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HALF-LINEAR EULER DIFFERENTIAL EQUATIONS IN THE CRITICAL CASE

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ABSTRACT. We investigate oscillatory properties of the perturbed half-linear Euler differential equation

$$\big(\Phi(x')\big)' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p.$$

A perturbation is also allowed in the coefficient involving derivative.

1. Introduction

The half-linear Euler differential equation

$$\big(\Phi(x')\big)' + \frac{\gamma}{t^p}\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \qquad p > 1,\ \gamma \in \mathbb{R}, \tag{1}$$

is one of a few half-linear second order differential equations which can be solved explicitly. Similarly to the linear case $p = 2$, if we look for a solution of (1) in the form $x(t) = t^\lambda$ substituting into (1), we find that λ has to be a solution of the algebraic equation

$$(p-1)\Phi(\lambda)(\lambda - 1) + \gamma = 0,$$

see also [5]. The function $F(\lambda) := (p-1)\Phi(\lambda)(\lambda-1)$ has a global minimum at $\lambda^* = \frac{p-1}{p}$ and the value of this minimum is $F(\lambda^*) = -\gamma_p := -\big(\frac{p-1}{p}\big)^p$. Consequently, the equation $F(\lambda) + \gamma = 0$ has two real roots if $\gamma < \gamma_p$, one double real root if $\gamma = \gamma_p$, and no real root if $\gamma > \gamma_p$.

Equation (1) is a particular case of the general half-linear second order differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \tag{2}$$

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where r, c are continuous functions and $r(t) > 0$. It is known that oscillation theory of (2) is almost the same as this theory for the second order Sturm-Liouville linear differential equation

$$\big(r(t)x'\big)' + c(t)x = 0 \tag{3}$$

which is a special case $p = 2$ in (2). In particular, all solutions of (2) are either oscillatory or nonoscillatory, see [3]. This means, in particular, that (1) is nonoscillatory if and only if $\gamma \le \gamma_p$. Also, equation (1) with the critical coefficient $\gamma = \gamma_p$ serves as a comparison equation for the Kneser-type (non)oscillation test which states that (2) with $r(t) = 1$ is oscillatory provided

$$\liminf_{t\to\infty} t^p c(t) > \gamma_p$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^p c(t) < \gamma_p.$$

The Kneser test does not apply when $\lim_{t\to\infty} t^p c(t) = \gamma_p$ and this situation is the principal concern of our paper.

2. Auxiliary results

In a general framework, we suppose that equation (2) is nonoscillatory and we study oscillatory properties of its perturbation

$$\Big[\big(r(t) + \tilde r(t)\big)\Phi(x')\Big]' + \big(c(t) + \tilde c(t)\big)\Phi(x) = 0. \tag{4}$$

We suppose that the perturbation terms $\tilde r, \tilde c$ are continuous functions such that $r(t) + \tilde r(t) > 0$ for large t.

Let x be a solution of (2) such that $x(t) \neq 0$, then $w = r\Phi\big(\frac{x'}{x}\big)$ is a solution of the Riccati type differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \qquad q := \frac{p}{p-1}.$$

In our research, an important role plays the concept of conditionally oscillatory half-linear equation. Following [3], equation (2) with $\lambda c(t)$ instead of $c(t)$ is said to be *conditionally oscillatory* if there exists a constant λ_0 such that this equation is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. The constant λ_0 is called the *oscillation constant* of (2). A typical example of a conditionally oscillatory equation is just Euler equation (1) and its oscillation constant is $\lambda_0 = \gamma_p$.

