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ON NONOSCILLATORY SOLUTIONS TENDING TO ZERO OF THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of this paper is to present some results concerning with the asymptotic behavior of solutions of nonlinear differential equations of the third-order with quasiderivatives. In particular, we state the necessary and sufficient conditions ensuring the existence of nonoscillatory solutions tending to zero as $t \to \infty$.

1. Introduction

This paper deals with the asymptotic behavior of nonoscillatory solutions of the third-order nonlinear differential equations with quasiderivatives of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f\big(x(t)\big) = 0, \qquad t \ge a \tag{N}$$

where

$$r, p, q \in C\big([a,\infty),\mathbb{R}\big),\ \ r(t) > 0,\ \ p(t) > 0,\ \ q(t) > 0 \qquad \text{on } [a,\infty),$$

$$f \in C(\mathbb{R},\mathbb{R}),\quad f(u)u > 0 \qquad \text{for} \quad u \ne 0\,.$$

For the sake of brevity, we introduce the following notation

$$x^{[0]} = x,\quad x^{[1]} = \frac{1}{r}x',\quad x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}\left(x^{[1]}\right)',\quad x^{[3]} = \left(x^{[2]}\right)'.$$

We call these functions $x^{[i]}$, $i = 0,1,2,3$, the *quasiderivatives* of x.

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By a *solution* of an equation of the form (N), we mean a function $w\colon[a,\infty)\to\mathbb{R}$ such that quasiderivatives $w^{[i]}(t)$, $0\le i\le 3$ exist and are continuous on the interval $[a,\infty)$ and w satisfies the equation (N) for all $t\ge a$. A solution w of equation (N) is said to be *proper* if it satisfies the following condition

$$\sup\{|w(s)| : t\le s<\infty\}>0 \qquad \text{for any} \quad t\ge a.$$

A proper solution is said to be *oscillatory* if it has a sequence of zeros converging to ∞; otherwise it is said to be *nonoscillatory*.

Fixed point theorems are important tools in the oscillation and nonoscillation theory of ordinary differential equations. Especially, when one proves the existence of nonoscillatory solutions with a specified asymptotic behavior as $t\to\infty$. We refer the reader to the books [1], [13] and to fairly comprehensive bibliography contained therein for various interesting results on this topic. Now, we state fixed point theorem that will be needed later.

Theorem 1.1 (Banach fixed point theorem)**.** *Any contraction mapping of a complete non-empty metric space $\mathcal{M}$ into $\mathcal{M}$ has a unique fixed point in $\mathcal{M}$.*

Let $\mathcal{N}(N)$ denote the set of all proper nonoscillatory solutions of equation (N). The set $\mathcal{N}(N)$ can be divided into the following four classes in the same way as in [5], [6]:

$$\mathcal{N}_0=\Big\{x\in\mathcal{N}(N),\ \exists\, t_x : x(t)x^{[1]}(t)<0,\ x(t)x^{[2]}(t)>0 \quad \text{for } t\ge t_x\Big\},$$

$$\mathcal{N}_1=\Big\{x\in\mathcal{N}(N),\ \exists\, t_x : x(t)x^{[1]}(t)>0,\ x(t)x^{[2]}(t)<0 \quad \text{for } t\ge t_x\Big\},$$

$$\mathcal{N}_2=\Big\{x\in\mathcal{N}(N),\ \exists\, t_x : x(t)x^{[1]}(t)>0,\ x(t)x^{[2]}(t)>0 \quad \text{for } t\ge t_x\Big\},$$

$$\mathcal{N}_3=\Big\{x\in\mathcal{N}(N),\ \exists\, t_x : x(t)x^{[1]}(t)<0,\ x(t)x^{[2]}(t)<0 \quad \text{for } t\ge t_x\Big\}.$$

Let us remark that the solutions in the class $\mathcal{N}_0$ satisfy for all sufficiently large t the inequality $x^{[i]}(t)x^{[i+1]}(t)<0$ for $i=0,1,2$ and in the literature they are called *Kneser solutions*. The following results regarding the asymptotic properties of Kneser solutions of equation (N) will be useful in the sequel. They are particular cases of more general results that have been presented in [16] for differential equations with deviating argument.

