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RESEARCH ARTICLE

Riemann's Philosophy of Geometry and Kant's Pure Intuition

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Abstract: The aim of this paper is twofold: first to explicate how Riemann's philosophy of geometry is organized around the concept of manifold. Second, to argue that Riemann's philosophy of geometry does not dismiss Kant's spatial intuition. To this end, first I analyse Riemann's *Habilitationsvortrag* with respect to interaction between philosophical, mathematical and physical perspectives. Then I will argue that although Riemann had no particular commitment to the truth of Euclidean geometry his alternative geometry does not necessarily dismiss Kant's spatial intuition.

Keywords: G.F.B. Riemann; Kant; pure intuition; non-Euclidean geometries.

1. Introduction

Although Georg Friedrich Bernhard Riemann's greatness in mathematics has been well acknowledged, and the importance and implications of his geometry studied widely by philosophers, the same does not seem to be true of his philosophy of geometry. In part, this paper is motivated by this very fact.

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In his *Habilitationsvortrag* of 1854, Riemann sets aside the usual approaches that had been taken until then, and instead tries out new ideas and approaches. Riemann wanted to depict nature from the perspective of its inner structures and one aspect of this endeavour entailed questioning the nature of space and geometry from heterogeneous points of view, such as mathematics, physics, and philosophy.[1] Riemann thought that while Euclidean geometry made an interesting proposal for the construction of a theory of space, there was in fact no *a priori* connection between the concept of space and the axioms of Euclidean geometry. He argued, then, that the fundamental concepts central to Euclidean geometry do not have to be part of every system of geometry imaginable. That is, the fundamental concepts of Euclidean geometry should not be thought of as necessary for the construction of all possible systems of geometry. In order to reach these conclusions about Euclidean geometry, and in order to introduce new concepts, it was necessary for Riemann to engage in the activity of conceptual clarification. The fundamental new concept he introduced was the concept of manifold. Describing this notion, Riemann explicitly refers to Johann Fredrich Herbart and Carl Fredrich Gauss. Since Herbart and Gauss were very critical of Kant's philosophy of geometry, Riemann—under their influence—also makes critical remarks about Kant's philosophy of geometry. However, in this paper I will argue that Riemann's alternative geometry does not necessarily dismiss Kant's spatial intuition.

2. The Architecture of Habilitationsvortrag of 1854

Riemann discusses the problem of what he calls 'multiply extended magnitude' in his famous lecture 'On the Hypotheses Which Lie at the

[1] Riemann gives a hint about his research project in an undated note 'My principal task is a new interpretation of the well laws of nature-their expression by means of other fundamental concepts-that would make possible the utilization of experimental data on the interaction of heat, light, magnetism, and electiricity for the investigation of their correlations'. For an exposition of Riemann's philosophy of nature see Bottazzini & Tazzioli (1995).

Foundation of Geometry'. Riemann's introduction clearly shows that he saw himself involved in a philosophical as well as mathematical enterprise:

> It is well known that geometry presupposes not only the concept of space but also the first fundamental notions for constructions in space as given in advance. It gives only nominal definitions for them, while the essential means of determining them appear in the form of axioms. The relation of these presuppositions is left in the dark; one sees neither whether nor in how far their connection is necessary, nor a priori whether it is possible. (1929/1959, 411)

Riemann claims that throughout history neither mathematicians nor philosophers shed light on the 'darkness' that lies at the foundations of geometry. In this regard, he thought that the reasons for this ambiguity lied in the fact that the general concept of multiply extended magnitudes had not been investigated, and that the ideas that properties depend on shape and that metrical properties depend on measure had not yet been properly separated. Accordingly, Riemann set himself two tasks: The first (a philosophical task) was to define a manifold extension. The second (an empirical task) was to give definitions of intrinsic curvature and measure determined from within extension. For the second he was indebted to Gauss, and the first to Herbart. Riemann neatly divides *Habilatitionsvortrag* into three parts, and as such one must analyse it in accordance with its philosophical, mathematical, and physical characteristics.