Here we will deal with conditionally oscillatory half-linear equations in a more general sense. We will consider the equation of the form

$$\Big[\big(r(t) + \lambda\tilde r(t)\big)\Phi(x')\Big]' + \big(c(t) + \mu\tilde c(t)\big)\Phi(x) = 0 \tag{5}$$

and we say that (5) is conditionally oscillatory if there exist constants $\alpha, \beta, \omega \in \mathbb{R}$, $\alpha \neq 0$, $\beta \neq 0$, such that (5) is oscillatory for $\alpha\lambda + \beta\mu > \omega$ and nonoscillatory for $\alpha\lambda + \beta\mu < \omega$. A typical example of conditionally oscillatory equation with two parameters is perturbed Euler equation (1) with the critical coefficient γ_p

$$\left[\left(1 + \frac{\lambda}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right]\Phi(x) = 0\,. \tag{6}$$

It is proved in [2] that (6) is oscillatory if $\mu - \lambda\gamma_p > \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$ and nonoscillatory if $\mu - \lambda\gamma_p < \mu_p$. It was conjectured in [2] that (6) is nonoscillatory also in the limiting case

$$\mu - \lambda\gamma_p = \mu_p\,. \tag{7}$$

In our paper we prove that this conjecture is true. We also discuss some general aspects of the problem.

Next we derive the modified Riccati equation in a more general setting than in [2]. Let $h(t) \neq 0$ be a positive differentiable function, denote

$$G(t) := r(t)h(t)\Phi\big(h'(t)\big) \tag{8}$$

and let

$$\Omega(t) := \left(1 + \frac{\tilde{r}(t)}{r(t)}\right) G(t).$$

Define the function

$$\mathcal{G}(t,z) := |z + \Omega(t)|^q - q\Phi^{-1}\big(\Omega(t)\big)z - |\Omega(t)|^q, \tag{9}$$

where $\Phi^{-1}(x) = |x|^{q-2}x$ is the inverse function of Φ, and put

$$z := h^p(w - w_h) - \frac{\tilde{r}}{r}G = h^p w - G - \tilde{G},$$

where w is a solution of the Riccati equation associated with (4)

$$w' + c(t) + \tilde{c}(t) + (p-1)\big(r(t) + \tilde{r}(t)\big)^{1-q}|w|^q = 0,$$

$w_h = r\Phi\big(\frac{h'}{h}\big)$, and $\tilde{G} = \tilde{r}h\Phi(h')$. Then z is a solution of the equation

$$z' + C(t) + (p-1)\big(r(t) + \tilde{r}(t)\big)^{1-q}h^{-q}(t)\mathcal{G}(t,z) = 0, \tag{10}$$

where

$$C(t) = h(t)\left[\Big(\big(r(t) + \tilde{r}(t)\big)\Phi\big(h'(t)\big)\Big)' + \big(c(t) + \tilde{c}(t)\big)\Phi\big(h(t)\big)\right]. \tag{11}$$

Indeed, by a direct computation we have

$$\begin{aligned}
z' &= p\Phi(h)h'w + h^p w' - r|h'|^p - h\big(r\Phi(h')\big)' - \tilde r|h'|^p - h\big(\tilde r\Phi(h')\big)' \\
&= p\Phi(h)h'w + h^p\Big[-(c+\tilde c) - (p-1)(r+\tilde r)^{1-q}|w|^q\Big] \\
&\quad - (r+\tilde r)|h'|^p - h\big[(r+\tilde r)\Phi(h')\big]' \\
&= p\frac{h'}{h}(z+G+\tilde G) - (r+\tilde r)|h'|^p - h\Big[\big((r+\tilde r)\Phi(h')\big)' + (c+\tilde c)\Phi(h)\Big] \\
&\quad - (p-1)h^p(r+\tilde r)^{1-q}\left|h^{-p}z + w_h + \tilde r\Phi\left(\frac{h'}{h}\right)\right|^q \\
&= -h\Big[\big((r+\tilde r)\Phi(h')\big)' + (c+\tilde c)\Phi(h)\Big] - (p-1)(r+\tilde r)^{1-q}h^{-q} \\
&\quad \times\left\{|z+\Omega|^q - \frac{p}{p-1}h'(r+\tilde r)^{q-1}h^{q-1}z - (r+\tilde r)^q h^q|h'|^p\right\} \\
&= -h\Big[\big((r+\tilde r)\Phi(h')\big)' + (c+\tilde c)\Phi(h)\Big] \\
&\quad - (p-1)(r+\tilde r)^{1-q}h^{-q}\Big\{|z+\Omega|^q - q\Phi^{-1}(\Omega)z - |\Omega|^q\Big\}.
\end{aligned}$$

Note that in contrast to [2] we do not suppose that h is a solution of (2), so the extra term

$$h\big[\big(r\Phi(h')\big)' + c\Phi(h)\big]$$

appears in the definition of the function C in (11).

