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ON THE BIFURCATION OF A TORUS IN A SMALL OPEN ECONOMY MODEL

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ABSTRACT. A four-dimensional macroeconomic model of a small open economy under fixed exchange rates is investigated. The model describes the development of national income, capital stock, interest rate and money stock. Sufficient conditions for the existence of an invariant torus are given. A numerical example illustrating the gained results is presented.

1. Introduction

In [4] a model of a small open economy under fixed exchange rate regime was formulated. It is the 4-dimensional nonlinear continuous dynamical model with three parameters, which describes the development of national income, capital stock, interest rate and money stock. The model is the following:

$$\begin{aligned} \dot{Y} &= \alpha\big[I(Y,K,R) + G - S(Y^D,R) - T(Y) + J(Y,\rho)\big], \\ \dot{K} &= I(Y,K,R), \\ \dot{R} &= \beta\big[L(Y,R) - M\big], \\ \dot{M} &= J(Y,\rho) + \gamma\ (R - R_f), \end{aligned} \tag{1}$$

where Y–net real national income, K–real physical capital stock, R–nominal rate of interest of domestic country, M–nominal money stock, I–net real private investment expenditure on physical capital, G–real government expenditure (fixed), S–savings, T–real income tax, J–net exports in real terms, ρ–exchange rate (the value of a unit of foreign currency in terms of domestic currency), L–liquidity preference function, R_f–nominal rate of interest of foreign country

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(fixed), α, β, γ–positive parameters, t–time and

$$Y^D = Y - T(Y),\ \dot{Y} = \frac{dY}{dt},\ \dot{K} = \frac{dK}{dt},\ \dot{R} = \frac{dR}{dt},\ \dot{M} = \frac{dM}{dt}\,.$$

The economic properties of the functions in (1) are expressed by the following partial derivatives

$$\frac{\partial I(Y,K,R)}{\partial Y} > 0,\ \frac{\partial I(Y,K,R)}{\partial K} < 0,\ \frac{\partial I(Y,K,R)}{\partial R} < 0,\ \frac{\partial S(Y^D,R)}{\partial Y^D} > 0,\ \frac{\partial S(Y^D,R)}{\partial R} > 0,$$
$$\frac{\partial T(Y)}{\partial Y} > 0,\ \frac{\partial J(Y,\rho)}{\partial Y} < 0,\ \frac{\partial J(Y,\rho)}{\partial \rho} > 0,\ \frac{\partial L(Y,R)}{\partial Y} > 0,\ \frac{\partial L(Y,R)}{\partial R} < 0.$$

The model deals with a small economy, what means that economic processes in this economy have negligible influence on those in the region with which it is connected through inter-regional trade and inter-regional capital movement (this region we call a foreign country).

Parameter α can take small values as income Y reacts rather slow to the changes of the functions involved in the model, while parameters β and γ can take large values as interest rate and money stock react to the changes of the functions very quickly. This knowledge about possible values of parameters α, β and γ will be used in the analysis of the model.

The model can be utilized at the forecasting of the development of basic macroeconomic processes as well as at the regulation of them in a favorable direction. It can be used for Slovakia within Eurozone (Eurozone can be looked at as the extreme case of economies with fixed exchange rates).

Remark 1. Model (1) can be looked at as an expansion of Schinasi's three--dimensional model of a closed economy [6] to the model of an open economy, and as a generalization of Asada's three-dimensional model of an open economy [1].

2. Bifurcation analysis

We assume the following form of the functions in model (1):

$$\begin{aligned} I(Y,K,R) &= f_1(Y) - i_2K - i_3R + i_0, \\ S\big(Y^D,R\big) &= f_2\big(Y^D\big) + s_3R + s_0, \\ T(Y) &= t_1Y - t_0, \qquad (2) \\ L(Y,R) &= f_3(Y) - l_3R + l_0, \\ J(Y,\rho) &= J(Y),\ \rho \text{ is constant}, \qquad (3) \end{aligned}$$

where $f_1(Y)$, $f_2\big(Y^D\big)$, $f_3(Y)$, $J(Y)$ are nonlinear functions with respect to Y of the type C^6, $\frac{df_1(Y)}{dY} > 0$, $\frac{df_2(Y^D)}{dY^D} > 0$, $\frac{df_3(Y)}{dY} > 0$, i_2, i_3, s_3, t_0, t_1, l_3 are positive constants, $0 < t_1 < 1$, and i_0, s_0, l_0 are real numbers. After substituting (2) into model (1) we get the model

