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ON THE KNESER-HUKUHARA PROPERTY FOR AN INTEGRO-DIFFERENTIAL EQUATION IN BANACH SPACES

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ABSTRACT. In this paper we investigate some topological properties of solutions sets of some integro-differential equations in Banach spaces. Our assumptions and proofs are expressed in terms of the measure of weak noncompactness.

1. Introduction.

Let $I = [0, a]$ be a compact interval in $\mathbb{R}$, $B = \{x \in E \colon \|x\| \leq b\}$ and let E be a sequentially weakly complete Banach space. Throughout this paper we shall assume that $f \colon I \times B \mapsto E$ and $g \colon I^2 \times B \mapsto E$ are functions continuous in the weak—weak sense, that is for every $t \in I$, $x \in B$ and arbitrary weak neighbourhood U of the point $f(t, x)$ there exists an $\varepsilon > 0$ and a weak neighbourhood V of x so that for every $y \in V \cap B$, $s \in I$, $|s - t| < \varepsilon$, $f(s, y) \in U$ is valid.

Consider the Cauchy problem

$$x^{(m)}(t) = f\big(t, x(t)\big) + \int_0^t g\big(t, s, x(s)\big)\, ds, \qquad x(0) = 0,\ x'(0) = \eta_1, \ldots, x^{(m-1)}(0) = \eta_{m-1}, \tag{1}$$

where $m \geq 1$ and $\eta_1, \ldots, \eta_{m-1} \in E$ and $x^{(m)}$ means the mth order derivative in the weak sense and integral denotes the weak Riemann integral. Let us recall that the weak Riemann integral of a weak continuous function $y(t)$ $(t \in I)$ with values in E is defined as the weak limit of Riemann sums (cf. [7]).

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In this paper we prove that the set of all weak solutions of this problem, defined on a compact subinterval $J = [0, d]$ of I, for some $d > 0$, is nonempty, compact and connected in the space $C_w(J, E)$ of weakly continuous functions $J \mapsto E$ with the topology of weak uniform convergence.

The method of the proof of our main result is suggested by the paper [9] concerning differential equations. Nevertheless the idea to consider the ε-approximate solutions set of Volterra integral equation

$$x(t) = f(t) + \int_0^t g\big(t, s, x(s)\big)\, ds$$

goes back to H u k u h a r a [6], who proved that this set is connected in $C(J, \mathbb{R}^n)$.

Our approach is to impose a weak compactness type conditions expressed in terms of the measure of weak noncompactness introduced by D e B l a s i [5].

Let A be a nonvoid bounded subset of E. The measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact set } K \text{ such that } A \subset K + \varepsilon B\},$$

where B is the norm unit ball.

We make use of the following properties of the measure of weak noncompactness β (for bounded nonvoid subsets A and B of E):

1° $A \subset B \Rightarrow \beta(A) \le \beta(B)$;

2° $\beta(\bar{A}^w) = \beta(A)$ where $\bar{A}^w$ denotes the weak closure of A;

3° $\beta(A) = 0 \Leftrightarrow \bar{A}^w$ is weakly compact;

4° $\beta(A \cup B) = \max\big(\beta(A), \beta(B)\big)$;

5° $\beta(\mathrm{conv} A) = \beta(A)$;

6° $\beta(A + B) \le \beta(A) + \beta(B)$;

7° $\beta(\lambda A) = |\lambda| \beta(A)$, $(\lambda \in \mathbb{R})$;

8° $\beta\big(\bigcup_{|\lambda| \le h} \lambda A\big) = h\beta(A)$.

2. Basic lemmas

Let V be a subset of $C_w(J, E)$. Put

$$V(t) = \big\{u(t) \colon u \in V\big\} \quad \text{and} \quad V(T) = \big\{u(t) \colon u \in V,\ t \in T\big\}.$$

In what follows we shall use the following Ambrosetti-type

Lemma 1. *If the set V is strongly equicontinuous and uniformly bounded, then*

(i) *the function* $t \mapsto \beta\big(V(t)\big)$ *is continuous on* J;

(ii) *for each compact subset* T *of* J

$$\beta\big(V(T)\big) = \sup\Big\{\beta\big(V(t)\big) \colon t \in T\Big\},$$

and Krasnoselskii-type.