2.1. Philosophy in Habilitationsvortrag

Riemann introduced certain new and fruitful concepts into the discussion about geometry and space. For example, the discrete and continuous structure of space and the problem of measurement related to it; intrinsic features of space (that is those features that could only be determined without considering the fact that space is embedded in a higher-dimensional space) and extrinsic features (that are properties of this embedding); the problem of metric; and intrinsic and extrinsic metric. All of these new concepts imply a new vision for geometry; however, the concept of manifold stands at the centre of Riemann's new understanding of the subject.[2] By means of this notion, Riemann developed a new ontology of space. His concept of manifold pre-exists in an epistemic sense and it is logically prior to the concept of space (Gray in Laugwitz 1999, 235; Scholz 1992, 23). Riemann's main concern was construction *of* space, rather than construction *in* space. Riemann reasons as follows: Take a concept from any field of investigation, then think of a concept 'whose mode of determination varies continuously'; if one proceeds in 'a well determined way' from one mode to another, one gets a simply extended manifold. If one proceeds to pass over from each point of a manifold to another this procedure will result in two-dimensional (doubly extended) manifold. If we continue this procedure from two-dimensional manifold to another we will get to a triply extended manifold. Here it is important to note that in the one-dimensional case we can only move in one direction: forwards and backwards. So, in order to define motion on two-dimensional manifolds (i.e. surfaces), we have to speak of two different directions; in the case of three-dimensional mani-

[2] Before Riemann, Kant had also used the concept of manifold in *Prolegomena*, *Metaphysical Foundations of Natural Science,* and *Critique of Pure Reason*. In this sense it can be argued that Riemann owes this concept to Kant (Plotnitsky 2009, 112). However, Ferreiros (2004, 4) argues that Riemann owes the term manifold to his teacher Gauss. I agree with Ferreiros' interpretatation. In the beginning of Habilatitionsvortrag, Riemann refers to Gauss' studies of biquadratic residues in the 1832 announcement of that paper, and his 1849 proof of the fundamental theorem of algebra. All of these works are related to complex numbers (Nowak 1989, 27, Ferreiros 2007, 44). Gauss speaks of 'a manifold of two dimensions' in his interpretation of complex numbers. Gauss understands 'manifold' as nothing but system of objects connected with relations. These relations have some interconnections and properties that determine the dimensionality of manifold. Hence, Gauss wants to pay attention to properties by means of which it would be possible to consider a physical system as a two-dimensional manifold (Ferreiros 2007, 44). Gauss makes use of geometric language in a non-geometric context. Separating possibility of mathematics based on abstract spatial concepts from a constrained approach derived from perception, he discusses the geometry of the complex numbers (Nowak 1989, 27). More importantly, he talks about continua of *n*-tuples of numbers. He takes points of a plane determined by the coordinates t, u, and introduced an algebraic structure of complex numbers. Similarly, Riemann was to introduce real *n*-tuples and to investigate a 'metric structure' (Laugwitz 1999, 226).

fold (space), three different directions. n-dimensional manifolds can be understood in a similar way (i.e., where we can move in n different directions). So, we can simply say that a manifold is constructed in the relation between 'a variable object' and its capability of taking different states ('forms or modes of determinations'); these different states comprise the 'points' of manifold.[3] Riemann coined the term "Mannigfaltigkeit" to describe a set where "Bestimmungsweisen" ("ways of determination") constitute its instances or specializations. There are various interpretations of Riemann's concept. For instance, one could consider the permitted singularities within a "Mannigfaltigkeit". Riemann illustrated his idea using the set of colours, suggesting it possessed three dimensions. Additionally, he referred to a "Riemann" surface as a mathematical example.