$$\begin{aligned}
\dot{Y} &= \alpha\Big[f_1(Y) - i_2K - i_3R + i_0 + G - f_2\big(Y^D\big) - s_3R - s_0 - t_1Y + t_0 + J(Y)\Big],\\
\dot{K} &= f_1(Y) - i_2K - i_3R + i_0,\\
\dot{R} &= \beta\big[f_3(Y) - l_3R + l_0 - M\big],\\
\dot{M} &= J(Y) + \gamma\ (R - R_f).
\end{aligned} \tag{4}$$

Suppose model (4) has the unique positive equilibrium

$$\begin{gathered}
E^*(\gamma) = \big(Y^*(\gamma), K^*(\gamma), R^*(\gamma), M^*(\gamma)\big),\\
Y^*(\gamma) > 0, K^*(\gamma) > 0, R^*(\gamma) > 0, M^*(\gamma) > 0,
\end{gathered}$$

for every positive γ.

Remark 2. Sufficient conditions for the existence of a positive equilibrium can be found in [4].

Let us transform the equilibrium $E^*(\gamma)$ into the origin

$$E_1^* = \big(Y_1^* = 0,\ K_1^* = 0,\ R_1^* = 0,\ M_1^* = 0\big)$$

by shifting

$$Y_1 = Y - Y^*,\ K_1 = K - K^*,\ R_1 = R - R^*,\ M_1 = M - M^*.$$

Then model (4) obtains the form

$$\begin{aligned}
\dot{Y}_1 &= \alpha\Big[f_1(Y_1 + Y^*) - f_2\big(Y_1^D + (Y^*)^D - t_0\big) + J(Y_1 + Y^*)\Big]\\
&\quad + \alpha\big[-t_1Y_1 - i_2K_1 - (i_3 + s_3)R_1\big]\\
&\quad + \alpha\big[-t_1Y^* - i_2K^* - (i_3 + s_3)R^* + i_0 - s_0 + t_0 + G\big],\\
\dot{K}_1 &= f_1(Y_1 + Y^*) - i_2K_1 - i_3R_1 - i_2K^* - i_3R^* + i_0,\\
\dot{R}_1 &= \beta\big[f_3(Y_1 + Y^*) - l_3R_1 - M_1 - l_3R^* + l_0 - M^*\big],\\
\dot{M}_1 &= J(Y_1 + Y^*) + \gamma\ (R_1 + R^* - R_f).
\end{aligned} \tag{5}$$

The Jacobian matrix $\mathbf{A} = \mathbf{A}(\alpha, \beta, \gamma)$ of (5) at the equilibrium E_1^* is

$$\mathbf{A}(\alpha, \beta, \gamma) = \begin{pmatrix} -\alpha A & -\alpha i_2 & -\alpha(i_3 + s_3) & 0\\ f_{1Y} & -i_2 & -i_3 & 0\\ \beta f_{3Y} & 0 & -\beta l_3 & -\beta\\ J_Y & 0 & \gamma & 0 \end{pmatrix}, \tag{6}$$

where $A = -f_{1Y} + f_{2Y} - J_Y + t_1$, $f_{1Y} = \frac{df_1(E^*)}{dY}$, $f_{2Y} = \frac{df_2((E^*)^D)}{dY^D}(1 - t_1)$, $f_{3Y} = \frac{df_3(E^*)}{dY}$, $J_Y = \frac{dJ(E^*)}{dY}$.

The characteristic equation of $\mathbf{A}(\alpha,\beta,\gamma)$ has the form

$$\lambda^4 + a_1(\alpha,\beta,\gamma)\lambda^3 + a_2(\alpha,\beta,\gamma)\lambda^2 + a_3(\alpha,\beta,\gamma)\lambda + a_4(\alpha,\beta,\gamma) = 0, \qquad (7)$$

where

$$\begin{aligned}
a_1 &= \alpha A + \beta l_3 + i_2,\\
a_2 &= \alpha\beta\big(l_3 A + f_{3Y}(i_3+s_3)\big) + \alpha i_2(A+f_{1Y}) + \beta(\gamma + i_2 l_3),\\
a_3 &= \beta\Big(\alpha\gamma A + \alpha\big(i_2 l_3(A+f_{1Y}) + i_2 s_3 f_{3Y} - J_Y(i_3+s_3)\big) + \gamma i_2\Big),\\
a_4 &= \alpha\beta\gamma i_2(A+f_{1Y}) - \alpha\beta i_2 s_3 J_Y.
\end{aligned}$$