Lemma 2 ([9]). *For any* $\varphi \in E^*$, $\varepsilon \geq 0$ *and for any weakly continuous function* $z \colon J \mapsto B$ *there exists a weak neighbourhood* U *of* 0 *in* E *such that* $\big\|\varphi\big(f(t,z(t)) - f(t,y(t))\big)\big\| \leq \varepsilon$ *for* $t \in J$ *and for every weakly continuous function* $y \colon J \mapsto B$ *such that* $y(s) - z(s) \in U$ *for all* $s \in J$.

In our considerations we apply the following

Lemma 3 ([8]). *Let* $m \geq 1$ *be a natural number and let* $w \colon [0,2b] \mapsto \mathbb{R}_+$ *be a continuous nondecreasing function such that* $w(0) = 0$, $w(r) > 0$ *for* $r > 0$ *and*

$$\int\limits_{0+} \frac{dr}{\sqrt[m]{r^{m-1}w(r)}} = \infty.$$

If $u \colon [0,c) \mapsto [0,2b]$ *is a* C^m *function satisfying the inequalities*

$$\begin{aligned} &u^{(j)}(t) \geq 0, && j = 0,1,\ldots,m,\\ &u^{(j)}(0) = 0, && j = 0,1,\ldots,m-1,\\ &u^{(m)}(t) \leq w\big(u(t)\big), && t \in [0,c), \end{aligned}$$

then $u = 0$.

3. The main result

Put

$$\begin{aligned} M_1 &= \sup\big\{\|f(t,x)\| \colon t \in I,\, x \in B\big\},\\ M_2 &= \sup\big\{\|g(t,s,x)\| \colon t,\, s \in I,\, x \in B\big\}. \end{aligned}$$

Choose a positive number d such that $d \leq a$ and

$$\sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + M_1 \frac{d^m}{m!} + M_2 \frac{d^{m+1}}{m!} < b. \tag{2}$$

Let $J = [0,d]$. By $C_w(J,E)$ we denote the space of weakly continuous functions $J \mapsto E$ endowed with the topology of weak uniform convergence and by E^* the space of continuous linear functionals on E.

Our main result is given by the following Kneser-Hukuhara-type

Theorem 1. *Let $w\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous nondecreasing function such that $w(0) = 0$ and*

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1}w(r)}} = \infty. \tag{3}$$

If

$$\beta\big(f(J \times X)\big) \le w\big(\beta(X)\big) \qquad \textit{for} \quad X \subset B, \tag{4}$$

and the set $g(I^2 \times B)$ is relatively weakly compact in E, then the set S of all weak solutions of (1) defined on J is nonempty, compact and connected in $C_w(J,E)$.

P r o o f. **1°** Let $\widetilde{B}$ denote the set of all weakly continuous functions $J \mapsto B$. We shall consider $\widetilde{B}$ as a topological subspace of $C_w(J,E)$. For $t \in J$ and $x \in \widetilde{B}$ put

$$\widetilde{g}(t,x) = \int_0^t g\big(t,s,x(s)\big)\,ds.$$

Fix $\tau \in J$ and $x \in \widetilde{B}$. As the set $J \times x(J)$ is weakly compact, from the weak continuity of g it follows that for each $\varepsilon > 0$ and $\phi \in E^\star$ such that $\|\phi\| \le 1$ there exists $\delta > 0$ such that

$$\phi\Big(g\big(t,s,x(s)\big) - g\big(\tau,s,x(s)\big)\Big) < \varepsilon \qquad \text{for} \quad t,s \in J \quad \text{with} \quad |t-\tau| < \delta.$$

In view of the inequality

$$\phi\Big(\widetilde{g}(t,x) - \widetilde{g}(\tau,x)\Big) \le M_2|t-\tau| + \int_0^\tau \phi\Big(g\big(t,s,x(s)\big) - g\big(\tau,s,x(s)\big)\Big)\,ds,$$

this implies the weak continuity of the function $t \to \widetilde{g}(t,x)$. On the other hand, applying Lemma 2, we can prove that for each fixed $t \in J$ the function $x \to \widetilde{g}(t,x)$ is weakly continuous on $\widetilde{B}$. Moreover

$$\|\widetilde{g}(t,x)\| \le M_2 t \qquad \text{for} \quad t \in J \quad \text{and} \quad x \in \widetilde{B}.$$

2° The initial value problem (1) is equivalent to the following integral equation

$$x(t) = p(t) + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}\Big[f\big(s,x(s)\big) + \widetilde{g}(s,x)\Big]ds \qquad (t \in J), \tag{5}$$

where $p(t) = \sum_{j=1}^{m-1} \eta_j \frac{t^j}{j!}$.

