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OSCILLATION RESULTS FOR SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATIONS OF MIXED TYPE

Tongxing Li — Blanka Baculíková — Jozef Džurina

ABSTRACT. Some oscillation theorems are established for the second-order linear neutral differential equations of mixed type

$\big(r(t)[x(t)+p_1(t)x(t-\sigma_1)+p_2(t)x(t+\sigma_2)]'\big)'+q_1(t)x(t-\sigma_3)+q_2(t)x(t+\sigma_4)=0.$

Several examples are also provided to illustrate the main results.

1. Introduction

This paper is concerned with the oscillatory behavior of the second-order linear neutral differential equation of mixed type

$$\Big(r(t)\big[x(t)+p_1(t)x(t-\sigma_1)+p_2(t)x(t+\sigma_2)\big]'\Big)' + q_1(t)x(t-\sigma_3)+q_2(t)x(t+\sigma_4)=0, \qquad t\ge t_0. \tag{1.1}$$

Throughout this paper, we will assume the following conditions hold:

(A_1) $r\in C^1\big([t_0,\infty),\mathbb{R}\big)$, $r(t)>0$ for $t\ge t_0$;

(A_2) $p_i\in C\big([t_0,\infty),[0,a_i]\big)$, where a_i are constants for $i=1,2$;

(A_3) $q_j\in C\big([t_0,\infty),[0,\infty)\big)$, for $j=1,2$;

(A_4) $\sigma_i\ge 0$ are constants, for $i=1,2,3,4$.

By a solution of Eq. (1.1), we mean a function $x\in C\big([T_x,\infty),\mathbb{R}\big)$ for some $T_x\ge t_0$ which has the properties $\big[x(t)+p_1(t)x(t-\sigma_1)+p_2(t)x(t+\sigma_2)\big]\in C^1\big([T_x,\infty),\mathbb{R}\big)$ and $r(t)\big[x(t)+p_1(t)x(t-\sigma_1)+p_2(t)x(t+\sigma_2)\big]\in C^1\big([T_x,\infty),\mathbb{R}\big)$ and satisfying Eq. (1.1) on $[T_x,\infty)$. As is customary, a solution of Eq. (1.1) is

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called oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$, otherwise, it is called nonoscillatory. Eq. (1.1) is said to be oscillatory if all its solutions are oscillatory.

Neutral functional differential equations have numerous applications in electric networks. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high speed computers where the lossless transmission lines are used to interconnect switching circuits; see [1].

In recent years, many results have been obtained on oscillation of nonneutral differential equations and neutral functional differential equations, we refer the reader to the papers [2]–[7] and [8]–[30], and the references cited therein.

Philos [2] established some Philos-type oscillation criteria for the second-order linear differential equation

$$\big(r(t)x'(t)\big)' + q(t)x(t) = 0.$$

In [3]–[5], the authors gave some sufficient conditions for oscillation of all solutions of second-order half-linear differential equation

$$\big(r(t)|x'(t)|^{\gamma-1}x'(t)\big)' + q(t)\big|x\big(\tau(t)\big)\big|^{\gamma-1}x\big(\tau(t)\big) = 0,$$

by employing a Riccati substitution technique.

Džurina [7] presented some sufficient conditions for the oscillation of the second-order differential equation with mixed argument

$$\left(\frac{1}{r(t)}u'(t)\right)' + p(t)u\big(\tau(t)\big) + q(t)u\big(\sigma(t)\big) = 0, \qquad t \ge t_0.$$

Han et al. [14], [15] examined the oscillation of second-order neutral differential equation

$$\left(r(t)\Big[x(t) + p(t)x\big(\tau(t)\big)\Big]'\right)' + q(t)x\big(\sigma(t)\big) = 0, \tag{1.2}$$

where $\tau'(t) = \tau_0 > 0, 0 \le p(t) \le p_0 < \infty$, and the authors obtained some oscillation criteria for (1.2) when

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\mathrm{d}t = \infty \tag{1.3}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\mathrm{d}t < \infty. \tag{1.4}$$

Some oscillation criteria for the following second-order neutral differential equation

$$\big(r(t)|z'(t)|^{\gamma-1}z'(t)\big)' + q(t)\big|x\big(\sigma(t)\big)\big|^{\gamma-1}x\big(\sigma(t)\big) = 0,$$

with $z(t) = x(t) + p(t)x\big(\tau(t)\big)$ were established by [16]–[20].