Definition 1. A triple $(\alpha_0,\beta_0,\gamma_0)$ of parameters α, β, γ is called the critical triple of model (5), if matrix $\mathbf{A} = \mathbf{A}(\alpha_0,\beta_0,\gamma_0)$ has two pairs of purely imaginary eigenvalues $\lambda_{1,2} = \pm \mathrm{i}\,\omega_1$, $\lambda_{3,4} = \pm\mathrm{i}\,\omega_2$.

Let us denote $B = \gamma A + i_2 l_3(A+f_{1Y}) + i_2 s_3 f_{3Y} - J_Y(i_3+s_3)$. The following statement on the existence of a critical triple, which is the necessary condition for a birth of a torus, was proved in [5].

Lemma 1. *Let $i_2 l_3 - s_3 \geq 0$ and $\gamma = \gamma_0$ be such that*

(i) $l_3 A + f_{3Y}(i_3+s_3) > 0$,

(ii) $B < 0$.

Then the triple of parameters $(\alpha_0,\beta_0,\gamma_0)$, where

$$\alpha_0 = -\frac{\gamma_0 i_2}{B}, \quad \beta_0 = -\frac{\alpha_0 A + i_2}{l_3},$$

is the critical triple of model (5).

Consider now a critical triple $(\alpha_0,\beta_0,\gamma_0)$, which existence is guaranteed by Lemma 1. Fix β_0, γ_0 and investigate model (5) on the interval $(\alpha_0-\varepsilon,\ \alpha_0+\varepsilon)$, $\varepsilon > 0$. Performing the Taylor expansion of the right-hand sides of (5) at E_1^*, and transforming parameter α_0 to zero by shifting $\alpha_1 = \alpha - \alpha_0$ we get

$$\dot{\mathbf{x}} = \mathbf{A}(\alpha_0,\beta_0,\gamma_0)\mathbf{x} + \tilde{\mathbf{X}}(\mathbf{x},\alpha_1),$$

where $\mathbf{x} = (Y_1,K_1,R_1,M_1)^T$, $\mathbf{A}$ is Jacobian matrix, $\tilde{\mathbf{X}}$ is the nonlinear part of the model.

By further transformation $\mathbf{x} = \mathbf{M}\mathbf{y}$, $\mathbf{y} = (Y_2,K_2,R_2,M_2)^T$, we obtain

$$\begin{aligned}
\dot{Y}_2 &= i\omega_1 Y_2 + F_1(Y_2,K_2,R_2,M_2,\alpha_1),\\
\dot{K}_2 &= -i\omega_1 K_2 + F_2(Y_2,K_2,R_2,M_2,\alpha_1),\\
\dot{R}_2 &= i\omega_2 R_2 + F_3(Y_2,K_2,R_2,M_2,\alpha_1),\\
\dot{M}_2 &= -i\omega_2 M_2 + F_4(Y_2,K_2,R_2,M_2,\alpha_1), \qquad (8)
\end{aligned}$$

where $K_2 = \overline{Y_2}$, $M_2 = \overline{R}_2$, $F_2 = \overline{F}_1$, $F_4 = \overline{F}_3$ (notation “$\overline{}$” means complex conjugate expression in the whole paper).

The following theorem gives the partial normal form of model (8) including the formulae for the calculation of its resonant coefficients.

Theorem 1. *There exists a polynomial transformation* $\mathbf{y} = \mathbf{u} + \mathbf{h}(\mathbf{u}, \alpha_1)$, $\mathbf{u} = (Y_3, K_3, R_3, M_3)^T$, *where* $\mathbf{h} = (h_1, h_2, h_3, h_4)^T$ *are nonlinear polynomials with constant coefficients of the kind*

$$h_j(\mathbf{u}, \alpha_1) = \sum_{\sum m_i + m \geq 2, m \in \{0,1\}}^{4-2m} h_j^{(m_1, m_2, m_3, m_4, m)} {Y_3}^{m_1} {K_3}^{m_2} {R_3}^{m_3} {M_3}^{m_4} {\alpha_1}^{m},$$

