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HOW TO FIND INITIAL DATA GENERATING BOUNDED SOLUTIONS OF DISCRETE EQUATIONS

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ABSTRACT. This contribution concerns the asymptotic behavior of solutions of a first-order difference equation. We are looking for a solution whose graph stays in a given domain. It is supposed that all the boundary points of this domain are the so called points of strict egress. Under this supposition, it has been already proved that the existence of at least one solution the graph of which stays in the given domain is guaranteed. The main aim of this article is to find the concrete value of the initial condition which generates such a solution. The method we introduce resembles the well-known bisection method for finding roots of equations.

1. Introduction

For integers s, q, $s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$ where possibilities $s = -\infty$ or $q = \infty$ are admitted, too.

We investigate the asymptotic behavior for $n \to \infty$ of the solutions of the equation

$$\Delta u(n) = f\big(n, u(n)\big), \tag{1}$$

where $n \in \mathbb{Z}_a^\infty$, $a \in \mathbb{N}$ is fixed, $\Delta u(n) = u(n+1) - u(n)$, and $f\colon \mathbb{Z}_a^\infty \times \mathbb{R} \to \mathbb{R}$.

If an initial condition

$$u(a) = u^a$$

is given, then for $n \in \mathbb{Z}_a^\infty$ there exists a unique solution $u = u(n)$ of (1) such that $u(a) = u^a$.

If the function f is continuous, then the solution depends continuously on initial data.

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The main problem

Let $b(n), c(n)$ be real functions defined on $\mathbb{Z}_a^\infty$ such that $b(n) < c(n)$ for every $n \in \mathbb{Z}_a^\infty$. Define the set ω as

$$\omega := \big\{(n,u) : n \in \mathbb{Z}_a^\infty,\, u \in \omega(n)\big\} \tag{2}$$

with

$$\omega(n) := \big\{u : b(n) < u < c(n)\big\},$$

its closure as

$$\overline{\omega} := \big\{(n,u) : n \in \mathbb{Z}_a^\infty,\, u \in \overline{\omega}(n)\big\}$$

with

$$\overline{\omega}(n) := \big\{u : b(n) \le u \le c(n)\big\},$$

and its boundary as

$$\partial\omega := \big\{(n,u) : n \in \mathbb{Z}_a^\infty,\, u = b(n) \ \text{ or } \ u = c(n)\big\}.$$

The aim is to find a solution $u = u(n)$ of equation (1) such that

$$u(n) \in \omega(n) \qquad \text{for every} \quad n \in \mathbb{Z}_a^\infty.$$

Similar problems are studied in many papers, e.g., [2], [4]–[11]. In those papers, various conditions concerning the set ω and the right-hand side of equation (1) are considered.

Here, we will assume that the boundary of the set ω consists of the so called points of strict egress.

Points of strict egress

We say that a point $(n,u) \in \partial\omega$ is a point of strict egress for the set ω with respect to equation (1) if and only if

$$u \;=\; b(n) \quad \text{and} \quad f(n,u) - b(n+1) + b(n) < 0, \tag{3}$$

or

$$u \;=\; c(n) \quad \text{and} \quad f(n,u) - c(n+1) + c(n) > 0. \tag{4}$$

Let us explain the geometrical meaning of conditions (3) and (4):

Consider a solution $u = u(n)$ of equation (1) such that $u(s) = b(s)$ for some $s \in \mathbb{Z}_a^\infty$. Then, due to equation (1), the next member of this solution, $u(s+1)$, is

$$u(s+1) = u(s) + f\big(s, u(s)\big) = b(s) + f\big(s, b(s)\big).$$

According to (3), $f\big(s, b(s)\big) - b(s+1) + b(s) < 0$, i.e., $u(s+1) < b(s+1)$. This means that $u(s+1) \notin \omega(s+1)$.

Analogously, we can show that if we have a solution $u = u(n)$ such that $u(s) = c(s)$ for some $s \in \mathbb{Z}_a^\infty$, then for this solution, $u(s+1) > c(s+1)$, i.e., $u(s+1) \notin \omega(s+1)$.

2. The existence theorem

The following theorem concerning asymptotic behavior of solutions of equation (1) is a particular case of more general results in [6, Theorem 2] and [10].