Lemma 1.2 ([16, Lemma 2.8])**.** *If $I(r)=I(p)=\infty$, then any solution x of equation* (N) *in the class $\mathcal{N}_0$ satisfies $\lim_{t\to\infty}x^{[i]}(t)=0$ for $i=1,2$.*

Theorem 1.3 ([16, Theorem 4.3])**.** *Assume that*

$$\limsup_{u\to 0}\frac{f(u)}{u}<\infty \quad \textit{and} \quad I(r)=I(p)=\infty.$$

If there exists a solution x of equation (N) *in the class $\mathcal{N}_0$ such that*

$$\lim_{t\to\infty}x^{[i]}(t)=0 \quad \textit{for} \quad i=0,1,2, \quad \textit{then} \quad I(q,p,r)=\infty.$$

The object of our interest are nonoscillatory solutions of equation (N) in the classes $\mathcal{N}_0$ and $\mathcal{N}_3$ tending to zero as $t \to \infty$, i.e., the solutions that belong to the following two subclasses

$$\mathcal{N}_0^0 = \left\{ x \in \mathcal{N}_0 : \lim_{t\to\infty} x(t) = 0 \right\}, \qquad \mathcal{N}_3^0 = \left\{ x \in \mathcal{N}_3 : \lim_{t\to\infty} x(t) = 0 \right\}.$$

Various types of differential equations (without or with deviating argument) of the third-order have been subject of intensive studying in the literature. The authors have obtained the sufficient conditions for oscillation and asymptotic behavior of solutions, conditions for existence or nonexistence some types of solutions and also many results for the classification of solutions according to their oscillatory and asymptotic properties. Among the extensive literature on these topics, we mention here [2], [4], [5], [6], [9], [17] for the differential equations without deviating argument and [3], [8], [9], [10], [11], [14], [15], [18], [19] for those with deviating argument.

The purpose of this paper is to study the existence and asymptotic behavior of some nonoscillatory solutions of equation of the form (N). Namely, we give the necessary and sufficient conditions for the existence of nonoscillatory solutions in the subclasses $\mathcal{N}_0^0$ and $\mathcal{N}_3^0$. The results are proved by means of a study of the asymptotic properties of considered solutions as well as a topological approach via the Banach fixed point theorem. Obtained results complement those in [17] where the existence of bounded nonoscillatory solutions of (N) in the classes $\mathcal{N}_1$ and $\mathcal{N}_2$ has been investigated. Moreover, our results complement and extend some other ones that have been stated in [7] and [12], respectively.

Finally, we introduce the following notation

$$I(u_i) = \int_a^\infty u_i(t)\, dt, \qquad I(u_i, u_j) = \int_a^\infty u_i(t) \int_a^t u_j(s)\, ds\, dt, \qquad i, j = 1, 2,$$

$$I(u_i, u_j, u_k) = \int_a^\infty u_i(t) \int_a^t u_j(s) \int_a^s u_k(z)\, dz\, ds\, dt, \qquad i, j, k = 1, 2, 3,$$

where u_i, $i = 1, 2, 3$ are continuous positive functions on the interval $[a, \infty)$.

2. Main results

We begin our investigation with the results concerning the nonoscillatory solutions of equation (N) in the class $\mathcal{N}_3$. The following result gives the sufficient condition for the existence of solutions in the class $\mathcal{N}_3^0$.

Theorem 2.1. *Let $I(r) < \infty$, $I(p,q) < \infty$ and assume that function f satisfies Lipschitz condition on the interval $\big[0; 2I(r)\big]$. Then equation* (N) *has a solution x in the class $\mathcal{N}_3$ such that*

$$\lim_{t\to\infty} x(t) = 0, \quad i.e., \ \mathcal{N}_3^0 \neq \emptyset.$$

P r o o f. We prove the existence of a positive solution of equation (N) in the class $\mathcal{N}_3$ which tends to zero as $t \to \infty$.