which transforms model (8) *to its partial normal form*

$$\begin{aligned}
\dot{Y}_3 &= i\omega_1 Y_3 + \delta_1 Y_3 \alpha_1 + \delta_2 Y_3^2 K_3 + \delta_3 Y_3 R_3 M_3 + U_1^*(\mathbf{u}, \alpha_1),\\
\dot{K}_3 &= -i\omega_1 K_3 + \bar{\delta}_1 K_3 \alpha_1 + \bar{\delta}_2 Y_3 K_3^2 + \bar{\delta}_3 K_3 R_3 M_3 + \bar{U}_1^*(\mathbf{u}, \alpha_1),\\
\dot{R}_3 &= i\omega_2 R_3 + \sigma_1 R_3 \alpha_1 + \sigma_2 R_3^2 M_3 + \sigma_3 Y_3 K_3 R_3 + U_3^*(\mathbf{u}, \alpha_1),\\
\dot{M}_3 &= -i\omega_2 M_3 + \bar{\sigma}_1 M_3 \alpha_1 + \bar{\sigma}_2 R_3 M_3^2 + \bar{\sigma}_3 Y_3 K_3 M_3 + \bar{U}_3^*(\mathbf{u}, \alpha_1),
\end{aligned} \tag{9}$$

where

$$U_k^*\big(\sqrt{\alpha_1} Y_3, \sqrt{\alpha_1} K_3, \sqrt{\alpha_1} R_3, \sqrt{\alpha_1} M_3, \alpha_1\big) = \mathcal{O}\big((\sqrt{\alpha_1})^5\big), \qquad k = 1, 2, 3, 4.$$

The resonant coefficients $\delta_1, \delta_2, \delta_3, \sigma_1, \sigma_2, \sigma_3$, *in* (9) *are determined by the formulae*

$$\begin{aligned}
\delta_1 &= \frac{\partial^2 F_1}{\partial Y_2 \partial \alpha_1},\\
\delta_2 &= -\frac{1}{2i\omega_1} \frac{\partial^2 F_1}{\partial Y_2^2} \frac{\partial^2 F_1}{\partial Y_2 \partial K_2} + \frac{1}{6i\omega_1} \frac{\partial^2 F_1}{\partial K_2^2} \frac{\partial^2 F_2}{\partial Y_2^2} + \frac{1}{i\omega_1} \frac{\partial^2 F_1}{\partial Y_2 \partial K_2} \frac{\partial^2 F_2}{\partial Y_2 \partial K_2}\\
&\quad - \frac{1}{i\omega_2} \frac{\partial^2 F_1}{\partial Y_2 \partial R_2} \frac{\partial^2 F_3}{\partial Y_2 \partial K_2} + \frac{1}{i\omega_2} \frac{\partial^2 F_1}{\partial Y_2 \partial M_2} \frac{\partial^2 F_4}{\partial Y_2 \partial K_2} + \frac{1}{2(2i\omega_1 - i\omega_2)}\\
&\quad \times \frac{\partial^2 F_1}{\partial K_2 \partial R_2} \frac{\partial^2 F_3}{\partial Y_2^2} + \frac{1}{2(2i\omega_1 + i\omega_2)} \frac{\partial^2 F_1}{\partial K_2 \partial M_2} \frac{\partial^2 F_4}{\partial Y_2^2} + \frac{1}{2} \frac{\partial^3 F_1}{\partial Y_2^2 \partial K_2},\\
\delta_3 &= \frac{-1}{i\omega_1} \frac{\partial^2 F_1}{\partial Y_2^2} \frac{\partial^2 F_1}{\partial R_2 \partial M_2} + \frac{1}{i\omega_1 - 2i\omega_2} \frac{\partial^2 F_1}{\partial R_2^2} \frac{\partial^2 F_3}{\partial Y_2 \partial M_2} + \frac{1}{i\omega_1 + 2i\omega_2} \frac{\partial^2 F_1}{\partial M_2^2} \frac{\partial^2 F_4}{\partial Y_2 \partial R_2}\\
&\quad + \frac{1}{i\omega_1} \frac{\partial^2 F_1}{\partial Y_2 \partial K_2} \frac{\partial^2 F_2}{\partial R_2 \partial M_2} - \frac{1}{i\omega_2} \frac{\partial^2 F_1}{\partial Y_2 \partial R_2} \left(\frac{\partial^2 F_3}{\partial R_2 \partial M_2} + \frac{\partial^2 F_1}{\partial Y_2 \partial M_2} \right)\\
&\quad + \frac{1}{i\omega_2} \frac{\partial^2 F_1}{\partial Y_2 \partial M_2} \left(\frac{\partial^2 F_4}{\partial R_2 \partial M_2} + \frac{\partial^2 F_1}{\partial Y_2 \partial R_2} \right) + \frac{1}{i\omega_1} \frac{\partial^2 F_1}{\partial R_2 \partial M_2}\\
&\quad \times \left(\frac{\partial^2 F_4}{\partial Y_2 \partial M_2} + \frac{\partial^2 F_3}{\partial Y_2 \partial R_2} \right) + \frac{1}{2i\omega_1 - i\omega_2} \frac{\partial^2 F_1}{\partial K_2 \partial R_2} \frac{\partial^2 F_2}{\partial Y_2 \partial M_2}\\
&\quad + \frac{1}{2i\omega_1 + i\omega_2} \frac{\partial^2 F_1}{\partial K_2 \partial M_2} \frac{\partial^2 F_2}{\partial Y_2 \partial R_2} + \frac{\partial^3 F_1}{\partial Y_2 \partial R_2 \partial M_2},
\end{aligned}$$