However, there are few results regarding the oscillatory problems of neutral differential equations with mixed arguments, see the papers [23]–[30]. In [23], the authors established some oscillation criteria for the following mixed neutral equation

$$\big(x(t) + p_1x(t-\tau_1) + p_2x(t+\tau_2)\big)'' = q_1(t)x(t-\sigma_1) + q_2(t)x(t+\sigma_2),$$

here q_1 and q_2 are nonnegative real-valued functions. Yan [24] considered the oscillation of even-order mixed neutral differential equation

$$\big(x(t) - c_1x(t-h_1) - c_2x(t+h_2)\big)^{(n)} + qx(t-g_1) + px(t+g_2) = 0,$$

where c_1 and c_2 are nonnegative, p and q are positive real numbers. Grace [25] obtained some oscillation theorems for the odd order neutral differential equation

$$\big(x(t) + p_1x(t-\tau_1) + p_2x(t+\tau_2)\big)^{(n)} = q_1x(t-\sigma_1) + q_2x(t+\sigma_2),$$

where $n \geq 1$ is odd. Grace [26] and Yan [27] obtained several sufficient conditions for the oscillation of solutions of higher order neutral functional differential equation of the form

$$\big(x(t) + cx(t-h) + Cx(t+H)\big)^{(n)} + qx(t-g) + Qx(t+G) = 0, \tag{1.5}$$

where q and Q are nonnegative real constants.

The purpose of this paper is to study the oscillation problem of (1.1). The organization of this paper is as follows: In Section 2, by using Riccati substitution technique, some oscillation criteria are obtained for (1.1). In Section 3, we give some examples to illustrate the main results.

In the sequel, for the sake of convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

2. Main results

In this section, we will establish some oscillation criteria for Eq. (1.1).

Throughout this paper, we let

$$\begin{aligned} Q(t) &= Q_1(t) + Q_2(t), Q_1(t) = \min\big\{q_1(t), q_1(t-\sigma_1), q_1(t+\sigma_2)\big\}, \\ Q_2(t) &= \min\big\{q_2(t), q_2(t-\sigma_1), q_2(t+\sigma_2)\big\}, \big(\rho'(t)\big)_+ = \max\big\{0, \rho'(t)\big\}, \end{aligned}$$

$$\delta(t) = \int_{t+\sigma_4}^{\infty} \frac{1}{r(s)}\mathrm{d}s \quad \text{and} \quad \zeta(t) = \delta(t+\sigma_2).$$

Theorem 2.1. *Suppose that* (1.3) *holds and* $\sigma_3 \geq \sigma_1$. *Moreover, assume that there exists* $\rho \in C^1\big([t_0,\infty),(0,\infty)\big)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)Q(s)-\frac{1+a_1+a_2}{4}\cdot\frac{r(s-\sigma_3)((\rho'(s))_+)^2}{\rho(s)}\right]\mathrm{d}s=\infty \tag{2.1}$$

holds. Then every solution of Eq. (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t)>0$, $x(t-\sigma_1)>0$, $x(t+\sigma_2)>0$, $x(t-\sigma_3)>0$ and $x(t+\sigma_4)>0$ for all $t \geq t_1$. Define

$$z(t)=x(t)+p_1(t)x(t-\sigma_1)+p_2(t)x(t+\sigma_2).$$

Then $z(t)>0$ for $t\geq t_1$. In view of (1.1), we obtain

$$\big(r(t)z'(t)\big)'=-q_1(t)x(t-\sigma_3)-q_2(t)x(t+\sigma_4)\leq 0, \qquad t\geq t_1. \tag{2.2}$$

Thus, $r(t)z'(t)$ is nonincreasing function. By (1.3), there exists a $t_2 \geq t_1$ such that

$$z'(t)>0,\ z'(t-\sigma_1)>0 \quad \text{and} \quad z'(t+\sigma_2)>0, \tag{2.3}$$

for $t\geq t_2$.

By applying (1.1), for all sufficiently large t, we obtain

$$\begin{aligned} a_1\big(r(t-\sigma_1)z'(t-\sigma_1)\big)' &+ a_1q_1(t-\sigma_1)x(t-\sigma_1-\sigma_3)\\ &+ a_1q_2(t-\sigma_1)x(t+\sigma_4-\sigma_1)=0 \end{aligned}$$

and

$$\begin{aligned} a_2\big(r(t+\sigma_2)z'(t+\sigma_2)\big)' &+ a_2q_1(t+\sigma_2)x(t+\sigma_2-\sigma_3)\\ &+ a_2q_2(t+\sigma_2)x(t+\sigma_2+\sigma_4)=0. \end{aligned}$$

Combining the previous two equalities with (1.1), we have

$$\begin{aligned} \big(r(t)z'(t)\big)' &+ a_1\big(r(t-\sigma_1)z'(t-\sigma_1)\big)' + a_2\big(r(t+\sigma_2)z'(t+\sigma_2)\big)'\\ &+ Q_1(t)z(t-\sigma_3)+Q_2(t)z(t+\sigma_4)\leq 0. \end{aligned} \tag{2.4}$$

Since $z'(t)>0$, we have $z(t+\sigma_4)\geq z(t-\sigma_3)$. Then,

$$\begin{aligned} \big(r(t)z'(t)\big)' &+ a_1\big(r(t-\sigma_1)z'(t-\sigma_1)\big)' + a_2\big(r(t+\sigma_2)z'(t+\sigma_2)\big)'\\ &+ Q(t)z(t-\sigma_3)\leq 0. \end{aligned} \tag{2.5}$$

Using the Riccati transformation

$$\omega_1(t)=\rho(t)\frac{r(t)z'(t)}{z(t-\sigma_3)}, \qquad t\geq t_2. \tag{2.6}$$

