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# ON A BOUNDARY VALUE PROBLEM FOR DIFFERENTIAL EQUATION WITH P-LAPLACIAN

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ABSTRACT. The existence of a solution of a boundary value problem for differential equation with p-Laplacian is proved by the technique of lower and upper solutions. A nonlocal boundary condition and a derivative dependent nonlinearity is assumed.

The paper deals with the boundary value problem for a second order differential equation with one dimensional p-Laplacian

$$\big(\varphi_p(x')\big)' = f\big(t, x, \varphi_p(x')\big), \tag{1}$$

$$x'(0) = 0, \qquad x(b) = \int_0^b x(s)dg(s) - k\varphi_p\big(x'(b)\big). \tag{2}$$

The second, nonlocal, boundary condition covers Dirichlet boundary condition as well as certain types of multipoint boundary conditions.

Several authors prove the existence and multiplicity of positive solutions for various types of boundary conditions using fixed point theorems on a positive cone. See [2], [5]. Here the nonlinearity $f$ doesn't depend on the derivative.

We prove the existence of a solution using a method of lower and upper solutions. Our ideas are motivated by the results derived for the classical second order boundary value problems [1], [4].

Further results and references concerning method of lower and upper solutions for regular and singular two point and periodic boundary value problems with p-Laplacian can be found in [3].

We use the function $\varphi_p(x) = |x|^{p-1}\operatorname{sgn}(x)$ with $p > 1$. Its inverse is $\varphi_q(x)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The function $g$ is a nondecreasing function of bounded variation and $k \geq 0$.

We assume $f\colon I \times R^2 \to R$ is a continuous function, $I = [a, b]$ and we seek for a classical solution $x(t) \in D$, $D = \big\{x \in C^1(I),\ \varphi_p(x') \in C^1(I)\big\}$.

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We don't assume differentiability of a lower and upper solution on the whole interval $I$. Set $I^0 = I \setminus \{t_i;\ 0 < t_i < b,\ i = 1 \dots n\}$.

**Definition 1.** A function $\alpha \in C(I) \cup C^1(I^0)$, with $\varphi(\alpha') \in C^1(I^0)$ is called a lower solution of (1), (2) if

$$\lim_{t\to t_i-} \alpha'(t) \le \lim_{t\to t_i+} \alpha'(t) \qquad \text{for} \quad i = 1, \dots, n,$$

$$\Big(\varphi_p\big(\alpha'(t)\big)\Big)' \ge f\Big(t, \alpha(t), \varphi_p\big(\alpha'(t)\big)\Big) \qquad \text{for} \quad t \in I^0,$$

$$\alpha'(0) \ge 0, \qquad \alpha(b) \le \int_0^b \alpha(s)\, dg(s) - k\varphi_p\big(\alpha'(b)\big).$$

Similarly, a function $\beta \in C(I) \cup C^1(I^0)$, with $\varphi(\beta') \in C^1(I^0)$ is called an upper solution of (1), (2) if

$$\lim_{t\to t_i-} \beta'(t) \ge \lim_{t\to t_i+} \beta'(t) \qquad \text{for} \quad i = 1, \dots, n,$$

$$\Big(\varphi_p\big(\beta'(t)\big)\Big)' \le f\big(t, \beta(t), \varphi_p(\beta')\big), \qquad \text{for} \quad t \in I^0,$$

$$\beta'(0) \le 0, \qquad \beta(b) \ge \int_0^b \beta(s) dg(s) - k\varphi_p\big(\beta'(b)\big).$$

In the case of strict inequalities for limits at $t_i$, for the equation on $I^0$ and for the second boundary condition we say that lower and upper solutions are strict.

**Lemma 2.** *Let $\alpha$, $\beta$ be a strict lower and upper solutions and $x(t)$ be a solution of the problem* (1), (2).

*Then $\alpha(t) \le x(t)$ implies $\alpha(t) < x(t)$ and $\beta(t) \ge x(t)$ implies $\beta(t) > x(t)$.*

Proof. Let $x(t) \ge \alpha(t)$ and suppose that $x(t_0) = \alpha(t_0)$.

Suppose $t_0 = t_i$, then $\lim_{t\to t_i-} x'(t) - \alpha'(t) > \lim_{t\to t_i+} x'(t) - \alpha'(t)$ which inequality is in a contradiction with minimum of $x - \alpha$ at $t_0$.