Let L denote Lipschitz constant of function f on the interval $\big[0; 2I(r)\big]$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^{\infty} r(s)\, ds \ \leq \frac{1}{L+1} \tag{1}$$

and

$$\int_{t_0}^{\infty} p(\tau) \int_{t_0}^{\tau} q(s)\, ds\, d\tau \ \leq \min\left\{\frac{1}{K}, 1\right\}, \tag{2}$$

where

$$K = \max\left\{ f(u) : u \in \left[0;\ 2\int_{t_0}^{\infty} r(s)\, ds\right]\right\}.$$

For the sake of convenience, we introduce the following notation

$$H_1(t) = \int_t^{\infty} r(s)\, ds, \qquad t \geq t_0.$$

Let us define the set

$$\Delta_1 = \Big\{u \in C\big([t_0,\infty),\mathbb{R}\big) : H_1(t) \leq u(t) \leq 2H_1(t)\Big\},$$

where $C\big([t_0,\infty),\mathbb{R}\big)$ denotes the Banach space of all continuous and bounded functions defined on the interval $[t_0,\infty)$ with the sup norm

$$\|u\| = \sup\{|u(t)|, \quad t \geq t_0\}.$$

It is clear that Δ_1 is a non-empty closed subset of space $C\big([t_0,\infty),\mathbb{R}\big)$ and so Δ_1 is a non-empty complete metric space. For every $u \in \Delta_1$, we consider a mapping

$$T_1 : \Delta_1 \to C\big([t_0,\infty),\mathbb{R}\big)$$

given by

$$x_u(t) = (T_1 u)(t) = H_1(t) + \int_t^{\infty} r(\tau) \int_{t_0}^{\tau} p(s) \int_{t_0}^{s} q(z) f\big(u(z)\big)\, dz\, ds\, d\tau, \qquad t \geq t_0.$$

In the following, we prove that T_1 maps Δ_1 into itself and T_1 is a contraction mapping in Δ_1 in order to apply to the mapping T_1 the Banach fixed point theorem (Theorem 1.1).

T_1 maps Δ_1 into Δ_1. Really, $x_u(t) \geq H_1(t)$ and in view of (2), we have

$$\begin{aligned} x_u(t) &= H_1(t) + \int_t^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z) f\big(u(z)\big)\, dz\, ds\, d\tau \\ &\leq H_1(t) + K \int_t^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z)\, dz\, ds\, d\tau \\ &\leq H_1(t) + K \left(\int_{t_0}^\infty p(s) \int_{t_0}^s q(z)\, dz\, ds \right) \left(\int_t^\infty r(\tau)\, d\tau \right) \\ &\leq H_1(t) + H_1(t) = 2H_1(t). \end{aligned}$$

Now, let $u_1, u_2 \in \Delta_1$ and $t \geq t_0$. Taking into account the fact that function f satisfies Lipschitz condition on the interval $[0; 2I(r)]$ and the inequalities (1) and (2), we obtain the following

$$\begin{aligned} |(T_1u_1)(t) - (T_1u_2)(t)| &\leq \int_t^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z) \big| f\big(u_1(z)\big) - f\big(u_2(z)\big) \big|\, dz\, ds\, d\tau \\ &\leq \int_{t_0}^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z) \big| f\big(u_1(z)\big) - f\big(u_2(z)\big) \big|\, dz\, ds\, d\tau \\ &\leq L \int_{t_0}^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z) |u_1(z) - u_2(z)|\, dz\, ds\, d\tau \\ &\leq L\|u_1 - u_2\| \int_{t_0}^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z)\, dz\, ds\, d\tau \\ &\leq L\|u_1 - u_2\| \left(\int_{t_0}^\infty r(\tau)\, d\tau \right) \left(\int_{t_0}^\infty p(s) \int_{t_0}^s q(z)\, dz\, ds \right) \\ &\leq \frac{L}{L+1} \|u_1 - u_2\| = Q_1 \|u_1 - u_2\|. \end{aligned}$$

These inequalities immediately imply that for every $u_1, u_2 \in \Delta_1$

$$\|T_1u_1 - T_1u_2\| \leq Q_1\|u_1 - u_2\|, \qquad \text{where} \quad 0 < Q_1 < 1.$$