Then $\omega_1(t) > 0$ for $t \geq t_2$. Differentiating (2.6), we obtain

$$\omega_1'(t) = \rho'(t)\frac{r(t)z'(t)}{z(t-\sigma_3)} + \rho(t)\frac{(r(t)z'(t))'}{z(t-\sigma_3)} - \rho(t)\frac{r(t)z'(t)z'(t-\sigma_3)}{z^2(t-\sigma_3)}.$$

By (2.2), we have $r(t-\sigma_3)z'(t-\sigma_3) \geq r(t)z'(t)$. Thus, from (2.6), we get

$$\omega_1'(t) \leq \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) + \rho(t)\frac{(r(t)z'(t))'}{z(t-\sigma_3)} - \frac{(\omega_1(t))^2}{\rho(t)r(t-\sigma_3)}. \tag{2.7}$$

Next, define function ω_2 by

$$\omega_2(t) = \rho(t)\frac{r(t-\sigma_1)z'(t-\sigma_1)}{z(t-\sigma_3)}, \qquad t \geq t_2. \tag{2.8}$$

Then $\omega_2(t) > 0$ for $t \geq t_2$. Differentiating (2.8), we see that

$$\begin{aligned}\omega_2'(t) =& \rho'(t)\frac{r(t-\sigma_1)z'(t-\sigma_1)}{z(t-\sigma_3)} + \rho(t)\frac{(r(t-\sigma_1)z'(t-\sigma_1))'}{z(t-\sigma_3)} \\ &- \rho(t)\frac{r(t-\sigma_1)z'(t-\sigma_1)z'(t-\sigma_3)}{z^2(t-\sigma_3)}.\end{aligned}$$

Note that $\sigma_3 \geq \sigma_1$. By (2.2), we have $r(t-\sigma_3)z'(t-\sigma_3) \geq r(t-\sigma_1)z'(t-\sigma_1)$. Hence, by (2.8), we get

$$\omega_2'(t) \leq \frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) + \rho(t)\frac{(r(t-\sigma_1)z'(t-\sigma_1))'}{z(t-\sigma_3)} - \frac{(\omega_2(t))^2}{\rho(t)r(t-\sigma_3)}. \tag{2.9}$$

In the following, we define another function ω_3 by

$$\omega_3(t) = \rho(t)\frac{r(t+\sigma_2)z'(t+\sigma_2)}{z(t-\sigma_3)}, \qquad t \geq t_2. \tag{2.10}$$

Then $\omega_3(t) > 0$ for $t \geq t_2$. Differentiating (2.10), we see that

$$\begin{aligned}\omega_3'(t) =& \rho'(t)\frac{r(t+\sigma_2)z'(t+\sigma_2)}{z(t-\sigma_3)} + \rho(t)\frac{(r(t+\sigma_2)z'(t+\sigma_2))'}{z(t-\sigma_3)} \\ &- \rho(t)\frac{r(t+\sigma_2)z'(t+\sigma_2)z'(t-\sigma_3)}{z^2(t-\sigma_3)}.\end{aligned}$$

By (2.2), we have $r(t-\sigma_3)z'(t-\sigma_3) \geq r(t+\sigma_2)z'(t+\sigma_2)$. Then, from (2.10), we get

$$\omega_3'(t) \leq \frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) + \rho(t)\frac{(r(t+\sigma_2)z'(t+\sigma_2))'}{z(t-\sigma_3)} - \frac{(\omega_3(t))^2}{\rho(t)r(t-\sigma_3)}. \tag{2.11}$$

Therefore, by (2.7), (2.9) and (2.11), we obtain

$$\begin{aligned}
&\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t) \\
&\le \rho(t)\left[\frac{(r(t)z'(t))' + a_1(r(t-\sigma_1)z'(t-\sigma_1))' + a_2(r(t+\sigma_2)z'(t+\sigma_2))'}{z(t-\sigma_3)}\right] \\
&\quad + \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) - \frac{(\omega_1(t))^2}{\rho(t)r(t-\sigma_3)} + a_1\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) - a_1\frac{(\omega_2(t))^2}{\rho(t)r(t-\sigma_3)} \\
&\quad + a_2\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) - a_2\frac{(\omega_3(t))^2}{\rho(t)r(t-\sigma_3)}. \qquad (2.12)
\end{aligned}$$

Thus, from (2.5) and (2.12), we get

$$\begin{aligned}
&\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t) \\
&\le -\rho(t)Q(t) + \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) - \frac{(\omega_1(t))^2}{\rho(t)r(t-\sigma_3)} + a_1\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) \\
&\quad - a_1\frac{(\omega_2(t))^2}{\rho(t)r(t-\sigma_3)} + a_2\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) - a_2\frac{(\omega_3(t))^2}{\rho(t)r(t-\sigma_3)}. \qquad (2.13)
\end{aligned}$$

Then, by (2.13), we find that

$$\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t) \le -\rho(t)Q(t) + \frac{1+a_1+a_2}{4}\frac{r(t-\sigma_3)((\rho'(t))_+)^2}{\rho(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$\begin{aligned}
&\int_{t_2}^{t}\left[\rho(s)Q(s) - \frac{1+a_1+a_2}{4}\frac{r(s-\sigma_3)((\rho'(s))_+)^2}{\rho(s)}\right]\mathrm{d}s \\
&\le \omega_1(t_2) + a_1\omega_2(t_2) + a_2\omega_3(t_2),
\end{aligned}$$

which contradicts (2.1). The proof is complete. □

As an immediate consequence of Theorem 2.1 we get the following.