Define the operator F by the formula

$$F(x)(t) = p(t) + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}\Big[f\big(s,x(s)\big) + \widetilde{g}(s,x)\Big]ds \qquad (t \in J,\ x \in \widetilde{B}).$$

For simplicity assume that $m \geq 2$.

Let us remark that if

$$y(t) = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f\big(s, x(s)\big)\, ds$$

and

$$z(t) = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \widetilde{g}(s, x)\, ds,$$

then

$$y'(t) = \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} f\big(s, x(s)\big)\, ds$$

and

$$z'(t) = \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} \widetilde{g}(s, x)\, ds,$$

so that

$$\|y'(t)\| \leq \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} M_1\, ds = M_1 \frac{t^{m-1}}{(m-1)!}$$

and

$$\|z'(t)\| \leq \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} M_2 t\, ds = M_2 \frac{t^m}{(m-1)!}.$$

Moreover,

$$\|p'(t)\| \leq \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^{j-1}}{(j-1)!}.$$

By the mean value theorem we obtain

$$\|F(x)(t) - F(x)(\tau)\| \leq K \mid t - \tau \mid \qquad (x \in \widetilde{B},\ t, \tau \in J), \tag{6}$$

where

$$K = \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^{j-1}}{(j-1)!} + M_1 \frac{d^{m-1}}{(m-1)!} + M_2 \frac{d^m}{(m-1)!}.$$

Since

$$\|y(t)\| \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} M_1\, ds = M_1 \frac{t^m}{m!}$$

and

$$\|z(t)\| \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} M_2 t\, ds = M_2 \frac{t^{m+1}}{m!},$$

$$\|F(x)(t)\| \le L \qquad (x \in \widetilde{B},\ t \in J), \tag{7}$$

where $L = \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + M_1 \frac{d^m}{m!} + M_2 \frac{d^{m+1}}{m!}$.

From (2), (6) and (7) it is clear that $F(\widetilde{B}) \subset \widetilde{B}$ and the set $F(\widetilde{B})$ is strongly equicontinuous. By Lemma 2 we can prove that F is a continuous.

Put $W = \bigcup_{0 \le \lambda \le d} \lambda\, \overline{\mathrm{conv}}\, g(I^2 \times B)$.

Since for convex subsets of E the closure in the norm topology coincides with the weak closure [4, Th. II. 1], it is clear that

$$\int_0^t (t-s)^{m-1} \widetilde{g}(s,x)\, ds \in t\, \overline{\mathrm{conv}} \Big\{ (t-s)^{m-1} \widetilde{g}(s,x) \colon s \in J,\ \ x \in \widetilde{B} \Big\}$$
$$= t\, \overline{\mathrm{conv}} \Big\{ (t-s)^{m-1} W \colon s \in [0,t] \Big\},$$

from the above and corresponding properties of β it follows that

$$\begin{aligned}
&\beta\left(\left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \widetilde{g}(s,x)\, ds \colon x \in \widetilde{B} \right\} \right) \\
&\le \beta\left(\frac{1}{(m-1)!} t\, \overline{\mathrm{conv}} \left\{ (t-s)^{m-1} W \colon s \in J \right\} \right) \\
&\le \beta\left(\frac{1}{(m-1)!} t \left\{ (t-s)^{m-1} W \colon s \in J \right\} \right) \\
&= \max_{s \in J} \left(\frac{1}{(m-1)!} t (t-s)^{m-1} \right) \beta(W) \\
&= \frac{1}{(m-1)!} t^m \beta(W) = 0.
\end{aligned} \tag{8}$$

3° For given $\varepsilon > 0$ denote by S_ε the set of all $z \in \widetilde{B}$ such that

$$\|z(t) - F(z)(t)\| < \varepsilon \quad \text{for all} t \in J.$$

The following lemma is proved in [9].