We will also need the following statement which is a slight modification of [2, Theorem 3] (here we do not require that the function h is a solution of (2)). So we omit its proof since it is the same as that of [2, Theorem 3] and it based on the fact that solvability of (10) implies nonoscillation of (4).

Theorem 1. *Let h be a positive differentiable function such that $h'(t) \neq 0$ for large t. Denote*

$$R(t) = \big(r(t) + \tilde r(t)\big)h^2(t)|h'(t)|^{p-2}, \tag{12}$$

and suppose that

$$\int_0^\infty \frac{dt}{R(t)} = \infty, \qquad \int_0^\infty C(t)\,dt \quad \text{is convergent,}$$

where C is given by (11), *and*

$$\liminf_{t\to\infty}\big(r(t)+\tilde r(t)\big)h(t)|h'(t)|^{p-1} > 0\,.$$

If

$$\limsup_{t\to\infty} \int_0^t \frac{ds}{R(s)} \int_t^\infty C(s)\, ds < \frac{1}{2q} \tag{13}$$

and

$$\liminf_{t\to\infty} \int_0^t \frac{ds}{R(s)} \int_t^\infty C(s)\, ds > -\frac{3}{2q}, \tag{14}$$

then equation (4) *is nonoscillatory.*

3. Equation (6) in the limiting case

Now we can prove the main result of our paper.

Theorem 2. *Suppose that* (7) *holds. Then the perturbed Euler equation with the critical coefficients* (6) *is nonoscillatory.*

P r o o f. We rewrite (6) into the form

$$\left[\left(1+\frac{\lambda}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t} + \frac{\mu-\mu_p}{t^p\log^2 t}\right]\Phi(x) = 0 \tag{15}$$

and we use the previous computation with

$$r(t)=1,\ \tilde r(t) = \frac{\lambda}{\log^2 t},\ c(t) = \frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p\log^2 t},\ \tilde c(t) = \frac{\mu-\mu_p}{t^p \log^2 t},\ h(t) = t^{\frac{p-1}{p}} \log^{\frac1p} t\,.$$

We have

$$h' = \frac{p-1}{p} t^{-\frac1p} \log^{\frac1p} t + \frac1p t^{-\frac1p}\log^{\frac1p - 1} t = \frac{p-1}{p} t^{-\frac1p}\log^{\frac1p} t\left(1+\frac{1}{(p-1)\log t}\right),$$

$$\Phi(h') = \left(\frac{p-1}{p}\right)^{p-1} t^{-\frac{p-1}{p}} \log^{\frac{p-1}{p}} t \left(1+\frac{1}{(p-1)\log t}\right)^{p-1},$$