Suppose $t_0 \in (0, b),\ t_0 \ne t_i$. Then there is $\varepsilon > 0$ such that $x'(t) \le \alpha'(t)$, for $t \in (t_0 - \varepsilon, t_0)$ and $x'(t) \ge \alpha'(t)$ for $t \in (t_0, t_0 + \varepsilon)$. As $\varphi_p$ is a strictly increasing function, $\varphi_p\big(x'(t)\big) \le \varphi_p\big(\alpha'(t)\big)$ for $t \in (t_0 - \varepsilon, t_0)$ and $\varphi_p\big(x'(t)\big) \ge \varphi_p\big(\alpha'(t)\big)$ for $t \in (t_0, t_0 + \varepsilon)$. Then $\big(\varphi_p(x'(t_0)) - \varphi_p\big(\alpha'(t_0)\big)\big)' \ge 0$. This is in a contradiction with

$$\varphi_p\big(x'(t_0)\big) - \varphi_p\big(\alpha'(t_0)\big)' < f\Big(t_0, x(t_0), \varphi_p\big(x'(t_0)\big)\Big) - f\Big(t_0, \alpha(t_0), \varphi_p\big(\alpha'(t_0)\big)\Big) = 0.$$

Case $t_0 = 0$ leads to the same contradiction arguing on the interval $(0, \varepsilon)$.

If $t_0 = b$, then $\varphi\big(x'(b)\big) \le \varphi_p\big(\alpha'(b)\big)$ and $\int_0^b x(s)\, dg\,(s) \ge \int_0^b \alpha(s)\, dg\,(s)$. Using the second boundary condition we obtain the contradiction $x(b) = \int_0^b x(s)\, dg\,(s) - k\varphi_p\big(x'(b)\big) \ge \int_0^b \alpha(s)\, dg\,(s) - k\varphi_p\big(\alpha'(b)\big) > \alpha(b)$. □

Our existence result below is based on the use of Leray-Schauder degree for an operator defined on $C^1$ space. Therefore we need a priori bound of a derivative of a solution. This can be achieved by the following version of Nagumo-Bernstein condition. (Compare with [1], [3].)

**Lemma 3.** *Let $f \in C(I \times R^2)$. Let for each $r > 0$ there exist $a_r > 0$ and a function $h_r \in C\big(R_0, [a_r, \infty]\big)$ satisfying*

$$\int_0^\infty \frac{\varphi_q(s)}{h_r(s)}\, ds = \infty$$

*such that*

$$|f(t, x, y)| < h_r(|y|) \qquad \text{for} \quad t \in I, \quad |x| < r, \quad y \in R.$$

*Then for each $r > 0$ there exists $\rho_r > 0$ such that for a solution $x$ of* (1), (2) *$|x| < r$ implies $|x'| < \rho_r$.*

P r o o f. Let $|x(t)| < r$ be a solution of (1), (2). Suppose that $x'(\tau) > 0$ on $(t_0, t)$. Then substitution $\varphi_p\big(x'(t)\big) = y(t)$ leads to

$$y'(t) = f(t, x, y) \le h_r(|y|)$$

and

$$\int_{t_0}^{t} \frac{\varphi_q(y)y'}{h_r(|y|)}\, d\tau \le \int_{t_0}^{t} x' d\tau \le 2r.$$

Substitution $y(t) = s$ leads to

$$\int_{y(t_0)}^{y(t)} \frac{\varphi_q(s)}{h_r(s)}\, ds \le 2r.$$

Divergence at infinity of the left hand side integral implies the existence of $\rho_r > 0$ such that $y(t) < \varphi_p(\rho_r)$. Similarly, we proceed in the case $x'(\tau) < 0$ on $(t_0, t)$ using substitution $-y(t) = s$. □

Our first existence theorem covers the case of constant lower and upper solutions. We denote $G(s) = \mathrm{var}_{[0,s]} g(\tau)$.

**Theorem 4.** *Let $r > 0$ be such that*

(i) *$f(t, r, 0) > 0$ and $f(t, -r, 0) < 0$ on $I$,*

(ii) *there exists a function $h_r \in C\big(R_0, [a_r, \infty]\big)$ with $a_r > 0$ satisfying $\int_0^\infty \frac{\varphi_q(s)}{h_r(s)}\, ds = \infty$ such that*

$$|f(t, x, y)| < h_r(|y|) \qquad \text{for} \quad t \in I, \quad |x| < r, \quad y \in R,$$

(iii) *$G(b) < 1$.*

*Then there exists a solution $x$ of* (1), (2) *such that $|x(t)| < r$.*

Proof. Set $X = C^1([0,b])$ and define an operator $T\colon X \to X$ by

$$Tx(t) = \frac{1}{G(b)-1}\left\{\int_0^b G(s)\varphi_q\big(F_x(s)\big)\,ds + k\big(F_x(b)\big)\right\} - \int_t^b \varphi_q\big(F_x(s)\big)\,ds,$$

where

$$F_x(s) = \int_0^s f\Big(\tau, x(\tau), \varphi_p\big(x'(\tau)\big)\Big)\,d\tau.$$

Then $Tx(t) \in D = \big\{x \in C^1(I),\ \varphi_p(x') \in C^1(I)\big\}$, and $(Tx)'(0) = 0$. We prove that $Tx$ satisfies also the second, nonlocal boundary condition.