Corollary 2.1. *Let assumption* (2.1) *in Theorem 2.1 be replaced by*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\rho(s)Q(s)\mathrm{d}s = \infty,$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{r(s-\sigma_3)\left((\rho'(s))_+\right)^2}{\rho(s)}\mathrm{d}s < \infty.$$

Then every solution of Eq. (1.1) *oscillates.*

From Theorem 2.1 by choosing the function ρ, appropriately, we can obtain different sufficient conditions for oscillation of Eq. (1.1), if we define a function ρ by $\rho(t) = 1$, and $\rho(t) = t$, we have the following oscillation results.

Corollary 2.2. *Suppose that* (1.3) *holds and* $\sigma_3 \geq \sigma_1$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} Q(s)\mathrm{d}s = \infty, \tag{2.14}$$

then every solution of Eq. (1.1) *oscillates.*

Corollary 2.3. *Suppose that* (1.3) *holds and* $\sigma_3 \geq \sigma_1$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[sQ(s) - \frac{1 + a_1 + a_2}{4} \frac{r(s - \sigma_3)}{s} \right] \mathrm{d}s = \infty, \tag{2.15}$$

then every solution of Eq. (1.1) *oscillates.*

In the following theorem, we present another oscillation criterion for Eq. (1.1) when $\sigma_1 \geq \sigma_3$.

Theorem 2.2. *Suppose that* (1.3) *holds and* $\sigma_1 \geq \sigma_3$. *Moreover, assume that there exists* $\rho \in C^1\big([t_0, \infty), (0, \infty)\big)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)Q(s) - \frac{1 + a_1 + a_2}{4} \frac{r(s - \sigma_1)((\rho'(s))_+)^2}{\rho(s)} \right] \mathrm{d}s = \infty \tag{2.16}$$

holds. Then every solution of Eq. (1.1) *oscillates.*

P r o o f. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t - \sigma_1) > 0$, $x(t + \sigma_2) > 0$, $x(t - \sigma_3) > 0$ and $x(t + \sigma_4) > 0$ for all $t \geq t_1$. Define

$$z(t) = x(t) + p_1(t)x(t - \sigma_1) + p_2(t)x(t + \sigma_2).$$

Then $z(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 2.1, we obtain (2.2)–(2.5), for $t \geq t_2 \geq t_1$.

Using the Riccati transformation

$$\omega_1(t) = \rho(t)\frac{r(t)z'(t)}{z(t - \sigma_1)}, \qquad t \geq t_2. \tag{2.17}$$

Then $\omega_1(t) > 0$ for $t \geq t_2$. Differentiating (2.17), we see that

$$\omega_1'(t) = \rho'(t)\frac{r(t)z'(t)}{z(t - \sigma_1)} + \rho(t)\frac{(r(t)z'(t))'}{z(t - \sigma_1)} - \rho(t)\frac{r(t)z'(t)z'(t - \sigma_1)}{z^2(t - \sigma_1)}.$$

By (2.2), we have $r(t-\sigma_1)z'(t-\sigma_1) \ge r(t)z'(t)$. Then, from (2.17), we get

$$\omega_1'(t) \le \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) + \rho(t)\frac{(r(t)z'(t))'}{z(t-\sigma_1)} - \frac{(\omega_1(t))^2}{\rho(t)r(t-\sigma_1)}. \tag{2.18}$$

Next, define function ω_2 by

$$\omega_2(t) = \rho(t)\frac{r(t-\sigma_1)z'(t-\sigma_1)}{z(t-\sigma_1)}, \qquad t \ge t_2. \tag{2.19}$$

Then $\omega_2(t) > 0$ for $t \ge t_2$. Differentiating (2.19), we find that

$$\begin{aligned}\omega_2'(t) = {}&\rho'(t)\frac{r(t-\sigma_1)z'(t-\sigma_1)}{z(t-\sigma_1)} + \rho(t)\frac{(r(t-\sigma_1)z'(t-\sigma_1))'}{z(t-\sigma_1)}\\ &- \rho(t)\frac{r(t-\sigma_1)z'(t-\sigma_1)z'(t-\sigma_1)}{z^2(t-\sigma_1)}.\end{aligned}$$

Hence, from (2.19), we get

$$\omega_2'(t) \le \frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) + \rho(t)\frac{(r(t-\sigma_1)z'(t-\sigma_1))'}{z(t-\sigma_1)} - \frac{(\omega_2(t))^2}{\rho(t)r(t-\sigma_1)}. \tag{2.20}$$

