u_\varepsilon\|_{L_1(I;V^*)} \le c_4. \tag{21}$$

Then there exist a sequence $\varepsilon_n \searrow 0$, a function $u \in \mathcal{V}$ and a functional $g \in \big(L_\infty(I)\big)^*$ such that

$$\ddot u \in \big(L_\infty(I, V)\big)^*, \quad \dot u \in L_\infty\big(I, L_2(0, L)\big) \cap C_w\big(\bar I, L_2(0, L)\big),$$

and for $u_n \equiv u_{\varepsilon_n}$ the following convergences hold

$$\begin{aligned}
\ddot{u}_n &\rightharpoonup^* \ddot{u} && \text{in } \big(L_\infty(I;V)\big)^*, \\
\dot{u}_n(t,\cdot) &\rightharpoonup \dot{u}(t,\cdot) && \text{in } L_2(0,L) && \text{for all } t \in [0,T], \\
\dot{u}_n &\rightharpoonup \dot{u} && \text{in } L_2(I;V), \\
\dot{u}_n &\rightharpoonup^* \dot{u} && \text{in } L_\infty\big(I;L_2(0,L)\big), \\
u_n &\rightharpoonup^* u && \text{in } L_\infty(I;V), \\
u_n(t,\cdot) &\rightharpoonup u(t,\cdot) && \text{in } V && \text{for all } t \in [0,T], \\
u_n(\cdot,L) &\to u(\cdot,L) && \text{in } C(\bar{I}), \\
u_n^-(\cdot,L) &\to 0 && \text{in } C(\bar{I}), \\
\varepsilon_n^{-1} u_n^-(\cdot,L) &\rightharpoonup^* g && \text{in } \big(L_\infty(I)\big)^*.
\end{aligned} \tag{22}$$

Let us define the operators $A_i(e)\colon V \mapsto V^*$ by

$$\big\langle A_i(e)u, y \big\rangle_* = d_i \int_0^L e^3(x) u_{xx} y_{xx}\, dx, \qquad u,y \in V, \quad i = 0,1. \tag{23}$$

The performed convergences imply that the limit function u satisfies in V^* the equation

$$e\ddot{u} + A_1(e)\dot{u} + A_0(e)u = f + g, \tag{24}$$

where $e\ddot{u} \in \big(L_\infty(I;V)\big)^*$ is defined by

$$\langle\langle e\ddot{u}, y\rangle\rangle = \langle\langle \ddot{u}, ey\rangle\rangle \qquad \text{for all} \quad y \in L_\infty(I;V).$$

The limit functional g represents a contact force acting at the right end of the beam and fulfils

$$\langle\langle g, v\rangle\rangle = \langle g, v(.,L)\rangle_I \geq 0 \qquad \text{for all} \quad v \in \mathcal{K}$$

due to the last convergence in (22).

It remains to prove $\langle g, u(.,L)\rangle_I = 0$. Applying the two last convergences in (22) we really obtain

$$\langle g, u(.,L)\rangle_I = \lim_{n\to\infty} \int_I \varepsilon_n^{-1} \|u_n^-(.,L)\|^2_{C(\bar{I})}\, dt = 0.$$

The initial condition (5) is fulfilled in the space V and (6) is satisfied in the weak sense due to the second limit in (22). Hence we have proved the next theorem.

Theorem 2.3. *Let*

$$u_0 \in K,\ v_0 \in L_2(0,L),\ f \in L_2(Q),\ e \in C^2[0,L],\ 0 < e_1 \le e(x) \le e_2,\ x \in [0,L].$$

Then there exists a weak solution of the state problem (1)–(6) *fulfilling the estimate*

$$\|\dot u_\varepsilon\|^2_{L_\infty(I,L_2(0,L))} + \|\dot u_\varepsilon\|^2_{L_2(I,V)} \le C(d_0,d_1,e_1,e_2,u_0,v_0,f) \tag{25}$$

with the constant $C(d_0,d_1,e_1,e_2,u_0,v_0,f)$ *defined in* (12).

3. Optimal control problem

We consider a cost functional

$$J\colon \mathcal{V} \times C^2\big([0,L]\big) \mapsto \mathbb{R}^+$$

fulfilling the assumption

$$u_n \rightharpoonup u \text{ in } \mathcal{V},\ e_n \to e \text{ in } C^2\big([0,L]\big) \Rightarrow J(u,e) \le \liminf_{n\to\infty} J(u_n,e_n).$$

Let

$$E_{ad} = \Big\{ e \in H^3(0,L):\ 0 < e_1 \le e(x) \le e_2 \quad \text{for all} \quad x \in [0,L],\ \|e\|_{H^3(0,L)} \le e_3 \Big\}$$

be the set of admissible thicknesses. We remark that E_{ad} is compact in $C^2\big([0,L]\big)$.