In contemporary language, concept of manifold should be understood as a set characterized by n -tuples of real numbers. However, no formal definition is provided at the beginning of the *Habilitationsvortrag* (Ohshika 2017, 300). Hermann Weyl provides examples of manifolds, illustrating that the distinct conditions of equilibrium of an ideal gas, characterized by two independent variables like pressure and temperature, constitute a two-dimensional manifold. Similarly, the points on a sphere, or the system of pure tones described in terms of intensity and pitch, represent other examples. Additionally, based on physiological theory, which posits that colour sensation is influenced by three chemical processes occurring on the retina—black-white, red-green, and yellow-blue—each with specific directions and intensities, colours form a three-dimensional manifold in terms of quality

[3] Ohshika (2017, 295) underlines that the concept of manifold stands as a crucial cornerstone in modern geometry, and even in contemporary mathematics as a whole. Its inception is commonly attributed to Riemann, with the term "Mannigfaltigkeit," translated into English as "manifold," making its debut in Riemann's renowned *Habilitationsvortrag*. He also explains that while "manifold" is the most common translation, alternative English renderings like "multiplicity" or "variety" can be found in literature. Furthermore, he adds that, "Mannigfaltigkeit" had been used in non-mathematical contexts prior to Riemann's work, including a poem by Schiller titled "Mannigfaltigkeit." To explore the inception, elaboration, and evolution of the manifold concept, beginning with Riemann's *Habilitationsvortrag*, refer to (Ohshika, 2017). Alternatively, for an examination of the historical trajectory of the manifold concept from Grassmann through Riemann to Husserl, see Morales (2019).

and intensity. However, colour qualities alone form a two-dimensional manifold. Weyl highlights that the defining feature of an n-dimensional manifold is that every element within it (whether individual points, gas conditions, colours, or tones) can be precisely described by providing n quantities, referred to as "coordinates," which are continuous functions within the manifold (1922, 84). The concept of manifolds hints at the prospect of conceptualizing and potentially defining a space based on its relationships with other spaces (Plotnisky 2017, 350). In the definition of an n-fold extended manifold it is crucial to note that n represents the count of independent directions available for movement and that manifold concept is related with local uniqueness of the way connecting two points.

Riemann's philosophical concerns had already appeared before *Habilitationsvortrag*, when he felt the need to outline a new approach to geometry (Scholz, 1982). This new approach was philosophical in character and Herbartian in spirit. Although there is little agreement concerning exactly to what extent Herbart influenced Riemann, in its main aspects Riemann's view of mathematics benefits from a comparison with certain points of Herbart's philosophy[4]. Riemann's published works contain philosophical fragments that shed some light upon his reflections about science and which provide evidence that Riemann was strongly influenced by Herbart. Specifically, Erhard Scholz's (1982) essay contains extracts from Riemann's *Nachlass* that indicate that mathematics from Riemann's point of view and philosophy as seen by Herbart share some fundamental similarities. Riemann's selections of passages from texts by Herbart suggest that he was particularly interested in the problem of change and the structure of reality. Herbart differentiates appearance and reality. In Herbart's view, experience shows us properties and bundles of properties, while the underlying reality must be searched for within the things to which properties are ascribed. This distinction between the phenomena and a more stable underlying reality, and an investigation of the relationship between them, is essential in Riemann's own reflections about the epistemology of science. Based on Herbart's distinction between changing phenomena and underlying reality, Riemann constructs his methodology of science. However, Herbart's idea of

4 See Russell (1956), Torretti (1978), Scholz (1982), Ferreiros (2007), Banks (2005), and Werner (2010).

the advancement of knowledge is modified by Riemann. While Herbart seems to explain the process of knowledge in metaphysical way, Riemann's modified view is closer to a form of scientific research.