In the following, we define another function ω_3 by

$$\omega_3(t) = \rho(t)\frac{r(t+\sigma_2)z'(t+\sigma_2)}{z(t-\sigma_1)}, \qquad t \ge t_2. \tag{2.21}$$

Then $\omega_3(t) > 0$ for $t \ge t_2$. Differentiating (2.21), we obtain

$$\begin{aligned}\omega_3'(t) = {}&\rho'(t)\frac{r(t+\sigma_2)z'(t+\sigma_2)}{z(t-\sigma_1)} + \rho(t)\frac{(r(t+\sigma_2)z'(t+\sigma_2))'}{z(t-\sigma_1)}\\ &- \rho(t)\frac{r(t+\sigma_2)z'(t+\sigma_2)z'(t-\sigma_1)}{z^2(t-\sigma_1)}.\end{aligned}$$

By (2.2), we have $r(t-\sigma_1)z'(t-\sigma_1) \ge r(t+\sigma_2)z'(t+\sigma_2)$. Thus, by (2.21), we get

$$\omega_3'(t) \le \frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) + \rho(t)\frac{(r(t+\sigma_2)z'(t+\sigma_2))'}{z(t-\sigma_1)} - \frac{(\omega_3(t))^2}{\rho(t)r(t-\sigma_1)}. \tag{2.22}$$

Therefore, by (2.18), (2.20) and (2.22), we obtain

$$\begin{aligned}&\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t)\\ &\le \rho(t)\left[\frac{(r(t)z'(t))' + a_1(r(t-\sigma_1)z'(t-\sigma_1))' + a_2(r(t+\sigma_2)z'(t+\sigma_2))'}{z(t-\sigma_1)}\right]\\ &\quad + \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) - \frac{(\omega_1(t))^2}{\rho(t)r(t-\sigma_1)} + a_1\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) - a_1\frac{(\omega_2(t))^2}{\rho(t)r(t-\sigma_1)}\\ &\quad + a_2\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) - a_2\frac{(\omega_3(t))^2}{\rho(t)r(t-\sigma_1)}.\end{aligned} \tag{2.23}$$

Thus, by (2.5) and (2.23), we get

$$\begin{aligned}
&\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t)\\
&\le -\rho(t)Q(t) + \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) - \frac{(\omega_1(t))^2}{\rho(t)r(t-\sigma_1)} + a_1\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t)\\
&\quad - a_1\frac{(\omega_2(t))^2}{\rho(t)r(t-\sigma_1)} + a_2\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) - a_2\frac{(\omega_3(t))^2}{\rho(t)r(t-\sigma_1)}. \qquad (2.24)
\end{aligned}$$

Then, by (2.24), we find that

$$\omega_1'(t) + a_1\omega_2'(t) + p_2\omega_3'(t) \le -\rho(t)Q(t) + \frac{1+a_1+a_2}{4}\cdot\frac{r(t-\sigma_1)((\rho'(t))_+)^2}{\rho(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$\begin{aligned}
&\int_{t_2}^{t}\left[\rho(s)Q(s) - \frac{1+a_1+a_2}{4}\frac{r(s-\sigma_1)((\rho'(s))_+)^2}{\rho(s)}\right]\mathrm{d}s\\
&\qquad \le \omega_1(t_2) + a_1\omega_2(t_2) + a_2\omega_3(t_2),
\end{aligned}$$

which contradicts (2.16). The proof is complete. □

Remark 2.1. From Theorem 2.2, we can obtain some oscillation criteria for Eq. (1.1) by choosing different ρ, the details are left to the reader.

Remark 2.2. By (2.12), (2.23) and the techniques given in [2], we can establish some Philos-type oscillation criteria for Eq. (1.1), the details are left to the reader.

Now, we will establish some oscillation results for Eq. (1.1) under the case when (1.4) holds.

Theorem 2.3. *Suppose that* (1.4) *holds and* $\sigma_3 \ge \sigma_1$. *Moreover, assume that there exists* $\rho \in C^1\big([t_0,\infty),(0,\infty)\big)$ *such that* (2.1) *holds. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\zeta(s)Q(s) - \frac{(1+a_1)r(s+\sigma_4)+a_2r(s+\sigma_2+\sigma_4)}{4r^2(s+\sigma_2+\sigma_4)\zeta(s)}\right]\mathrm{d}s = \infty, \qquad (2.25)$$

then every solution of Eq. (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(t-\sigma_1) > 0$, $x(t+\sigma_2) > 0$, $x(t-\sigma_3) > 0$ and $x(t+\sigma_4) > 0$ for all $t \ge t_1$. Define

$$z(t) = x(t) + p_1(t)x(t-\sigma_1) + p_2(t)x(t+\sigma_2).$$

Then $z(t) > 0$ for $t \ge t_1$. In view of (1.1), we obtain that (2.2) holds. From (2.2), we see that there exist two possible cases for the sign of $z'(t)$.

Assume that $z'(t) > 0$, $z'(t-\sigma_1) > 0$ and $z'(t+\sigma_2) > 0$, for $t \ge t_2 \ge t_1$. Then (2.5) holds. Proceeding as in the proof of Theorem 2.1, we can obtain a contradiction with (2.1).