$$
\begin{aligned}
\sigma_1 &= \frac{\partial^2 F_3}{\partial R_2 \partial \alpha_1}\,,\\
\sigma_2 &= -\frac{1}{2i\omega_2}\frac{\partial^2 F_3}{\partial R_2^2}\frac{\partial^2 F_3}{\partial R_2 \partial M_2} + \frac{1}{6i\omega_2}\frac{\partial^2 F_3}{\partial M_2^2}\frac{\partial^2 F_4}{\partial R_2^2} + \frac{1}{i\omega_2}\frac{\partial^2 F_3}{\partial R_2 \partial M_2}\frac{\partial^2 F_4}{\partial R_2 \partial M_2}\\
&\quad - \frac{1}{i\omega_1}\frac{\partial^2 F_1}{\partial R_2 \partial M_2}\frac{\partial^2 F_3}{\partial Y_2 \partial R_2} + \frac{1}{i\omega_1}\frac{\partial^2 F_2}{\partial R_2 \partial M_2}\frac{\partial^2 F_3}{\partial K_2 \partial R_2} + \frac{1}{2(2i\omega_2 - i\omega_1)}\\
&\quad \times \frac{\partial^2 F_1}{\partial R_2^2}\frac{\partial^2 F_3}{\partial Y_2 \partial M_2} + \frac{1}{2(2i\omega_2 + i\omega_1)}\frac{\partial^2 F_2}{\partial R_2^2}\frac{\partial^2 F_3}{\partial K_2 \partial M_2} + \frac{1}{2}\frac{\partial^3 F_3}{\partial R_2^2 \partial M_2}\,,\\
\sigma_3 &= \frac{-1}{i\omega_2}\frac{\partial^2 F_3}{\partial R_2^2}\frac{\partial^2 F_3}{\partial Y_2 \partial K_2} + \frac{1}{i\omega_2 - 2i\omega_1}\frac{\partial^2 F_3}{\partial Y_2^2}\frac{\partial^2 F_1}{\partial K_2 \partial R_2} + \frac{1}{i\omega_2 + 2i\omega_1}\frac{\partial^2 F_3}{\partial K_2^2}\frac{\partial^2 F_2}{\partial Y_2 \partial R_2}\\
&\quad + \frac{1}{i\omega_2}\frac{\partial^2 F_3}{\partial R_2 \partial M_2}\frac{\partial^2 F_4}{\partial Y_2 \partial K_2} - \frac{1}{i\omega_1}\frac{\partial^2 F_3}{\partial Y_2 \partial R_2}\left(\frac{\partial^2 F_1}{\partial Y_2 \partial K_2} + \frac{\partial^2 F_3}{\partial K_2 \partial R_2}\right)\\
&\quad + \frac{1}{i\omega_1}\frac{\partial^2 F_3}{\partial K_2 \partial R_2}\left(\frac{\partial^2 F_2}{\partial Y_2 \partial K_2} + \frac{\partial^2 F_3}{\partial Y_2 \partial R_2}\right) + \frac{1}{i\omega_2}\frac{\partial^2 F_3}{\partial Y_2 \partial K_2}\\
&\quad \times \left(\frac{\partial^2 F_2}{\partial K_2 \partial R_2} + \frac{\partial^2 F_1}{\partial Y_2 \partial R_2}\right) + \frac{1}{2i\omega_2 - i\omega_1}\frac{\partial^2 F_3}{\partial Y_2 \partial M_2}\frac{\partial^2 F_4}{\partial K_2 \partial R_2}\\
&\quad + \frac{1}{2i\omega_2 + i\omega_1}\frac{\partial^2 F_3}{\partial K_2 \partial M_2}\frac{\partial^2 F_4}{\partial Y_2 \partial R_2} + \frac{\partial^3 F_3}{\partial Y_2 \partial K_2 \partial R_2}\,,
\end{aligned}
$$