Hence, we proved that T_1 is a contraction mapping in Δ_1. Now, according to the Banach fixed point theorem, there exists the unique fixed point $x \in \Delta_1$ such that

$$x(t) = H_1(t) + \int_t^\infty r(\tau) \int_{t_0}^\tau p(s) \int_{t_0}^s q(z) f\big(x(z)\big)\, dz\, ds\, d\tau, \qquad t \geq t_0.$$

As

$$x^{[1]}(t) = -1 - \int_{t_0}^{t} p(s) \int_{t_0}^{s} q(z) f\big(x(z)\big)\, dz\, ds < 0$$

and

$$x^{[2]}(t) = -\int_{t_0}^{t} q(z) f\big(x(z)\big)\, dz < 0,$$

it is clear that x is a positive solution of the equation (N) in the class $\mathcal{N}_3$ which approaches to zero as $t \to \infty$, i.e., $x \in \mathcal{N}_3^0$. □

The following theorem for the solutions in the class $\mathcal{N}_3$ holds.

Theorem 2.2. *If $I(r) = \infty$ or $I(r,p) = \infty$, then $\mathcal{N}_3 = \emptyset$.*

Proof. Let $x \in \mathcal{N}_3$. Without loss of generality, we suppose that there exists $t_0 \ge a$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \ge t_0$. Because

$$\Big(x^{[2]}(t)\Big)' = -q(t) f\big(x(t)\big) < 0 \quad \text{for all} \quad t \ge t_0,\ x^{[2]}(t)$$

is a negative decreasing function and so

$$\Big(x^{[1]}(t)\Big)' \le x^{[2]}(t_0) p(t) \qquad \text{for all} \quad t \ge t_0.$$

Integrating this inequality twice in the interval $[t_0, t]$, we obtain

$$x(t) \le x(t_0) + x^{[1]}(t_0) \int_{t_0}^{t} r(s)\, ds + x^{[2]}(t_0) \int_{t_0}^{t} r(s) \int_{t_0}^{s} p(\tau)\, d\tau\, ds.$$

When $t \to \infty$, we get a contradiction because function $x(t)$ is a positive for all $t \ge t_0$. The case $x(t) < 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \ge t_1$ (where $t_1 \ge a$) can be treated similarly. □

As a consequence of Theorems 2.1 and 2.2, we get the following result.

Corollary 2.3. *Let function f satisfy Lipschitz condition on the interval $\big[0; 2I(r)\big]$ and $I(p,q) < \infty$. Then a necessary and sufficient condition for equation* (N) *to have a solution x in the class $\mathcal{N}_3^0$ is that $I(r) < \infty$.*

Evidently, the following also holds.

Corollary 2.4. *Let function f satisfy Lipschitz condition on the interval $\big[0; 2I(r)\big]$ and $I(p,q) < \infty$. Then a necessary and sufficient condition for equation* (N) *to have a solution x in the class $\mathcal{N}_3$ is that $I(r) < \infty$.*

In the sequel, we turn our attention to the solutions of the equation (N) in the class $\mathcal{N}_0$. More precisely, the sufficient and also necessary condition guaranteeing the existence of solutions in the class $\mathcal{N}_0^0$ are stated.

Theorem 2.5. *Let $I(p,r) < \infty$, $I(q) < \infty$ and assume that function f satisfies Lipschitz condition on the interval $\big[0; 2I(p,r)\big]$. Then equation* (N) *has a solution x in the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x(t) = 0$, i.e., $\mathcal{N}_0^0 \neq \emptyset$.*

P r o o f. In the following, we prove the existence of a positive solution of equation (N) in the class $\mathcal{N}_0$ which tends to zero as $t \to \infty$.