Suppose that $z'(t) < 0$, $z'(t-\sigma_1) < 0$ and $z'(t+\sigma_2) < 0$, for $t \ge t_2 \ge t_1$. Also, we have (2.4). From $z'(t) < 0$, we have $z(t+\sigma_4) \le z(t-\sigma_3)$. Then,

$$\begin{aligned}\big(r(t)z'(t)\big)' &+ a_1\big(r(t-\sigma_1)z'(t-\sigma_1)\big)' \\ &+ a_2\big(r(t+\sigma_2)z'(t+\sigma_2)\big)' + Q(t)z(t+\sigma_4) \le 0. \end{aligned} \tag{2.26}$$

Define function ω_1 by

$$\omega_1(t) = \frac{r(t)z'(t)}{z(t+\sigma_4)}, \qquad t \ge t_2. \tag{2.27}$$

Clearly, $\omega_1(t) < 0$ for $t \ge t_2$. Noting that $r(t)z'(t)$ is nonincreasing, we have

$$z'(s) \le \frac{r(t)z'(t)}{r(s)}, \qquad s \ge t \ge t_2.$$

Integrating it from $t+\sigma_4$ to l, we obtain

$$z(l) \le z(t+\sigma_4) + r(t)z'(t)\int_{t+\sigma_4}^{l} \frac{\mathrm{d}s}{r(s)}, \qquad l \ge t+\sigma_4.$$

Note that $\lim_{l\to\infty} z(l) \ge 0$. Letting $l \to \infty$ in the above inequality, we have

$$0 \le z(t+\sigma_4) + r(t)z'(t)\delta(t), \qquad t \ge t_2.$$

Therefore,

$$\frac{r(t)z'(t)}{z(t+\sigma_4)}\delta(t) \ge -1, \qquad t \ge t_2.$$

From (2.27), we have

$$-1 \le \omega_1(t)\delta(t) \le 0, \qquad t \ge t_2. \tag{2.28}$$

By (2.2), we obtain

$$z'(t+\sigma_4) \le \frac{r(t)z'(t)}{r(t+\sigma_4)}.$$

Differentiating (2.27), we get

$$\omega_1'(t) \le \frac{(r(t)z'(t))'}{z(t+\sigma_4)} - \frac{(\omega_1(t))^2}{r(t+\sigma_4)}. \tag{2.29}$$

Next, we introduce another function

$$\omega_2(t) = \frac{r(t-\sigma_1)z'(t-\sigma_1)}{z(t+\sigma_4)}, \qquad t \ge t_2. \tag{2.30}$$

Obviously, $\omega_2(t) < 0$ for $t \geq t_2$. Noting that $r(t)z'(t)$ is nonincreasing for $t \geq t_1$, we get $r(t-\sigma_1)z'(t-\sigma_1) \geq r(t)z'(t)$, for $t \geq t_2$. Thus $\omega_2(t) \geq \omega_1(t)$, for $t \geq t_2$. By (2.28), we obtain

$$-1 \leq \omega_2(t)\delta(t) \leq 0, \qquad t \geq t_2. \tag{2.31}$$

It follows from (2.2) that

$$z'(t+\sigma_4) \leq \frac{r(t-\sigma_1)z'(t-\sigma_1)}{r(t+\sigma_4)}.$$

Differentiating (2.30), we have

$$\omega_2'(t) \leq \frac{(r(t-\sigma_1)z'(t-\sigma_1))'}{z(t+\sigma_4)} - \frac{(\omega_2(t))^2}{r(t+\sigma_4)}. \tag{2.32}$$

Similarly, we introduce substitution

$$\omega_3(t) = \frac{r(t+\sigma_2)z'(t+\sigma_2)}{z(t+\sigma_2+\sigma_4)}, \qquad t \geq t_2. \tag{2.33}$$

Clearly, $\omega_3(t) < 0$ for $t \geq t_2$. In view of the definition of ω_1 and (2.28), we find that $\omega_3(t) = \omega_1(t+\sigma_2)$ and

$$-1 \leq \omega_3(t)\delta(t+\sigma_2) \leq 0, \qquad t \geq t_2. \tag{2.34}$$

By (2.2), we have $z'(t+\sigma_2+\sigma_4) \leq r(t+\sigma_2)z'(t+\sigma_2)/r(t+\sigma_2+\sigma_4)$. Differentiating (2.33), we get

$$\begin{aligned} \omega_3'(t) &\leq \frac{(r(t+\sigma_2)z'(t+\sigma_2))'}{z(t+\sigma_2+\sigma_4)} - \frac{(\omega_3(t))^2}{r(t+\sigma_2+\sigma_4)} \\ &\leq \frac{(r(t+\sigma_2)z'(t+\sigma_2))'}{z(t+\sigma_4)} - \frac{(\omega_3(t))^2}{r(t+\sigma_2+\sigma_4)}. \end{aligned} \tag{2.35}$$