where all derivatives are calculated at

$$(Y_2, K_2, R_2, M_2) = (0,0,0,0), \qquad \alpha_1 = 0.$$

P r o o f. The unknown terms $h_j^{(m_1,m_2,m_3,m_4,m)}$, $j = 1,2,3,4$, and the resonant terms $\delta_1, \delta_2, \delta_3, \sigma_1, \sigma_2, \sigma_3$, can be found by the standard procedure, which is described, e.g., in [2]. As the whole process of finding them is rather elaborated and longwinded, we do not present it here. □

In polar coordinates $Y_3 = \rho_1 e^{i\varphi_1}$, $K_3 = \rho_1 e^{-i\varphi_1}$, $R_3 = \rho_2 e^{i\varphi_2}$, $\mathrm{M}_3 = \rho_2 e^{-i\varphi_2}$, the model is

$$
\begin{aligned}
\dot{\rho_1} &= \rho_1\big(a_{11}\rho_1^2 + a_{12}\rho_2^2 + c_1\alpha_1\big) + R_1^*(\rho_1,\rho_2,\varphi_1,\varphi_2,\alpha_1),\\
\dot{\rho_2} &= \rho_2\big(a_{21}\rho_1^2 + a_{22}\rho_2^2 + c_2\alpha_1\big) + R_2^*(\rho_1,\rho_2,\varphi_1,\varphi_2,\alpha_1),\\
\dot{\varphi_1} &= \omega_1 + b_{11}\rho_1^2 + b_{12}\rho_2^2 + d_1\alpha_1 + \frac{1}{\rho_1}\Phi_1^*(\rho_1,\rho_2,\varphi_1,\varphi_2,\alpha_1),\\
\dot{\varphi_2} &= \omega_2 + b_{21}\rho_1^2 + b_{22}\rho_2^2 + d_2\alpha_1 + \frac{1}{\rho_2}\Phi_2^*(\rho_1,\rho_2,\varphi_1,\varphi_2,\alpha_1),
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
a_{11} &= \operatorname{Re}\delta_2,\ a_{12} = \operatorname{Re}\delta_3,\ a_{21} = \operatorname{Re}\sigma_2,\ a_{22} = \operatorname{Re}\sigma_3,\\
c_1 &= \operatorname{Re}\delta_1,\ c_2 = \operatorname{Re}\sigma_1,\\
b_{11} &= \operatorname{Im}\delta_2,\ b_{12} = \operatorname{Im}\delta_3,\ b_{21} = \operatorname{Im}\sigma_2,\ b_{22} = \operatorname{Im}\sigma_3,\\
d_1 &= \operatorname{Im}\delta_1,\ d_2 = \operatorname{Im}\sigma_1.
\end{aligned}
$$

The equation

$$\mathcal{A}\rho^2 + \alpha_1 \mathbf{c} = \mathbf{0}, \tag{11}$$

where

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \rho^2 = \begin{pmatrix} \rho_1^2 \\ \rho_2^2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

is the bifurcation equation of model (10). Suppose that $\det \mathcal{A} \neq 0$. Denote the solution of (11) as

$$\rho^2 = \alpha_1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

and consider the matrix

$$\mathcal{P} = \begin{pmatrix} a_{11}|w_1| & a_{12}\sqrt{w_1 w_2} \\ a_{21}\sqrt{w_1 w_2} & a_{22}|w_2| \end{pmatrix}.$$

By [3, Theorem 1, p. 25], we can formulate the following statement on sufficient conditions for the existence of a torus in the model.