Theorem 2.1. *Suppose that f is continuous. If, moreover, all the boundary points of the set ω defined by* (2) *are points of strict egress, then there exists an initial condition*

$$u(a) = u^* \in \omega(a) \tag{5}$$

such that the corresponding solution $u = u^*(n)$ satisfies the relation

$$u^*(n) \in \omega(n) \qquad \text{for every} \quad n \in \mathbb{Z}_a^\infty. \tag{6}$$

The proof of Theorem 2.1 is done by contradiction. It is supposed that no solution stays in ω and under this supposition a continuous mapping of the interval $\bigl[b(a), c(a)\bigr]$ onto the set $\bigl\{b(a), c(a)\bigr\}$ is found which is impossible.

Unfortunately, Theorem 2.1 just states that there exists a solution staying in the domain ω but it does not give us any recipe how to find the appropriate initial condition (5). This gap is particularly filled, e.g., in [3] where the case of linear equation is studied. Here we present another approach which is more general. Our method will be applicable to any equation satisfying the conditions of Theorem 2.1. The following algorithm has been already sketched in [12] but without a rigorous mathematical justification which will be set right now.

3. Bisection method for finding the initial data

We will describe an algorithm how to find u^* so that the solution generated by the initial condition (5) stays in the domain ω. Similarly as in the well-known bisection method for finding roots of functions, we will construct a sequence of intervals $\bigl[u^a_{L,i}, u^a_{U,i}\bigr]$ (L as "lower", U as "upper" bound), $i = 1, 2, \dots$, each of them half the width of the previous one. In the classical bisection method, the existence of a root of a continuous function f in the interval $[a, b]$ is guaranteed by the opposite signs of $f(a)$ and $f(b)$. Here, this role will be played by the fact that the solution of equation (1) starting at the point $u^a_{L,i}$ exceeds the lower bound of the set ω and the solution starting at $u^a_{U,i}$ exceeds the upper bound. Theorem 3.1 below states that this property will ensure that the sought u^* is between $u^a_{L,i}$ and $u^a_{U,i}$. In its proof we will use Lemma 3.1 which could be proved by means of basic mathematical analysis.

Lemma 3.1. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and $[a, b], [c, d]$ two non-empty intervals. If*

$$g\bigl([a, b]\bigr) \supseteq [c, d\,],$$

then there exists an interval $[\tilde{a}, \tilde{b}] \subseteq [a, b]$ *such that*

$$g\big([\tilde{a}, \tilde{b}]\big) = [c, d].$$

Theorem 3.1. *Suppose that the function* f *in equation* (1) *is continuous and that all the boundary points of the set* ω *defined by* (2) *are points of strict egress.*

Let $u = u_1(n)$ *and* $u = u_2(n)$ *be two solutions of equation* (1) *given by the initial conditions*

$$u(a) = u_1^a \in \omega(a) \quad \text{and} \quad u(a) = u_2^a \in \omega(a),$$

respectively.

If there exist integers $r, s \in \mathbb{Z}_{a+1}^{\infty}$ *such that*

$$u_1(n) \in \overline{\omega(n)} \quad \text{for } n \in \mathbb{Z}_a^{r-1} \quad \text{and} \quad u_1(r) < b(r) \tag{7}$$

and

$$u_2(n) \in \overline{\omega(n)} \quad \text{for } n \in \mathbb{Z}_a^{s-1} \quad \text{and} \quad u_2(s) > c(s) \tag{8}$$

then there exists an initial condition

$$u(a) = u^* \in (u_1^a, u_2^a)$$

such that the corresponding solution $u = u^*(n)$ *satisfies the relation* (6).

P r o o f. Without the loss of generality we may suppose that $s \geq r$. The opposite case would be analogous. It is easy to see that if we take s as the initial value of n instead of a, Theorem 2.1 remains valid, i.e., if f is continuous and all the points $(n, u) \in \partial\omega$, $n \in \mathbb{Z}_s^{\infty}$, are points of strict egress, then there exists a initial condition

$$u(s) = u^{**} \in \omega(s) \tag{9}$$

such that the corresponding solution $u = u^{**}(n)$ satisfies the relation

$$u^{**}(n) \in \omega(n) \qquad \text{for every} \quad n \in \mathbb{Z}_s^{\infty}. \tag{10}$$