Note that $\delta(t) \geq \delta(t+\sigma_2)$. Then, we have

$$-1 \leq \omega_1(t)\delta(t+\sigma_2) \leq 0, \qquad t \geq t_2, \tag{2.36}$$

and

$$-1 \leq \omega_2(t)\delta(t+\sigma_2) \leq 0, \qquad t \geq t_2. \tag{2.37}$$

From (2.29), (2.32) and (2.35), we can obtain

$$\begin{aligned} &\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t) \\ &\leq \frac{(r(t)z'(t))' + a_1(r(t-\sigma_1)z'(t-\sigma_1))' + a_2(r(t+\sigma_2)z'(t+\sigma_2))'}{z(t+\sigma_4)} \\ &\quad - \frac{(\omega_1(t))^2}{r(t+\sigma_4)} - a_1\frac{(\omega_2(t))^2}{r(t+\sigma_4)} - a_2\frac{(\omega_3(t))^2}{r(t+\sigma_2+\sigma_4)}. \end{aligned} \tag{2.38}$$

Therefore, by (2.26) and (2.38), we have

$$\begin{aligned}
&\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t)\\
&\le -Q(t) - \frac{(\omega_1(t))^2}{r(t+\sigma_4)} - a_1\frac{(\omega_2(t))^2}{r(t+\sigma_4)} - a_2\frac{(\omega_3(t))^2}{r(t+\sigma_2+\sigma_4)}.
\end{aligned} \tag{2.39}$$

Multiplying (2.39) by $\zeta(t)$, and integrating over $[t_2, t]$ implies

$$\begin{aligned}
&\zeta(t)\omega_1(t) - \zeta(t_2)\omega_1(t_2) + \int_{t_2}^{t}\frac{\omega_1(s)}{r(s+\sigma_2+\sigma_4)}\,\mathrm{d}s + \int_{t_2}^{t}\frac{(\omega_1(s))^2\zeta(s)}{r(s+\sigma_4)}\,\mathrm{d}s\\
&+ a_1\zeta(t)\omega_2(t) - a_1\zeta(t_2)\omega_2(t_2) + a_1\int_{t_2}^{t}\frac{\omega_2(s)}{r(s+\sigma_2+\sigma_4)}\,\mathrm{d}s\\
&+ a_1\int_{t_2}^{t}\frac{(\omega_2(s))^2\zeta(s)}{r(s+\sigma_4)}\,\mathrm{d}s + a_2\zeta(t)\omega_3(t) - a_2\zeta(t_2)\omega_3(t_2)\\
&+ a_2\int_{t_2}^{t}\frac{\omega_3(s)}{r(s+\sigma_2+\sigma_4)}\,\mathrm{d}s + a_2\int_{t_2}^{t}\frac{(\omega_3(s))^2\zeta(s)}{r(s+\sigma_2+\sigma_4)}\,\mathrm{d}s + \int_{t_2}^{t}\zeta(s)Q(s)\mathrm{d}s \le 0.
\end{aligned}$$

From the above inequality, we obtain

$$\begin{aligned}
&\int_{t_2}^{t}\left[\zeta(s)Q(s) - \frac{(1+a_1)r(s+\sigma_4) + a_2 r(s+\sigma_2+\sigma_4)}{4r^2(s+\sigma_2+\sigma_4)\zeta(s)}\right]\mathrm{d}s\\
&\quad\le -\big[\zeta(t)\omega_1(t) + a_1\zeta(t)\omega_2(t) + a_2\zeta(t)\omega_3(t)\big]\\
&\quad\le 1 + a_1 + a_2
\end{aligned}$$

due to (2.34), (2.36) and (2.37). This contradicts (2.25) and finishes the proof. □

Combining Theorem 2.2 with Theorem 2.3, we give the following criterion for the oscillation of Eq. (1.1) when the conditions $\sigma_1 \ge \sigma_3$ and (1.4) hold.

Theorem 2.4. *Suppose that* (1.4) *holds and* $\sigma_1 \ge \sigma_3$. *Moreover, assume that there exists* $\rho \in C^1\big([t_0,\infty),(0,\infty)\big)$ *such that* (2.16) *holds. If* (2.25) *holds, then every solution of Eq.* (1.1) *oscillates.*

Remark 2.3. The technique used in this paper can be extended to even-order mixed neutral differential equations.

3. Applications and examples

Grace [26] and Yan [27] investigated the oscillation behavior of Eq. (1.5). The authors gave some oscillation criteria when $n \ge 1$ is odd, c, C and Q are nonnegative real constants, and g, G, h, H, and q are positive real constants. It is easy to find that the results given in [26], [27] cannot be applied to Eq. (1.1). Also, the results obtained in [23]–[25], [28]–[30] do not apply to Eq. (1.1).

In order to illustrate the main results, we will give the following examples.

EXAMPLE 3.1. Consider the second-order Euler differential equation:

$$x''(t) + \frac{\gamma}{t^2}x(t) = 0, \qquad t \ge t_0. \tag{3.1}$$

Now $a_1 = a_2 = 0$ and $Q(t) = \frac{\gamma}{t^2}$. Applying Corollary 2.3, we can obtain that Eq. (3.1) is oscillatory for $\gamma > \frac{1}{4}$, which is a sharp condition for the oscillation of Eq. (3.1).