Before formulating the Optimal control problem we introduce the space of functions

$$\mathcal{W} = \Big\{ v \in L_\infty\big(I;L_2(0,L)\big) : \exists\, \dot v \in L_\infty(I;V)^* \quad \text{and} \quad \{v_n\} \subset H^1\big(I;L_2(0,L)\big)$$
$$\text{such that} \quad v_n \rightharpoonup^* v \text{ in } L_\infty\big(I;L_2(0,L)\big), \quad \dot v_n \rightharpoonup^* \dot v \text{ in } L_\infty(I;V)^* \Big\}.$$

Optimal control problem $\mathcal{P}$. To find a couple $(u_*,e_*) \in U_{ad}(e_*) \times E_{ad}$ such that

$$J(u_*,e_*) \le J(u,e) \qquad \text{for all} \quad (u,e) \in U_{ad}(e) \times E_{ad}, \tag{26}$$

$$U_{ad}(e) = \Big\{ u \in \mathcal{K}:\ \dot u \in \mathcal{W},\ u \text{ is a weak solution of (1)–(6)},$$
$$\|\dot u\|^2_{L_\infty(I;L_2(0,L))} + \|u\|^2_{L_\infty(I;V)} \le C_1 \Big\} \tag{27}$$

with $C_1 \ge C(d_0,d_1,e_1,e_2,u_0,v_0,f)$ - a positive constant defined in (15).

The construction of a solution $u \in \mathcal{K}$ using the penalization method in Theorem 2.3 implies that $U_{ad}(e) \neq \emptyset$ for every $e \in E_{ad}$.

Theorem 3.1. *There exists a solution of the Optimal control problem $\mathcal{P}$.*

P r o o f. Let $\{(u_n, e_n)\} \in U_{ad}(e_n) \times E_{ad}$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} J(u_n, e_n) = \inf_{U_{ad}(e)\times E_{ad}} J(u, e).$$

Due to the boundedness and compactness of the sequence in the appropriate spaces there exist $(u_*, e_*) \in \mathcal{K}\times E_{ad}$ and a subsequence denoted again by (u_n, e_n) such that

$$e_n \to e_* \text{ in } C^2\big([0,T]\big), \quad u_n \rightharpoonup u_* \text{ in } \mathcal{V}. \tag{28}$$

The elements $u_n \in U_{ad}(e_n)$ are weak solutions of the State problem (1)–(6) with $e \equiv e_n$ and satisfy

$$\begin{aligned}\langle\langle \ddot{u}_n, e_n y\rangle\rangle + &\int_Q \big[e_n^3(x)(d_1\dot{u}_n + d_0 u_n)_{xx} y_{xx} - f(t,x) y(t,x)\big]\, dx\, dt \\ = \ &\langle g_n, y(\cdot, L)\rangle_I \qquad \text{for all} \quad y \in L_\infty(I; V) \end{aligned}\tag{29}$$

with functionals $g_n \in \big(L_\infty(I)\big)^*$, $n \in \mathbb{N}$ fulfilling

$$\langle g_n, v(\cdot, L)\rangle_I \ge 0, \qquad \text{for all} \quad v \in \mathcal{K}, \tag{30}$$

$$\langle g_n, u_n(\cdot, L)\rangle_I = 0. \tag{31}$$

We use the similar approach as in the proof of the existence in Theorem 2.3 in order to verify $u_* \in U_{ad}(e_*)$. We have, due to the fact that $\dot{u}_n \in \mathcal{W}$, the relation

$$\langle\langle \ddot{u}_n, z\rangle\rangle = \big(\dot{u}_n(T), z\big) - \big(\dot{u}_n(0), z\big) = \big(\dot{u}_n(T) - v_0, z\big) \quad \text{for all} \quad z \in V.$$

After inserting $y(x) \equiv x^2$ in (29) we obtain

$$\begin{aligned}\int_0^L \Big[\big(\dot{u}_n(T,x) - v_0(x)\big) e_n(x) x^2\, dx + 2d_1 e_n^3(x)\big(u_n(T,x) - u_0(x)\big)_{xx}\Big]\, dx \\ + \int_Q \big[2d_0 e_n^3(x) u_n(t,x)_{xx} - f(t,x)x^2\big]\, dx\, dt = L^2\langle g_n, 1\rangle_I.\end{aligned}$$

Using the definition of the admissible set U_{ad} and the property (30) of the functionals g_n we arrive to the estimates

$$\|g_n\|_{\left(L_\infty(I)\right)^*} \le c_5$$

and

$$\|\ddot{u}_n\|_{\left(L_\infty(I,V)\right)^*} \le c_6.$$

Then there exists the subsequence of $\{u_n, e_n, g_n\}$ (denoted by $\{u_n, e_n, g_n\}$) fulfilling the convergence (28) and $g_n \rightharpoonup^* g$ in $\big(L_\infty(I)\big)^*$ such that $u_* \in U_{ad}(e_*)$ with a contact functional $g \equiv g_*$.

Lower semicontinuity properties of the functional J imply

$$J(u_*, e_*) \leq \liminf_{n\to\infty} J(u_n, e_n) = \inf_{U_{ad}(e)\times E_{ad}} J(u, e).$$

Then

$$J(u_*, e_*) = \min_{U_{ad}(e)\times E_{ad}} J(u, e)$$

and the proof is complete. □

Remark 3.2. We have chosen an admissible set $U_{ad}(e)$ in a form (27) because there are no uniqueness and no a priori estimates of solutions of the state variational inequality. The smoothness assumption $e \in H^3(0, L)$ is inevitable due to the appearance of the control parameter e in the term connected with the second derivative $\ddot{u} \in \big(L_\infty(I, V)\big)^*$.

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