We will show that there exists a value $u^* \in (u_1^a, u_2^a)$ such that the solution $u = u^*(n)$ given by the initial condition $u(a) = u^*$ satisfies the conditions

$$u^*(s) = u^{**}.$$

and

$$u^*(n) \in \omega(n) \qquad \text{for} \quad n \in \mathbb{Z}_a^{s-1}.$$

For next considerations, define for every $n \in \mathbb{Z}_a^{\infty}$ the function $g_n \colon \mathbb{R} \to \mathbb{R}$ as

$$g_n(u) := u + f(n, u).$$

As f is a continuous function, g_n is continuous as well. Notice that for every solution $u = u(n)$ of equation (1),

$$u(n+1) = g_n\big(u(n)\big)$$

and that the conditions (3) and (4) are equivalent to

$$\begin{aligned} g_n\big(b(n)\big) &< b(n+1),\\ g_n\big(c(n)\big) &> c(n+1). \end{aligned}$$

Further, define auxiliary functions

$$\hat{b}(n) := \begin{cases} u_1(n) & \text{for } n \in \mathbb{Z}_a^{r-1},\\ b(n) & \text{for } n \in \mathbb{Z}_r^s \end{cases}$$

and

$$\hat{c}(n) := \begin{cases} u_2(n) & \text{for } n \in \mathbb{Z}_a^{s-1},\\ c(s) & \text{for } n = s. \end{cases}$$

Remark that $\hat{b}(a) = u_1(a) = u_1^*$, $\hat{c}(a) = u_2(a) = u_2^*$. Further, notice that, as we do not know anything about the monotonicity of the functions g_n, it is not guaranteed that $\hat{b}(n) < \hat{c}(n)$. However, the equality $\hat{b}(n) = \hat{c}(n)$ is ruled out by the uniqueness of solution of equation (1) and by (3) and (4). For $n \in \mathbb{Z}_a^s$, define the interval

$$\hat{I}_n := \Big[\min\{\hat{b}(n), \hat{c}(n)\}, \max\big\{\hat{b}(n), \hat{c}(n)\big\}\Big].$$

As the functions g_n are continuous and due to (3), (4), (7) and (8), we know that for $n \in \mathbb{Z}_a^{s-1}$,

$$g_n\left(\hat{I}_n\right) \supseteq \hat{I}_{n+1}. \tag{11}$$

Thus, we are able to find a sequence of intervals I_n, $n \in \mathbb{Z}_a^s$ such that

(i) $I_s = \hat{I}_s = \omega(s)$,

(ii) $I_n \subseteq \hat{I}_n$ for $n \in \mathbb{Z}_a^{s-1}$,

(iii) $g_n(I_n) = I_{n+1}$ for $n \in \mathbb{Z}_a^{s-1}$.

Constructing these intervals, we proceed backwards: we begin with the interval $I_s := \hat{I}_s = \big[b(s), c(s)\big]$. Due to (11) and Lemma 3.1, we can find an interval $I_{s-1} \subseteq \hat{I}_{s-1}$ such that $g_{s-1}(I_{s-1}) = I_s$. Recursively, we find $I_{s-2}, \dots, I_a$.

Having this sequence of intervals, we can find a sequence of points $\tilde{u}_n$, $n \in \mathbb{Z}_a^s$, such that

(i) $\tilde{u}_s = u^{**} \in I_s$,

(ii) $\tilde{u}_n \in I_n$ for $n \in \mathbb{Z}_a^{s-1}$,

(iii) $g_n(\tilde{u}_n) = \tilde{u}_{n+1}$ for $n \in \mathbb{Z}_a^{s-1}$.

Again, we proceed backwards: put $\tilde{u}_s := u^{**}$. Then there exists a point $\tilde{u}_{s-1} \in I_{s-1}$ such that $g_{s-1}(\tilde{u}_{s-1}) = \tilde{u}_s$, etc., until we come to $\tilde{u}_a \in I_a$. Now, prescribing the initial condition

$$u(a) = u^* := \tilde{u}_a,$$

we get the solution $u = u^*(n)$ of equation (1) for which

(i) $u^*(n) = \tilde{u}_n \in I_n \subset \omega(n)$ for $n \in \mathbb{Z}_a^{s-1}$,

(ii) $u^*(s) = u^{**} \in \omega(s)$,

(iii) $u^*(n) = u^{**}(n) \in \omega(n)$ for $n \in \mathbb{Z}_{s+1}^{\infty}$.