Changing the order of integration we obtain

$$Tx(b) = \frac{1}{G(b)-1}\left\{\int_0^b\int_s^b \varphi_q\big(F_x(\tau)\big)\,d\tau\,dg(s) + k\big(F_x(b)\big)\right\}.$$

Then

$$Tx(b) = G(b)Tx(b) - \int_0^b\int_s^b \varphi_q\big(F_x(\tau)\big)\,d\tau\,dg(s) - k\big(F_x(b)\big). \tag{3}$$

Integrating

$$Tx(t) = Tx(b) - \int_t^b \varphi_q\big(F_x(s)\big)\,ds,$$

we obtain

$$\int_0^b Tx(s)\,dg(s) = G(b)Tx(b) - \int_0^b\int_s^b \varphi_q\big(F_x(\tau)\big)\,d\tau\,dg(s). \tag{4}$$

As

$$F_x(b) = \varphi_p\big((Tx)'(b)\big) \tag{5}$$

substituting (4), (5) into (3), we obtain that $Tx$ satisfies the nonlocal boundary condition.

Operator $T\colon X \to X$ is completely continuous and a fixed point of $T$ is a solution of (1), (2).

Functions $\alpha(t) = -r$ and $\beta(t) = r$ are strict lower and upper solutions of (1), (2) and moreover of a perturbed boundary value problem

$$\big(\varphi_p(x')\big)' = \lambda f\big(t, x, \varphi_p(x')\big) + (1-\lambda)x(t), \tag{6}$$

$$x'(0) = 0, \qquad x(b) = \int_0^b x(s)\,dg(s) - k\varphi_p\big(x'(b)\big), \tag{7}$$

with $\lambda \in [0,1]$.

The associated homotopy operator

$$H(x,\lambda) = \frac{1}{G(b)-1}\left\{\int_0^b G(s)\varphi_q\big(F_{x,\lambda}(s)\big)ds + k\big(F_{x,\lambda}(b)\big)\right\} - \int_t^b \varphi_q\big(F_{x,\lambda}(s)\big)ds,$$

with $F_{x,\lambda}(s) = \int_0^s \lambda f\big(\tau, x(\tau), \varphi_p(x'(\tau))\big) + (1-\lambda)x(\tau)\,d\tau$ is completely continuous. We set $\Omega = \big\{x \in X;\ |x| < r,\ |x'| < \varrho_r\big\}$. As a fixed point of $H$ is a solution of (6), (7) and $-r$, $r$ are strict lower and upper solutions, Lemma 2 and Lemma 3 imply that there is no solution on the boundary of $\Omega$. Then the Leray-Schauder degree of $H(.,\lambda)$ is well defined and independent on $\lambda$.

For $\lambda = 0$ there is $H(x,0)$ an odd operator. Then

$$d(I-T,\Omega,0) = d(I-H(x,0),\Omega,0) = 1 \ (\mathrm{mod}\ 2)$$

which implies the existence of a fixed point $x \in \Omega$ of $T$. □

**Theorem 5.** *Let*

(i) $\alpha(t) \le \beta(t)$ *be a lower and upper solution of* (1), (2),

(ii) *there exist a function* $h \in C(R_0, [a,\infty])$ *with* $a > 0$ *satisfying* $\int_0^\infty \frac{\varphi_q(s)}{h(s)}\,ds = \infty$ *such that*

$$|f(t,x,y)| < h(|y|) \qquad \text{for} \quad t \in I, \quad \alpha(t) \le x \le \beta(t), \quad y \in R,$$

(iii) $G(b) < 1$.

*Then there exists a solution* $x$ *of* (1), (2) *such that* $\alpha(t) \le x(t) \le \beta(t)$.