Lemma 4. *For each ε, $0 < \varepsilon < b - L$, the set S_ε is nonempty and connected in $C_w(J,E)$.*

For any positive integer n we define

$$F_n(x)(t) = \begin{cases} p(t) & \text{if } 0 \le t \le \frac{d}{n}, \\ p(t) + \frac{1}{(m-1)!} \int_0^{t-\frac{d}{n}} (t-s)^{m-1} \Big[f\big(s, x(s)\big) + \widetilde{g}(s,x) \Big] ds & \text{if } \frac{d}{n} \le t \le d \end{cases}$$

for $x \in \widetilde{B}$, $t \in J$. Analogously as for F, by inequalities (6) and (7), we can prove that F_n maps continuosly $\widetilde{B}$ into itself and

$$\|F_n(x)(t) - F(x)(t)\| \le K\frac{d}{n} \qquad (x \in \widetilde{B},\ t \in J). \tag{9}$$

Moreover, there exists a unique $z_n \in \widetilde{B}$ such that $z_n = F_n(z_n)$. It is clear from (9) that $z_n \in S_\varepsilon$ for sufficiently large n.

Next we shall show that the set S is nonempty. From the above it follows that there exists a sequence (u_n) such that $u_n \in \widetilde{B}$ and

$$\lim_{n\to\infty} \sup_{t\in J} \|u_n(t) - F(u_n)(t)\| = 0. \tag{10}$$

Let $V = \{u_n : n \in N\}$. From (6) and (10) we deduce that the set V is strongly equicontinuous and

$$\beta\big(V(t)\big) = \beta\big(F(V)(t)\big) \qquad \text{for} \quad t \in J. \tag{11}$$

Hence, by Lemma 1, the function $t \mapsto v(t) = \beta\big(V(t)\big)$ is continuous on J.

Fix $t \in J$ and $\varepsilon > 0$. Choose $\delta > 0$ in such a way that

$$\big|(t-\tau)^{m-1}w\big(v(q)\big) - (t-s)^{m-1}w\big(v(s)\big)\big| < \varepsilon \tag{12}$$

if $|\tau - s| < \delta$, $|q - s| < \delta$, $q, s, \tau \in J$. Divide the interval $[0,t]$ into n parts $0 = t_0 < t_1 < \dots < t_n = t$ in such way that $\Delta t_i = t_i - t_{i-1} < \delta$ for $i = 1, \dots, n$. Let $T_i = [t_{i-1}, t_i]$. By Lemma 1 for each i there exists $s_i \in T_i$ such that

$$\beta\big(V(T_i)\big) = v(s_i) \qquad (i = 1, \dots, n).$$

By (4) we obtain

$$\begin{aligned}
&\beta\Big(\Big\{(t-s)^{m-1}f\big(s, x(s)\big) : x \in V,\ s \in T_i\Big\}\Big)\\
&\le (t - t_{i-1})^{m-1}\beta\Big(f\big(T_i \times V(T_i)\big)\Big)\\
&\le (t - t_{i-1})^{m-1}w\Big(\beta\big(V(T_i)\big)\Big)\\
&= (t - t_{i-1})^{m-1}w\big(v(s_i)\big).
\end{aligned}$$

Since

$$\begin{aligned}
F(V)(t) \subset p(t) &+ \frac{1}{(m-1)!}\sum_{i=1}^{n} \Delta t_i\, \overline{\operatorname{conv}}\Big\{(t-s)^{m-1}f\big(s, x(s)\big) : x \in V,\ s \in T_i\Big\}\\
&+ \frac{1}{(m-1)!}\left\{\int_0^t (t-s)^{m-1}\widetilde{g}(s,x)\,ds : x \in V\right\},
\end{aligned}$$