The conditions (i) and (ii) are fulfilled due to the construction of the sequence $\tilde{u}_n$ and the condition (iii) is fulfilled thanks to (9) and (10). Thus we have found the value of $u^* \in (u_1^a, u_2^a)$ for which the corresponding solution satisfies (6). $\square$

The algorithm of the bisection method for finding u^*

The value u^* will be found as a limit of an infinite sequence $\{u_i^a\}_{i=1}^{\infty}$ (although sometimes the process can be finite).

As it has been said above, the method of finding u^* will be similar to the bisection method for solving nonlinear equations of the form $f(x) = 0$. Let us start with an interval that certainly contains the sought "root" u^*. According to Theorem 2.1, it is the interval $\big[b(a), c(a)\big]$. Denote

$$u_{L,1}^a := b(a) \quad \text{and} \quad u_{U,1}^a := c(a).$$

Further, we will construct a sequence of intervals $\big[u_{L,i}^a, u_{U,i}^a\big]$, $i = 1, 2, \dots$, containing the "root" u^*. The next interval will be obtained by bisecting the previous one and choosing the correct half of it.

Denote the solutions of equation (1) given by the initial conditions $u(a) = u_{L,i}^a$ and $u(a) = u_{U,i}^a$ as $u = u_{L,i}(n)$ and $u = u_{U,i}(n)$, respectively.

Due to conditions (3) and (4), we have

$$u_{L,1}(a+1) < b(a+1) \quad \text{and} \quad u_{U,1}(a+1) > c(a+1).$$

Now we will bisect the interval $\big[u_{L,1}^a, u_{U,1}^a\big]$. Denote its center as

$$u_1^a := \frac{u_{L,1}^a + u_{U,1}^a}{2}\,.$$

Consider the solution $u = u_1(n)$ of equation (1) given by the initial condition $u(a) = u_1^a$. There are three possibilities:

I. $u_1(n) \in \omega(n)$ for every $n \in \mathbb{Z}_a^{\infty}$. In this case $u^* = u_1^a$, we have a solution with the desired property (6) and we can stop the process.

II. There exists an $r \in \mathbb{Z}_a^{\infty}$ such that $u_1(n) \in \omega(n)$ for $n = a, \dots, r-1$, but $u_1(r) \le b(r)$, i.e., $u_1(r) \notin \omega(r)$. In this case we set

$$u_{L,2}^a := u_1^a, \quad u_{U,2}^a := u_{U,1}^a.$$

III. There exists an $s \in \mathbb{Z}_a^{\infty}$ such that $u_1(n) \in \omega(n)$ for $n = a, \dots, s-1$, but $u_1(s) \ge c(s)$. This time we change the upper bound of the interval:

$$u_{U,2}^a := u_1^a, \quad u_{L,2}^a := u_{L,1}^a.$$

Now, either we have the desired u^*, or we have a new interval $\left[u^a_{L,2}, u^a_{U,2}\right]$ with the property that the solution $u = u_{L,2}(k)$ exceeds the lower bound $b(r)$ of the domain ω for some $r \in \mathbb{Z}_a^\infty$, meanwhile the solution $u = u_{U,2}(k)$ exceeds the upper bound $c(s)$ for some $s \in \mathbb{Z}_a^\infty$. Due to Theorem 3.1, this interval has to contain the value of u^* for which the corresponding solution $u = u^*(n)$ stays in ω.

Further, we will proceed inductively. Having the interval $\left[u^a_{L,i}, u^a_{U,i}\right]$, we bisect it and denote its center as

$$u^a_i := \frac{u^a_{L,i} + u^a_{U,i}}{2}\,.$$

For the solution $u = u_i(n)$ given by the initial condition $u(a) = u^a_i$, we have three possibilities: either it stays in ω, or it exceeds its lower bound, or it exceeds its upper bound. According to this, either we have found $u^* = u^a_i$, or we set $u^a_{L,i+1} := u^a_i, u^a_{U,i+1} := u^a_{U,i}$, or we set $u^a_{U,i+1} := u^a_i, u^a_{L,i+1} := u^a_{L,i}$, respectively.