Proof. Set $r = \max\{||\alpha||, ||\beta||\}$, and choose $M > \max\big\{|f(t,x,y)|;\ t \in I,\ \alpha(t) \le x \le \beta(t),\ |y| < \varrho_r\big\}$ and consider a perturbation (see [4])

$$\big(\varphi_p(x')\big)' = f^*\big(t, x, \varphi_p(x')\big), \tag{8}$$

$$x'(0) = 0, \qquad x(b) = \int_0^b x(s)\,dg\,(s) - k\varphi_p\big(x'(b)\big) \tag{9}$$

of the problem (1), (2) with

$$f^*(t,x,y) = \begin{cases} f\big(t,\beta(t),y\big) + M\big(r-\beta(t)\big) + M, & x > r+1, \\ f\big(t,\beta(t),y\big) + M\big(x-\beta(t)\big), & \beta(t) < x \le r+1, \\ f(t,x,y), & \alpha(t) \le x \le \beta(t), \\ f\big(t,\alpha(t),y\big) - M\big(\alpha(t)-x\big), & -r-1 \le x < \alpha(t), \\ f\big(t,\alpha(t),y\big) - M - M\big(\alpha(t)+r\big), & x < -r-1. \end{cases}$$

Then for each $\epsilon > 0$

$$\Big(\varphi_p\big(\alpha'(t)\big)\Big)' > f^*\big(t,\alpha(t)\big) - \epsilon, \varphi_p\big(\alpha'(t)\big).$$

That means $\alpha(t) - \epsilon$ is a strict lower solution of the BVP (8), (9).

Similarly, $\beta(t) + \epsilon$ is a strict upper solution of (8), (9).

Moreover, $-(r+1)$, $r+1$ are also strict lower and upper solutions of (8), (9) and $f^*$ satisfies (ii) of Theorem 4 with $h_{r+1}(s) = h(s) + (2r+1)M$.

Theorem 4 implies the existence of a solution $x$ of (8), (9) satisfying $|x(t)| < r+1$.

We prove that $x(t) \geq \alpha(t)$. Assuming the contrary we suppose that $\max\bigl(\alpha(t) - x(t)\bigr) = \epsilon > 0$. But $\alpha(t) - \epsilon$ is a strict lower solution which is in a contradiction with $\alpha(t_0) - \epsilon = x(t_0)$ due to Lemma 2. Then $\alpha(t) \leq x(t)$. Similarly, $x(t) \leq \beta(t)$. That means $f^*\bigl(t, x, \varphi_p(x')\bigr) = f\bigl(t, x, \varphi_p(x')\bigr)$ and $x(t)$ is also a solution of (1), (2). □

Example 6. We consider a three-point boundary value problem

$$\bigl(\varphi_p(x')\bigr)' = f\bigl(t, x, \varphi_p(x')\bigr), \tag{10}$$

$$x'(0) = 0, \quad x(1) = \frac{1}{2} x\left(\frac{1}{2}\right). \tag{11}$$

Assume that $f$ is a continuous function and

(i) there is a constant $y_0 > 0$ such that $|f(t,x,y)| < |y|^q$ for $t \in I$, $x \geq 0$, $|y| \geq y_0$,

(ii) $-M \leq f(t,x,y)$ for $t \in I$, $x \geq 0$, $y \leq 0$,

(iii) $f(t,0,0) \leq 0$ for $t \in I$.

Then $\alpha(t) = 0$ is a lower solution and

$$\beta(t) = \frac{1}{q}(aM)^{\frac{q}{p}}(a^q - t^q),$$

where $a = (2 - 2^{-q})^{\frac{1}{q}}$, is an upper solution of (10), (11).

We set

$$m = \max\bigl\{|f(t,x,y)|;\ \text{ for } t \in I,\ 0 \leq x \leq \beta(t),\ |y| \leq y_0\bigr\}.$$

Then Nagumo condition (ii) of Theorem 5 is satisfied with the function $h(s) = \max\{m, |s|^q\}$ and Theorem 5 implies the existence of a nonnegative solution of (10), (11).

REFERENCES

[1] GAINES, R. E.—MAWHIN, J.: *Coincidence Degree and Nonlinear Differential Equations.* Lecture Notes in Math., Vol. 568, Springer-Verlag, Berlin, 1977.

[2] LIU, Y.: *The existence of multiple positive solutions of p-Laplacian boundary value problems*, Math. Slovaca **57** (2007), 225–242.

[3] RACHŮNKOVÁ, I.—STANĚK, S.—TVRDÝ, M.: *Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations*, in: Handbook of differential equations, Vol. 3, Elsevier, Amsterdam, 2006.

[4] RUDOLF, B.: *On the generalized boundary value problem*, Arch. Math. (Brno) **36** (2000), 125–137.

[5] WANG, Y.—GE, W.: *Positive solutions for multipoint boundary value problems with a one-dimensional p-Laplacian*, Nonlinear Anal. **66** (2007), 1246–1256.

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