from (8) and corresponding properties of β we have

$$\beta\bigl(F(V)(t)\bigr) \le \frac{1}{(m-1)!}\beta\left(\sum_{i=1}^{n}\Delta t_i\,\overline{\mathrm{conv}}\,\{(t-s)^{m-1}f\bigl(s,x(s)\bigr)\colon x\in V,\ s\in T_i\}\right)$$
$$+\frac{1}{(m-1)!}\beta\left(\left\{\int_0^t (t-s)^{m-1}\widetilde{g}(s,x)\,ds\colon x\in V\right\}\right)$$
$$=\frac{1}{(m-1)!}\sum_{i=1}^{n}\Delta t_i\beta\Bigl\{(t-s)^{m-1}f\bigl(s,x(s)\bigr)\colon x\in V,\ s\in T_i\Bigr\}$$
$$\le\frac{1}{(m-1)!}\sum_{i=1}^{n}\Delta t_i(t-t_{i-1})^{m-1}w\bigl(v(s_i)\bigr).$$

Furthermore, from (12) we infer that

$$\frac{1}{(m-1)!}\sum_{i=1}^{n}(t-t_{i-1})^{m-1}w\bigl(v(s_i)\bigr)\Delta t_i$$
$$\le\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}w\bigl(v(s)\bigr)\,ds+\frac{\varepsilon t}{(m-1)!}.$$

Therefore

$$\beta\bigl(F(V)(t)\bigr)\le\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}w\bigl(v(s)\bigr)\,ds+\frac{\varepsilon t}{(m-1)!}.$$

Because ε is arbitrary

$$\beta\bigl(F(V)(t)\bigr)\le\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}w\bigl(v(s)\bigr)\,ds\,.$$

Thus, by (11),

$$v(t)\le\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}w\bigl(v(s)\bigr)\,ds\qquad\text{for}\quad t\in J.$$

Putting $u(t)=\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}w\bigl(v(s)\bigr)ds$ we see that $u\in C^m$, $v(t)\le u(t)$, $u^{(j)}(t)\ge 0$ for $j=0,1,\dots,m$, $u^{(j)}(0)=0$ for $j=0,1,\dots,m-1$ and

$$u^{(m)}(t)=w\bigl(v(t)\bigr)\le w\bigl(u(t)\bigr)\qquad\text{for}\quad t\in J.$$

As $u(0)=0$, from Lemma 3 we deduce that $u(t)=0$ for $t\in J$. Consequently, $\beta\bigl(V(t)\bigr)=v(t)=0$ for $t\in J$, i.e., $V(t)$ is relatively weakly compact for $t\in J$. Hence Ascoli's theorem implies that V is relatively compact in $C_w(J,E)$. Therefore the sequence (u_n) has a limit point x. From (10) and the continuity of F it follows that $x=F(x)$, i.e., $x\in S$.

4° Now we shall prove that the set S is compact and then that it is connected. Since F is continuous, S is closed in $C_w(J,E)$. As $S = F(S)$, we have $\beta\big(S(t)\big) = \beta\big(F(S)(t)\big)$ for $t \in J$. Therefore, repeating the argument from 3°, we can show that S is compact in $C_w(J,E)$. Suppose that S is not connected in $C_w(J,E)$. As S is compact, there are nonempty compact sets S_1, S_2 such that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, and consequently there are two disjoint open sets U_1, U_2 such that $S_1 \subset U_1$, $S_2 \subset U_2$. Let $U = U_1 \cup U_2$. We choose n_0 such that $\frac{1}{n_0} < b - L$. Suppose that for each $n \geq n_0$ there exists $u_n \in S_{\frac{1}{n}} \backslash U$. Put $V = \{u_n \colon n \in N\}$. Because $\lim_{n\to\infty} \sup_{t\in J} \|u_n(t) - F(u_n)(t)\| = 0$, using once more similar arguments as in 3°, we can prove that there exists $u_0 \in \overline{V}$ such that $u_0 = F(u_0)$, i.e., $u_0 \in S$. Furthermore, $\overline{V} \subset C_w(J,E) \backslash U$, as U is open, so that $u_0 \in S \backslash U$, a contradiction. Therefore there exists $k \in N$ such that $S_{\frac{1}{k}} \subset U$. Since $U_1 \cap S_{\frac{1}{k}} \neq \emptyset \neq U_2 \cap S_{\frac{1}{k}}$, this shows that $S_{\frac{1}{k}}$ is not connected, which contradicts Lemma 4. Hence S is connected. □

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