and

$$\begin{aligned}(\Phi(h'))' = \left(\frac{p-1}{p}\right)^{p-1} \Bigg[& -\frac{p-1}{p} t^{-2+\frac1p} \log^{\frac{p-1}{p}} t \left(1+\frac{1}{(p-1)\log t}\right)^{p-1} \\ & + \frac{p-1}{p} t^{-2+\frac1p} \log^{-\frac1p} t \left(1+\frac{1}{(p-1)\log t}\right)^{p-1} \\ & - t^{-2+\frac1p} \log^{\frac{p-1}{p}} t \log^{-2} t\left(1+\frac{1}{(p-1)\log t}\right)^{p-2}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{p-1}{p}\right)^{p-1} \frac{p-1}{p} t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \left(1+\frac{1}{(p-1)\log t}\right)^{p-2} \\
&\quad \times \left[-1-\frac{1}{(p-1)\log t}+\frac{1}{\log t}\left(1+\frac{1}{(p-1)\log t}\right)-\frac{p}{p-1}\frac{1}{\log^2 t}\right] \\
&= \gamma_p t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \left[1+\frac{p-2}{(p-1)\log t}+\binom{p-2}{2}\frac{1}{(p-1)^2\log^2 t}\right. \\
&\quad \left.+\binom{p-2}{3}\frac{1}{(p-1)^3\log^3 t}+o\Big(\log^{-3} t\Big)\right]\left[-1+\frac{1}{\log t}\left(1-\frac{1}{p-1}\right)\right. \\
&\quad \left.+\frac{1}{\log^2 t}\left(\frac{1}{p-1}-\frac{p}{p-1}\right)\right] \\
&= \gamma_p t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \left[-1+\frac{1}{\log t}\left(-\frac{p-2}{p-1}+\frac{p-2}{p-1}\right)\right. \\
&\quad \left.+\frac{1}{\log^2 t}\left(-\frac{(p-2)(p-3)}{2(p-1)^2}+\frac{(p-2)^2}{(p-1)^2}-1\right)+o\Big(\log^{-2} t\Big)\right] \\
&= \gamma_p t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t\left[-1-\frac{p}{2(p-1)}\frac{1}{\log^2 t}+O\Big(\log^{-3} t\Big)\right] \\
&= t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t\left[-\gamma_p-\frac{\mu_p}{\log^2 t}+O\Big(\log^{-3} t\Big)\right], \qquad \text{as } \ t\to\infty.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left(\frac{\lambda}{\log^2 t}\Phi(h')\right)' = \lambda\left(\frac{p-1}{p}\right)^{p-1}\left[t^{-\frac{p-1}{p}}\log^{-1-\frac{1}{p}} t\left(1+\frac{1}{(p-1)\log t}\right)^{p-1}\right]' \\
&= \lambda\left(\frac{p-1}{p}\right)^{p-1} t^{-2+\frac{1}{p}}\log^{-1-\frac{1}{p}} t\left[1+\frac{p-2}{(p-1)\log t}+o\Big(\log^{-1} t\Big)\right] \\
&\quad \times\left[-\frac{p-1}{p}\left(1+\frac{1}{(p-1)\log t}\right)-\left(1+\frac{1}{p}\right)\frac{1}{\log t}\left(1+\frac{1}{(p-1)\log t}\right)-\frac{1}{\log^2 t}\right] \\
&= \lambda\left(\frac{p-1}{p}\right)^{p-1} t^{-2+\frac{1}{p}}\log^{-1-\frac{1}{p}} t\left[1+\frac{p-2}{(p-1)\log t}+o\Big(\log^{-1} t\Big)\right] \\
&\quad \times\left[-\frac{p-1}{p}-\frac{p+2}{p\log t}+o\Big(\log^{-1}\Big)\right] \\
&= \lambda\left(\frac{p-1}{p}\right)^{p-1} t^{-2+\frac{1}{p}}\log^{-1-\frac{1}{p}} t\left[-\frac{p-1}{p}-\frac{2}{\log t}+o\Big(\log^{-1} t\Big)\right] \qquad \text{as } t\to\infty.
\end{aligned}
$$

Hence, in the limiting case (7) it holds

$$\begin{aligned}
&\Big[\big(\tilde r\Phi(h')\big)' + \tilde c\Phi(h)\Big] \\
&= t^{-2+\frac{1}{p}}\log^{-1-\frac{1}{p}} t\left[-\lambda\gamma_p + \mu - \mu_p - 2\lambda\Big(\frac{p-1}{p}\Big)^{p-1}\frac{1}{\log t} + o\Big(\log^{-1} t\Big)\right] \\
&\quad - 2\lambda\left(\frac{p}{p-1}\right)^{p-1} t^{-2+\frac{1}{p}}\log^{-2-\frac{1}{p}} t\big(1+o(1)\big) \qquad \text{as } \ t\to\infty.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&h\Big[\big(\tilde r\Phi(h')\big)' + \tilde c\Phi(h)\Big] \\
&= -2\lambda t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t\left(\frac{p}{p-1}\right)^{p-1} t^{-2+\frac{1}{p}}\log^{-2-\frac{1}{p}} t\big(1+o(1)\big) \\
&= O\Big(t^{-1}\log^{-2} t\Big) \qquad \text{as } \ t\to\infty.
\end{aligned}$$