Let L denote Lipschitz constant of function f on the interval $\big[0; 2I(p,r)\big]$ and let $t_0 \geq a$ be such that

$$\int_{t_0}^{\infty} p(s) \int_{t_0}^{s} r(\tau)\, d\tau\, ds \leq \frac{1}{L+1} \tag{3}$$

and

$$\int_{t_0}^{\infty} q(s)\, ds \leq \min\left\{\frac{1}{K}, 1\right\}, \tag{4}$$

where

$$K = \max\left\{ f(u) : u \in \left[0;\ 2\int_{t_0}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau\right]\right\}.$$

We observe that

$$\int_{t_0}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau = \int_{t_0}^{\infty} p(s) \int_{t_0}^{s} r(\tau)\, d\tau\, ds.$$

Further, for the sake of convenience, we introduce the following notation

$$H_2(t) = \int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s)\, ds\, d\tau, \qquad t \geq t_0.$$

Let us define the set

$$\Delta_2 = \Big\{ u \in C\big([t_0,\infty),\mathbb{R}\big) : H_2(t) \leq u(t) \leq 2H_2(t) \Big\},$$

where $C\big([t_0,\infty),\mathbb{R}\big)$ denotes the Banach space of all continuous and bounded functions defined on the interval $[t_0,\infty)$ with the sup norm $\|u\| = \sup\{|u(t)|, t \geq t_0\}$. Clearly, Δ_2 is a non-empty closed subset of space $C\big([t_0,\infty),\mathbb{R}\big)$ and so Δ_2 is a non-empty complete metric space. For every $u \in \Delta_2$, we consider a mapping $T_2 : \Delta_2 \to C\big([t_0,\infty),\mathbb{R}\big)$ given by

$$x_u(t) = (T_2 u)(t) = H_2(t) + \int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z) f\big(u(z)\big)\, dz\, ds\, d\tau, \qquad t \geq t_0.$$

Easy computation gives the following inequality

$$\int_{t}^{\infty} r(\tau) \int_{\tau}^{\infty} p(s) \int_{s}^{\infty} q(z)\, dz\, ds\, d\tau \leq H_2(t) \int_{t_0}^{\infty} q(z)\, dz, \qquad t \geq t_0. \tag{5}$$

The fact that function f satisfies Lipschitz condition on interval $\left[0; 2I(p,r)\right]$, the inequalities (3), (4) and (5) and similar arguments as in the proof of Theorem 2.1 enable us to verify that T_2 maps Δ_2 into itself and T_2 is a contraction mapping in Δ_2. Consequently, the Banach fixed point theorem ensures the existence of the unique fixed point $x \in \Delta_2$ such that

$$x(t) = H_2(t) + \int_t^\infty r(\tau) \int_\tau^\infty p(s) \int_s^\infty q(z) f\bigl(x(z)\bigr)\, dz\, ds\, d\tau, \qquad t \geq t_0.$$

It is evident that x is a positive solution of the equation (N) in the class $\mathcal{N}_0$ which approaches to zero as $t \to \infty$, i.e., $x \in \mathcal{N}_0^0$. □

Theorem 2.6. *Assume that*

$$\limsup_{u \to 0} \frac{f(u)}{u} < \infty.$$

If

$$I(q,p,r) < \infty \quad \textit{and} \quad I(r) = I(p) = \infty, \quad \textit{then} \quad \mathcal{N}_0^0 = \emptyset.$$

Proof. Let $x \in \mathcal{N}_0^0$. It means that we have a solution of the equation (N) in the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x(t) = 0$. Moreover, Lemma 1.2 secures that $\lim_{t\to\infty} x^{[i]}(t) = 0$ for $i = 1, 2$. Consequently, Theorem 1.3 gives that $I(q,p,r) = \infty$, which is a contradiction. □

Remark 2.7. Theorem 2.1 (Theorem 2.5) is still valid if instead of the assumption that function f satisfies Lipschitz condition on the interval $\left[0; 2I(r)\right]$ ($[0; 2I(p,r)]$), we will require that function f satisfies Lipschitz condition on the interval $\left[-2I(r); 0\right]$ ($[-2I(p,r); 0]$). Taking into account this assumption and using similar arguments as in the proof of Theorem 2.1 (Theorem 2.5), we can prove the existence of a negative solution of equation (N) in the class $\mathcal{N}_3$ ($\mathcal{N}_0$) which approaches to zero as $t \to \infty$.

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