Theorem 2. *Suppose that the coordinates w_1, w_2 have the same sign. If the eigenvalues of matrix $\mathcal{P}$ are not purely imaginary, then model* (10) *has an invariant torus for every $\alpha_1 \in (0, \varepsilon)$ if $w_1 > 0$, and for every $\alpha_1 \in (-\varepsilon, 0)$ if $w_1 < 0$, where ε is sufficiently small. The torus is given by equations*

$$\begin{aligned} \rho_1 &= \sqrt{\alpha_1 w_1} + \sqrt{|\alpha_1^3|} f_1(\varphi_1, \varphi_2, \alpha_1), \\ \rho_2 &= \sqrt{\alpha_1 w_2} + \sqrt{|\alpha_1^3|} f_2(\varphi_1, \varphi_2, \alpha_1), \end{aligned}$$

where f_1, f_2 are continuous functions with respect to $\varphi_1, \varphi_2, \alpha_1$ for arbitrary φ_1, φ_2 and $\alpha_1 \in (0, \varepsilon)$, resp. $\alpha_1 \in (-\varepsilon, 0)$, and 2π-periodic in φ_1, φ_2.

3. Numerical example

Consider the functions

$$\begin{aligned} I &= 0.1\sqrt{Y^3} - 10K - 200R + 150.79892, \\ S &= 0.8\sqrt{0.7Y + 4.2} + 100R - 1.35946, \\ T &= 0.19Y, \\ L &= 10\sqrt{Y} - 10R - 72.900054, \\ J &= -0.075\sqrt{Y} + 6, \end{aligned}$$

and assume the constants $G = 12$, $R_f = 0.001$.

Then model (4) has the form

$$\begin{aligned} \dot{Y} &= \alpha\left(\frac{1}{10}\sqrt{Y^3} - \frac{19}{100}Y - \frac{3}{40}\sqrt{Y} - \frac{4}{5}\sqrt{\frac{7}{10}Y + \frac{21}{5}} - 10K - 300R\right) \\ &\quad + \alpha\,\frac{8507919}{50000}, \end{aligned}$$

$$
\begin{aligned}
\dot{K} &= \frac{1}{10}\sqrt{Y^3} - 10K - 200R + \frac{3769973}{25000},\\
\dot{R} &= \beta\Big(10\sqrt{Y} - 10R - M - \frac{36450027}{500000}\Big),\\
\dot{M} &= -\frac{3}{40}\sqrt{Y} + \gamma(R - \frac{1}{100}) + 6.
\end{aligned}
\tag{12}
$$

The equilibrium E^* of (12) depends on parameter γ. For $\gamma_0 = 10^6$ the equilibrium is

$$(Y^* = 64,\ K^* = 20,\ R^* = 0.0099946,\ M^* = 7).$$

Let us verify the conditions of Lemma 1:

(1) $i_2 l_3 - s_3 = 10\cdot 10 - 100 = 0 \geq 0$;

(2) $l_3 A + f_{3Y}(i_3 + s_3) = 10\cdot\big(-\frac{6}{5} + \frac{1}{25} - \big(-\frac{3}{640}\big) + \frac{19}{100}\big) + \frac{5}{8}\cdot(200+100)$
$= 10\cdot\big(-\frac{3089}{3200}\big) + \frac{5}{8}\cdot 300 = \frac{56911}{320} > 0$;

(3) $B = 10^6\cdot\big(-\frac{3089}{3200}\big) + 10\cdot 10\cdot\big(-\frac{3089}{3200} + \frac{6}{5}\big) + 10\cdot 100\cdot\frac{5}{8} - \big(-\frac{3}{640}\big)(200+100)$
$= -\frac{7717301}{8} < 0$.

Thus, the sufficient condition for the existence of a critical triple in (12) is satisfied.