Continuing this process, either we get the sought initial point u^* in a finite number of steps, or we get infinite sequences

$$\left\{u^a_{L,i}\right\}_{i=1}^\infty,\ \left\{u^a_{U,i}\right\}_{i=1}^\infty \quad \text{and} \quad \left\{u^a_i\right\}_{i=1}^\infty.$$

These sequences are obviously convergent as $\left\{u^a_{L,i}\right\}_{i=1}^\infty$ is a nondecreasing sequence bounded from above by $c(a)$, $\left\{u^a_{U,i}\right\}_{i=1}^\infty$ is a nonincreasing sequence bounded from below by $b(a)$ and $u^a_{L,i} < u^a_i < u^a_{U,i}$ for every $i \in \mathbb{N}$. In this case,

$$u^* = \lim_{i\to\infty} u^a_i.$$

Problems with practical implementation of this method

Programming the above described method, we are limited by the possibilities of computers. In the ideal case, we would bisect the intervals until either we find a solution with property (6), or the length of the interval $\left[u^a_{L,i}, u^a_{U,i}\right]$ is less than some chosen $\varepsilon > 0$. But, practically, for a given initial condition, we can compute the values of the corresponding solution of equation (1) for $n = a, a+1, \dots$, but it is clear that it is impossible to compute to infinity. We have to stop sometimes. Thus, given a fixed $N \in \mathbb{Z}_a^\infty$, we are able to find a point $\tilde{u}^*$ such that the solution $u = \tilde{u}^*(k)$ satisfies the condition

$$\tilde{u}^*(n) \in \omega(n), \qquad n \in \mathbb{Z}_a^N.$$

But the validity of (6) for every $n \in \mathbb{Z}_a^\infty$ cannot be guaranteed.

Another limitation of the bisection method is that it cannot be generalized for systems of equations.

REFERENCES

[1] AGARWAL, R. P.: *Difference Equations and Inequalities, Theory, Methods, and Applications* (2nd ed.), Marcel Dekker, New York, 2000.

[2] BAŠTINEC, J.—DIBLÍK, J.: *Asymptotic formulae for a particular solution of linear nonhomogeneous discrete equations*, Comput. Math. Appl. **45** (2003), 1163–1169.

[3] BAŠTINEC, J.—DIBLÍK, J.—RŮŽIČKOVÁ, M.: *Initial data generating bounded solutions of linear discrete equations*, Opuscula Math. **26** (2006), 395–406.

[4] DIBLÍK, J.: *Discrete retract principle for systems of discrete equations,* Comput. Math. Appl. **42** (2001), 515–528.

[5] DIBLÍK, J.: *Anti-Lyapunov method for systems of discrete equations,* Nonlinear Anal. **57** (2004), 1043–1057.

[6] DIBLÍK, J.: *Asymptotic behaviour of solutions of discrete equations,* Funct. Differ. Equ. **11** (2004), 37–48.

[7] DIBLÍK, J.—MIGDA, M.—SCHMEIDEL, E.: *Bounded solutions of nonlinear discrete equations,* Nonlinear Anal. **65** (2006), 845–853.

[8] DIBLÍK, J.—RŮŽIČKOVÁ, M.—VÁCLAVÍKOVÁ, B.: *A retract principle on discrete time scales,* Opuscula Math. **26** (2006), 445–455.

[9] DIBLÍK, J.—RŮŽIČKOVÁ, I.—RŮŽIČKOVÁ, M.: *A general version of the retract method for discrete equations,* Acta Math. Sin. (Engl. Ser.) **23** (2007), 341–348.

[10] DIBLÍK, J.—HLAVIČKOVÁ, I.: *Combination of Liapunov and retract methods in the investigation of the asymptotic behavior of solutions of systems of discrete equations*, Dynam. Systems Appl. **18** (2009), 507–537.

[11] DIBLÍK, J.—RŮŽIČKOVÁ, M.—ŠMARDA, Z.: *Ważewski's method for systems of dynamic equations on time scales*, Nonlinear Anal. **71** (2009), 1124–1131.

[12] HLAVIČKOVÁ, I.: *Bisection method for finding initial data generating bounded solutions of discrete equations,* Aplimat – J. Appl. Math. **3** (2010), 99–105.

[13] WAŻEWSKI, T.: *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires,* Ann. Soc. Polon. Math. **20** (1948), 279–313.

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