2.2. Herbart on Space(s), Avoiding a priorism, and Orientation of Mathematical Research

Herbart treats the Kantian understanding of space and time as 'a completely shallow, meaningless, and inappropriate [*völlig gehaltlose, nichtssagende, unpassende*] hypothesis'; that is, as naming them 'inborn' and 'empty containers' (Scholz 1982, 421). According to Herbart, spatial concepts are no different from all other concepts, which serve as 'forms of experience'.[5] Like all concepts, the origin of spatial concepts is found in experience. Yet, through philosophical and scientific thinking we give to shape spatial concepts. Space and time are departures from which Herbart produces more broad 'continuous serial forms' (*continuierliche Reihenformen*). Herbart sees things as 'bundles of properties', so for him any property can be considered a 'qualitative continuum'. Thus, for Herbart continuous serial forms mean a pure flux of instantaneous, space-less sensations that undergo dynamical, reciprocal changes among themselves. Herbart's main examples are the 'line of sound' and a coloured triangle with blue, red, and the yellow at the corners and mixed colours in the two-dimensional continuum in between.[6] The basic idea of continuous serial forms is to transfer spatial concepts into a non-geometric context. It seems likely that Herbart's theory of Serial forms (*Reihenformen*) was stimulating for Riemann, and played a role in the formation of the concept of manifold. In *Habilitationsvortrag*, Riemann, mentioning colour when talking about continuous positions in space, and using the word 'transition' between modes of determination when introducing concept of manifold, evokes a Herbartian

[5] In this sense it can be argued that for Herbart spatial concepts are like Kant's space and time as pure forms of intuition.

[6] In this context, it is possible to refer to some important works on a "space of colour" accomplished during the 18th and 19th centuries by Johann Wolfgang von Goethe and Philipp Otto Runge. See Barsan & Merticariu (2016).

construction of extended magnitude by means of continuous transitions between qualities (Banks 2005, 228; Scholz 1982, 422).[7]

Based on his theory of psychological space, Herbart wanted to discuss space with respect to the difference between *intelligible* and *phenomenal* space. The quote from Herbart below is very similar to something Riemann says in the beginning of the *Habilitationsvortrag*:

> Geometry assumes space as given; and it makes its constituents, lines and angles, through construction. But for the simple essences (and natural philosophy must be reduced to them in order to find solid ground of the real) no space is given. It together with all its determinations must be produced. The standpoint of geometry is too low for metaphysics. Metaphysics must first make clear the possibility and validity of geometry before she can make use of it. This transpires in the construction of intelligible space. (Herbart in Lenoir 2006, 152)

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[7] Riemann prefers continuous manifolds over discrete manifolds. About the reasons for this preference see Laugwitz (1999, 307–308), who argues that continuous manifolds derive their existential quality from the realm of the conceptual. On the other hand, Ferreiros (2007, 58) holds the view that continuous manifolds are firm basis for the generalization of Gauss' differential geometry. Riemann's preference also seems to be compatible with an understanding of Herbart's philosophical speculations. Riemann seems to suggest a Herbartian construction of extended magnitude by means of a continuous transition between qualities (Banks 2005, 228). According to Scholz (1992, 22), since analyzing the concept of the continuity came after the emergence of formal definitions of real numbers and the formulation of set theoretical ideas, his preference must be interpreted in an 'intuitive sense'. Russell (1956, 14) on the contrary claims that Riemann prefers the discrete above the continuous. In opposition to Russel's claim, Torretti (1978, 108) argues that 'I do not know what Russell had in mind when he spoke of "Herbart's his general preference for the discrete above the continuous", so that I cannot judge wherein such preference shows up in Riemann's writings'. I think Torretti is right on this since we see clear evidence when Riemann explicitly stresses the importance of the continuous over the discrete. In *Habilatitionsvortrag* there are a number of places in which Riemann stresses this point. He particularly underlines that we can find many examples for discrete manifolds, whereas continuous manifolds are rare. Yet, the latter shapes the field of higher mathematics in which Riemann's notion of manifold serves a fundamental role.