The critical values of parameters α, β are

$$\alpha_0 = -\frac{\gamma_0 i_2}{B} = -\frac{10^6\cdot 10}{-\frac{7717301}{8}} = \frac{80000000}{7717301},$$

$$\beta_0 = -\frac{\alpha_0 A + i_2}{l_3} = -\frac{\frac{80000000}{7717301}\big(-\frac{3089}{3200}\big) + 10}{10} = \frac{5199}{7717301}.$$

Hence the critical triple is

$$(\alpha_0, \beta_0, \gamma_0) = \Big(\frac{8}{7717301}\cdot 10^7,\ \frac{5199}{7717301},\ 10^6\Big).$$

Matrix of linear approximation at $(\alpha_0, \beta_0, \gamma_0)$ has the form

$$\mathbf{A} = \begin{pmatrix} \frac{77225000}{7717301} & -\frac{800000000}{7717301} & -\frac{24000000000}{7717301} & 0\\ \frac{6}{5} & -10 & -200 & 0\\ \frac{25995}{61738408} & 0 & -\frac{51990}{7717301} & -\frac{5199}{7717301}\\ -\frac{3}{640} & 0 & 1000000 & 0 \end{pmatrix}.$$

Matrix A has two pairs of purely imaginary eigenvalues

$$
\begin{aligned}
\lambda_1 &\simeq -4.92743\, i,\\
\lambda_2 &\simeq 4.92743\, i,\\
\lambda_3 &\simeq -25.9815\, i,\\
\lambda_4 &\simeq 25.9815\, i.
\end{aligned}
$$

The formulae for the calculation of the resonant coefficients, which are introduced in Theorem 1, give the values

$$\begin{aligned}
&\delta_1 \simeq 0.481312 - 0.237194\, i,\\
&\delta_2 \simeq -0.000410092 + 0.000733545\, i,\\
&\delta_3 \simeq -3.66379 \cdot 10^{-9} - 1.61251 \cdot 10^{-9}\, i,\\
&\sigma_1 \simeq 0.00134379 - 0.00248592\, i,\\
&\sigma_2 \simeq -2.44592 \cdot 10^{-12} - 2.20892 \cdot 10^{-11}\, i,\\
&\sigma_3 \simeq -4.72101 \cdot 10^{-6} - 9.49541 \cdot 10^{-6}\, i.
\end{aligned}$$

The partial normal form on the center manifold in polar coordinates of (12) is

$$\begin{aligned}
\dot{\rho_1} &= \rho_1\big(-0.000410092\rho_1^2 - 3.66379 \cdot 10^{-9}\rho_2^2 + 0.481312\alpha_1\big)\\
&\quad + R_1^*(\rho_1, \rho_2, \varphi_1, \varphi_2, \alpha_1),\\
\dot{\rho_2} &= \rho_2\big(-2.44592 \cdot 10^{-12}\rho_1^2 - 4.72101 \cdot 10^{-6}\rho_2^2 + 0.00134379\alpha_1\big)\\
&\quad + R_2^*(\rho_1, \rho_2, \varphi_1, \varphi_2, \alpha_1),\\
\dot{\varphi_1} &= -4.92743 + 0.000733545\rho_1^2 - 1.61251 \cdot 10^{-9}\rho_2^2 - 0.237194\alpha_1\\
&\quad + \frac{1}{\rho_1}\Phi_1^*(\rho_1, \rho_2, \varphi_1, \varphi_2, \alpha_1),\\
\dot{\varphi_2} &= -25.9815 - 2.20892 \cdot 10^{-11}\rho_1^2 - 9.49541 \cdot 10^{-6}\rho_2^2 - 0.00248592\alpha_1\\
&\quad + \frac{1}{\rho_2}\Phi_2^*(\rho_1, \rho_2, \varphi_1, \varphi_2, \alpha_1).
\end{aligned} \tag{13}$$

The bifurcation equation of (13) is given by

$$\begin{pmatrix} -0.000410092 & -3.66379 \cdot 10^{-9} \\ -2.44592 \cdot 10^{-12} & -4.72101 \cdot 10^{-6} \end{pmatrix} \begin{pmatrix} \rho_1^2 \\ \rho_2^2 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0.481312 \\ 0.00134379 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The solution of the bifurcation equation is

$$\rho^2 = \alpha_1 \begin{pmatrix} 1173.67 \\ 284.641 \end{pmatrix},$$

and the matrix $\mathcal{P}$ has the form

$$\mathcal{P} = \begin{pmatrix} -0.481313 & -2.11764 \cdot 10^{-6} \\ -1.41372 \cdot 10^{-9} & -1.34379 \cdot 10^{-3} \end{pmatrix}.$$

According to Theorem 2, model (13) has an invariant torus for every $\alpha_1 \in (0, \varepsilon)$, where ε is sufficiently small.

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