EXAMPLE 3.2. Consider the following linear neutral equation:

$$\begin{aligned}\big(x(t) + x(t-(2n+1)\pi) + x(t+(2n-1)\pi)\big)''&\\ + \frac{1}{2}x(t-(2m+1)\pi) + \frac{1}{2}x(t+(2m-1)\pi) = 0,&\end{aligned} \tag{3.2}$$

for $t \ge t_0 > 0$, where n and m are positive integers, $m \ge n$.

Obviously, all the conditions of Corollary 2.2 hold. Thus, by Corollary 2.2, every solution of Eq. (3.2) is oscillatory. It is easy to verify that $x(t) = \sin t$ is a solution of Eq. (3.2).

EXAMPLE 3.3. Consider the following linear neutral equation:

$$\begin{aligned}\Big(t\big[x(t) + p_1(t)x(t-\sigma_1) + p_2(t)x(t+\sigma_2)\big]'\Big)'&\\ + \frac{\beta}{t}x(t-\sigma_3) + \frac{\gamma}{t}x(t+\sigma_4) = 0,&\end{aligned} \tag{3.3}$$

for $t \ge t_0 > 0$, where $\sigma_3 \ge \sigma_1$, $0 \le p_i(t) \le a_i$ for $i = 1,\ 2$, $a_1 + a_2 \le 3$, β and γ are positive constants.

We see that (1.3) holds and $Q_1(t) = \beta/(t+\sigma_2)$, $Q_2(t) = \gamma/(t+\sigma_2)$, and $Q(t) = (\beta+\gamma)/(t+\sigma_2)$. On the other hand, (2.14) reduces to

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{(\beta+\gamma-1)s^2 + s(\sigma_3-\sigma_2) + \sigma_2\sigma_3}{s(s+\sigma_2)}\mathrm{d}s = \infty,$$

which holds for $\beta+\gamma > 1$. Therefore, by Corollary 2.3, every solution of Eq. (3.3) is oscillatory provided that $\beta+\gamma > 1$.

Example 3.4. Consider the following linear neutral equation:

$$\begin{aligned}\left(\frac{1}{t}\left[x(t)+p_1(t)x(t-\sigma_1)+p_2(t)x(t+\sigma_2)\right]'\right)'&\\ +\frac{\beta}{t^2}x(t-\sigma_3)+\frac{\gamma}{t^2}x(t+\sigma_4)=0,&\qquad(3.4)\end{aligned}$$

for $t \geq t_0 > 0$, where $\sigma_1 \geq \sigma_3$, $0 \leq p_i(t) \leq a_i$, for $i = 1,\ 2$, $a_1 + a_2 \leq 3$, β and γ are positive constants.

We see that (1.3) holds and $Q_1(t) = \beta/(t+\sigma_2)^2$, $Q_2(t) = \gamma/(t+\sigma_2)^2$, and $Q(t) = (\beta+\gamma)/(t+\sigma_2)^2$. Take $\rho(t) = t$, then (2.16) takes the form

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{(\beta+\gamma)s}{(s+\sigma_2)^2}-\frac{1}{s(s-\sigma_1)}\right]\mathrm{d}s=\infty,$$

which is evidently true. Therefore, by Theorem 2.2, every solution of Eq. (3.4) is oscillatory.

Example 3.5. Consider the following linear neutral equation:

$$\begin{aligned}\left(t^2\left[x(t)+x(t-2\pi)+x(t+2\pi)\right]'\right)'&\\ +3t^2x(t-4\pi)+6tx\left(t+\frac{3\pi}{2}\right)=0,&\qquad(3.5)\end{aligned}$$

for $t \geq t_0 > 0$.

The condition (1.4) is fulfilled and $a_1 = a_2 = 1$, $Q_1(t) = 3(t-2\pi)^2$, $Q_2(t) = 6(t-2\pi)$, $Q(t) = 3(t-2\pi)^2+6(t-2\pi)$, $\delta(t) = 2/(2t+3\pi)$ and $\zeta(t) = 2/(2t+7\pi)$. Set $\rho(t) = 1$. Obviously, we get that (2.1) and (2.25) hold. That is, all the assumptions of Theorem 2.3 are satisfied. Hence, by Theorem 2.3, every solution of Eq. (3.5) is oscillatory. It is easy to see that $x(t) = \sin t$ is a solution of Eq. (3.5).

Remark 3.1. It is easy to find another example to illustrate Theorem 2.4, the details are left to the reader.

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Tongxing Li
School of Control Science and Engineering
Shandong University, Jinan
Shandong 250061
P. R. CHINA
E-mail: litongx2007@163.com

Blanka Baculíková
Jozef Džurina
Department of Mathematics
Faculty of Electrical Engineering and Informatics
Technical University of Košice
Letná 9
SK–042-00 Košice
SLOVAKIA
E-mail: blanka.baculikova@tuke.sk
jozef.dzurina@tuke.sk