Here Herbart offers a form of intelligible geometry that is compatible with Riemann's approach to investigating the foundations of geometry. Both philosophers think that in order to investigate the foundations of geometry we have to avoid considering space as given; instead they claim that we have to construct geometry starting from basic concepts. While Herbart claimed that this construction was possible from any continuum, Riemann adopts a scientist's point of view and says that we can view any space as an n-fold extended manifold, where n is the number of independent directions in which we can travel. Riemann's ontology concerning mathematics can best be understood in connection with Herbart's view of mathematics. Herbart regarded mathematics as part of philosophy because he thought that, like philosophy, mathematics turns its *concepts* to *its subjects*; this is a process that goes far beyond the manipulation of formulas (Scholz 1982, 425). Riemann uses the term *speculation* in trying to solve problems. Philosophy makes use of speculation, and its subjects are concepts. In the context of formation, development, and extension of scientific concepts Riemann sees the position of mathematics similarly to the role Herbart ascribes to philosophy. Herbart thought that the sciences developed their central concepts with respect to their contexts; however, philosophical studies of the sciences require more; they must form unifying concepts that transcend this or that specific context (Scholz 1982, 424). These ideas seem to influence Riemann's ideas about the methodology of mathematics. Riemann's studies in different fields of mathematics (complex function theory, geometry, and integration) show that he wanted to develop and use his geometric ideas on n-dimensional manifolds. Diversity in geometric thought could be kept together or, to put it in more philosophical language, it could be represented as 'a unity in diversity'. Riemann does this with the concept of manifold, for it could admit different enrichments in order to show the *possibilities* and *conceptual freedom* of geometric thought (Scholz 1992, 4). Riemann understands science as 'the attempt to perceive nature through accurate concepts'. Riemann's understanding of *concept* must be interpreted in accordance with his main aim, which was to perceive nature that is dynamic in character. The only way to grasp nature and its changing character is to study it, that is, by adjusting and modifying our concepts with respect to nature—which means that our concepts cannot be given, fixed, or necessary. In this sense the concept of space cannot be an exception; rather it must be an instance of 'multiply extended magnitude' that is capable of change and variation. For Riemann, this means abandoning *a priorism* and emphasizing the role of *hypotheses.*

Herbart's influence on Riemann is seen in the epistemology and conceptual methodology of mathematics. Thus, I think that the crucial point Riemann took from Herbart when liberating geometrical thought is the idea that we do not necessarily identify physical space with the space of the senses. Riemann aimed at founding geometry anew on our perception and on the construction of space. Hence, the first part of the *Habilitationsvortrag* lecture of 1854 reflects the philosophical investigations that influenced Riemann. The concept of manifold is philosophical since it is a concept that enabled Riemann to show the possibility of other geometries and examine the *necessity* and *a priority* of Euclidean geometry.

2.3. Gauss on the Nature of the Space

Gauss was opposed to the Kantian conception of space and geometry. For Gauss, space must have a real meaning. He made this point in a letter in response to Bolyai:

> Precisely in the impossibility of deciding a priori between *Σ* (the Euclidean system) and *S* (the system of the science of space) that we find the clearest demonstration that Kant was wrong to state that space is only a form of our intuition. Another and just as strong reason I have had occasion to point out in a short note in the *Göttingischen gelehrten Anzeigen 1831.* (Gauss in Bottazzini 1994, 23)

'Strong reason', Gauss points out, refers to Kant's *incongruent counterparts*; an argument Kant thinks shows *a priori* nature of space.[8] According to Gauss:

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[8] In his transition to his *Critical* period, advancing this argument Kant claims that since our right and left hands have Leibnizian internal spatial relations we can think of them as equal. However, since we cannot superimpose our one hand upon the other they are incongruent. Then there arises a difference between them concerning not to the spatial relations among their parts but space itself. Kant seems

> This difference between right and left is in itself completely determined soon as a random front and back have been fixed on a plane and an above and below in relation to the surfaces of the plane; only if we change our intuition of this difference can we communicate it by indicating really existing material objects. (Gauss in Bottazzini 1994, 23)

Although Gauss agrees with the premises of Kant's argument, he believes that, contrary to Kant, they prove that 'space is not an a priori form of intuition'. That is, if anything, these premises prove that space must have a real physical meaning, i.e., 'space, regardless of our capacity of intuition, must have a real meaning' (Gauss in Bottazzini 1994, 23)