Now we use Theorem 1. In this theorem

$$R = (r+\tilde r)h^2|h'|^{p-2} = t\log t\big(1+o(1)\big) \sim t\log t$$

(here $f(t)\sim g(t)$ for a pair of functions f, g means $\lim_{t\to\infty}\frac{f(t)}{g(t)} = 1$),

$$\begin{aligned}
&(r+\tilde r)h\Phi(h') \\
&= \left(\frac{p-1}{p}\right)^{p-1}\left(1+\frac{\lambda}{\log^2 t}\right)\log t\left(1+\frac{1}{(p-1)\log t}\right)^{p-1} \to \infty \qquad \text{as } \ t\to\infty,
\end{aligned}$$

and using the previous computations

$$\begin{aligned}
C &= h\Big[\big((r+\tilde r)\Phi(h')\big)' + (c+\tilde c)\Phi(h)\Big] \\
&= O\Big(t^{-1}\log^{-2} t\Big) \qquad \text{as } \ t\to\infty,
\end{aligned}$$

i.e., there exists a constant $M>0$ such that $|C(t)|\le Mt^{-1}\log^{-2} t$ for large t. Now, by a direct computation

$$\lim_{t\to\infty}\left|\int_0^t R^{-1}(s)\,ds\int_t^\infty C(s)\,ds\right| \le M\lim_{t\to\infty}\frac{\log(\log t)}{\log t} = 0,$$

so by Theorem 1 equation (6) with λ and μ satisfying (7) is nonoscillatory. □

4. Open problems

(i) In equation (6), the functions $\tilde{r}(t)=\frac{1}{\log^2 t}$, $\tilde{c}(t)=\frac{1}{t^p\log^2 t}$ "match together", i.e., for $r(t)=1$ and $c(t)=\gamma_p t^{-p}$ they have such asymptotic growth for $t\to\infty$ that equation (6) is conditionally oscillatory. This fact is likely a special case of the general situation which is a subject of the present investigation. More precisely, given the functions r,c, we look for functions $\tilde{r},\tilde{c}$ with such asymptotic growth that equation (5) is conditionally oscillatory. For $\tilde{r}=0$, this problem has been studied in [4], where conditions on unperturbed equation (2) are found under which its perturbation

$$\big(r(t)\Phi(x')\big)'+\left[c(t)+\frac{\mu}{h^p(t)R(t)\big(\int_0^t R^{-1}(s)\,ds\big)^2}\right]\Phi(x)=0$$

is conditionally oscillatory (and its oscillation constant is $\mu_0=\frac{1}{2q}$, where q is the conjugate exponent to p, i.e., $\frac{1}{p}+\frac{1}{q}=1$). Here h is the so called principal solution of (2) and $R=rh^2|h'|^{p-2}$. The subject of the present investigation is to find an explicit formula for the function $\tilde{r}$ in such a way that together with the function

$$\tilde{c}(t)=\frac{1}{h^p(t)R(t)\big(\int_0^t R^{-1}(s)\,ds\big)^2}$$

equation (5) is conditionally oscillatory.

(ii) In [10] the authors establish a "power comparison theorem" for the Riemann-Weber half-linear equation

$$\big(\Phi(x')\big)'+\left[\frac{\gamma_p}{t^p}+\frac{\mu}{t^p\log^2 t}\right]\Phi(x)=0\,. \tag{16}$$

They proved a (non)oscillation criterion for this equation, where this equation is compared with an equation of the same form, but with a different power in the function Φ and other functions and constants appearing in (16). It suggests to investigate a similar problem for the more general equation (6).

(iii) In [1], motivated by the linear case treated in [7], [8], [9], we have investigated oscillatory properties of the equation

$$\big(r(t)\Phi(x')\big)'+\left[\frac{c(t)}{t^p}+\frac{d(t)}{t^p\log^2 t}\right]\Phi(x)=0 \tag{17}$$

with positive α-periodic functions r,c,d. It was shown, similarly to the case when these periodic functions are constants, that (17) is conditionally oscillatory and an explicit formula for oscillation constants has been found.

This result suggests to establish a similar result for (6), where the constants in numerators of the fractions $\frac{\lambda}{\log^2 t}$, $\frac{\gamma_p}{t^p}$, and $\frac{\mu}{t^p \log^2 t}$ are replaced by periodic functions.

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