To search for evidence that the geometry of space is non-Euclidean, in the early 1820s Gauss measured the angles of a large triangle formed by light rays joining three peaks.[9] In emphasizing the importance of empirical investigation in *Habilitationsvortrag*, Riemann clearly reflects this Gaussian heritage. In addition, Riemann's strong interest in physics is clear from the fact that he was the physicist Weber's assistant for eighteen months. Although Riemann was under the influence of Gauss and therefore Gauss' empirical approach to geometry, it is nevertheless hard to call him a thoroughgoing empiricist. Before defining any metrical relations, the possibility of different geometries had to be investigated on the basis of the concept of manifold. In doing so, axioms of Euclidean Geometry are not only the 'most important', they are also empirically contingent rather than logically necessary, so that 'one may therefore inquire into their probability'. This shows that Riemann's main aim was not to reinterpret or to modify previously given geometric knowledge, and nor was it to examine the classical questions; rather, his main aim was to expand the domain of geometry—by which he would open new vistas for physical thought (Ferreiros 2007, 60-

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to argue in favor of Newton's absolute as opposed to Leibniz's relational space. However, there is no agreement concerning whether or not this is a valid interpretation of the purpose of argument. See, for example DiSalle (2006, 62-63).

9 This issue sparks significant controversy among historians of science, as evident in the debate between Arthur Miller's "The Myth of Gauss' Experiment on the Euclidean Nature of Physical Space" (1972) and Erhard Scholz's (2006) response to it.

61). That is why Riemann wants to speak of *hypotheses*, rather than *axioms*, in his lecture.

2.4. Mathematics in Habilatitionsvortrag

In the mathematical part of his *Habilatitionsvortrag*, Riemann follows Gauss' most fundamental steps, by extending Gaussian concepts and results for surfaces to n-dimensional manifolds, such as the measure of curvature and some properties of geodesic lines. Like Gauss, Riemann's approach is metric; the concept of distance plays a fundamental role both in the theory of curved surfaces and in Riemannian manifolds; in addition, the essential properties of manifolds are expressed by means of the linear element. Gauss' treatment of curved surfaces is of special importance in Riemann's *Habilitationsvortrag.*[10] Specifically, *Disquisitiones Generales Circa Superficies Curvas* of 1828, in which Gauss introduces his *Theorema Egregium* ('Remarkable Theorem'), includes all the results and concepts that are later developed and extended by Riemann. In his studies on surfaces, Gauss had already reached a formula that is then developed and extended by Riemann. Riemann's starting point was the equation $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ where a point determined by coordinates u and v on a surface in Euclidean space and E, F, and G are functions of the variables u and v. Based on this formula, we say that if we know the curvature then all the measure relations can be determined universally, that is, by means of Gaussian curvature we reach what can be called 'invariant structure'. Metric coefficients' behaviour on a surface contains all the information about the geometry of the surface. Without reference to any space outside the surface it is possible to know the measure of the curvature at the point determined by u and v, as a function of E, F, G, and their differentials. To put it another way: Gauss

[10] Riemann, potentially influenced by his mentor Gauss, pioneered the study of curved surfaces coordinated by parameters and introduced metrical-differential concepts. However, it is important to note that while Gauss played a significant role in advancing intrinsic geometry, it is worth noting a pre-existing tradition of mathematical endeavours focused on metrics, curved surfaces, and differential concepts. For instance, Leonhard Euler's studies could be considered within this tradition. See Papadopoulos (2017).

showed that we can do geometry on a surface (two-dimensional) independently of surrounding Euclidean (three-dimensional) space.

In *Habilitationsvortrag*, Riemann generalizes the Gaussian theory of curved spaces to n-dimensions. Such manifolds are characterized by the fact that each point within them can be uniquely specified by n real numbers. The introduction of the concept of distance into a manifold follows the Gaussian model. Analogously to the two-dimensional case, infinitesimal distances are expressed by processing differentials given in terms of some internal coordinate system, u, with the help of the metric tensor g_{ij}. Thus, Riemann arrives at a formula that is identical to the Gaussian expression for the surfaces:

$$ds^2 = \sum_{ij} g_{ij} dx^i dx^j \quad (1)$$

where g_{ij} are functions of coordinates, and $x^1, \ldots, x^n$ are coordinates on manifold. This quadratic form satisfies the following conditions:

a) the metric is symmetric, ($g_{ij} = g_{ji}$)
b) Positive definite matrix for all $1 \leq i, j \leq n$, which are the basic conditions to measure the distance in Euclidean space.

Although Gauss' studies of surfaces led to the discovery of the intrinsic aspects of space, he had nevertheless worked with Euclidean space—in fact, his surfaces were embedded in Euclidean space. However, Riemann—by attaching to each point of the manifold a Euclidean tangent space—in contradistinction to Gauss did not make use of the notion of an embedding space of higher dimensionality. Equation *(1)* brings about important results. A manifold thus allows for a distinction between neighbouring points—or events, in case of a space-time manifold—and distant points or events. The first point comes in relation to methodology of work. It brings to geometry the approach of theoretical physics: since differential expression *(1)* allows point by point analysis, it is useful for constructing basic laws that govern complex processes occurring within infinitely small elements of space or time. Second, *(1)* shows the possibility of different geometric systems, each of which depends on the chosen metrical system employed on the same manifold. In these different geometric systems of *n*-dimensions, geometric relations are no different from classical geometry. Although *(1)* is very general and also includes manifolds whose curvature is variable from one point to another, since Riemann finds these manifolds more interesting from a theoretical point of view, he chooses to consider a manifold of constant curvature—any portion of which can be continuously superposed anywhere on the manifold.

2.5. Physics in Habilatitionsvortrag

The last part of the *Habilitationsvortrag*, 'Applications to Space', develops an analysis of Euclidean space that Riemann characterizes as a three-dimensional flat space with curvature equal to zero. In this section, Riemann considers the necessary and sufficient conditions for determining metric relations in space. Riemann's three conditions are not independent but offer alternative characterizations of Euclidean Space:

1) The angles of triangles define it (at every point and in all planar directions).
2) It can be defined by the concept of "free mobility" of a rigid body at each point and in all directions, where assuming zero curvature at one point suffices.
3) It can also be characterized by the integrability of direction and length, making it not just a Riemannian manifold but a global vector space, where vectors' directions and lengths have meaningful distinctions.

The crucial aspect is not merely the existence of lines or bodies but their ability to be rigidly moved or to have their measures (length, area, volume) defined independently of their position and orientation in the manifold.

'Applications to Space' shows that for Riemann geometrical structures have fundamental physical significance in that they allow us to perceive nature in a more intelligible manner. Properly chosen, an infinite variety of geometric systems can be defined, which can function as tools for studying natural phenomena through their geometric representations. Thus, Riemann saw a strong relationship between geometry and the image of the physical universe. The physical importance of the concept of manifold is proven by a ground-breaking General Theory of Relativity. In General Relativity, space is conceived as a four-dimensional differentiable spacetime manifold (simply our ‘world’), in which metric is determined by the matter (Boi 1992, 198). As a result, Einstein’s principle of equivalence unifies metric and gravitation. We see the line element of a Riemannian manifold again:

$$ds^2 = \sum_{i,k=1}^{n} g_{ik} dx^i dx^k \quad (g_{ik} = g_{ki})$$

In General Relativity the function g_{ik} denotes the gravitational field. However, although the mathematics of Gauss and Riemann paved the way for Einstein’s Theory of General Relativity, it would be an overstatement to say that Riemann had foreseen the meaning, in physical terms, of his generalization of geometry. Riemann did not foresee what Einstein later accomplished. What he saw was not the emergence of a four-dimensional spacetime, but rather an understanding of the usual three dimensions of physical space as a particular case of n-dimensional space (Ferreiros 2004, 1). In *Habilitationsvortrag*[7] he says the main applications of his ideas would not be found in the large, but rather in the extremely small, since for him most of the physical phenomena on the microscopic level could not be explained by Euclidean light rays and rigid body, as at this level bodies would no longer exist independently of place, and because curvature of space would no longer be constant.[11]

2.6. Riemann’s Philosophy of Geometry

Interestingly, in *Habilatitionsvortrag* Riemann does not use the term ‘non-Euclidean geometry’, and he does not refer to the studies of János Bolyai or Nikolai I. Lobatchevsky; nor does he try to compare his views to Kant’s philosophy of space.[12] Although Riemann probably knew the studies

[11] For a discussion of older literature on Riemann’s *Habilitationsvortrag*, see Nowak (1989).

[12] Laugwitz connects Riemann’s avoidance of a direct attack on Kant with the presence of Rudolph Hermann Lotze among the audience of the *Habilitationsvortrag.* Lotze, as a follower of Kantian tradition, opposed non-Euclidean geometry—arguing that it is nonsense. See Laugwitz (1999, 222). Botazzini (1994, 25) claims that the *Habilitationsvortrag* was delivered in order for Riemann to qualify as a *Privatdozen*t, hence in such a delicate examination it would be better for Riemann to not enter into a discussion on such a controversial subject.

of Bolyai or Lobatchevsky (and of course Gauss), he cautiously refrains from discussing Bolyai's and Lobatchevsky's approaches to constructing non-Euclidean geometry, which involved an axiomatic method that negated Euclid's fifth postulate.[13] Yet Riemann's geometry requires us to at least re-evaluate our philosophical theories of geometry based on Euclidean geometry. For Kant, space and time are not concepts, they are pure forms of intuition (*Anschaauung*), and space is uniquely determined by three-dimensional Euclidean geometry and its propositions. Let us recall Kant's core arguments in the *Critique of Pure Reason*, about space. Space is the source of all synthetic a priori propositions of geometry. It is empirically real, but transcendentally ideal. It is a necessary condition of all objective experience, but it has no existence outside of our experience. All experience of objects in spatial relationships presupposes a space in which they are ordered. Space is an a priori intuition. We cannot represent to ourselves the absence of space. Space is not a concept of the relation of things. (a) There is only one space; and (b) the parts cannot precede this whole since they must exist within it. In addition, the synthetic a priori propositions of geometry are only possible if space is an a priori intuition. Space is an infinite given magnitude. No concept of relations can give rise to infinitude and no concept can contain an infinite number of representations within it.

It seems that Riemann could not agree with any of these propositions. Riemann's point was, as the structure of the *Habilitationsvortrag* clearly shows, that instead of postulating the axioms of Euclidean geometry, we should consider the conjunction of those axioms with a physical interpretation, and ask whether they were in point of fact really true. In Riemann's view, space has a physical reality. It is something given in experience together with metric determination. For pragmatic reasons he creates different spaces, and the question of what kind of geometry is true of space is a question of empirical determination, and is thus *a posteriori*.

[13] Although he does not mention these names he probably knew their works. One of the works of the Bolyai was presented in Crelle's journal *The Journal für die reine und angewandte Mathematik* (Journal for pure and applied mathematics) in 1837 (Bottazzini & Tazzioli 1995, 27). In addition, it is highly probable that he could have gained knowledge about the geometries of Bolyai and Lobatchevsky through Gauss (Laugwitz 1999, 224).

In my view, the *Habilitationsvortrag* is a perfect example of the interplay between philosophy and mathematics. Herbart's constructive approach to space inspired Riemann to create a fruitful combination of higher-dimensional geometry and Gauss' differential geometry (Banks, 2013). Riemann also followed Herbart and Gauss in rejecting Kant's view of space as an a priori form of intuition. Riemann regards space as a concept with meaning for the physical realm and as capable of change and variation. In accordance with Herbartian psychological theory, Riemann adopts a materialist criterion of truth and wants to answer the question: 'When is our conception of the world true?' with 'When the coherence of our concepts corresponds to the coherence among things', and when the 'connection of things' is deduced from 'connections of phenomena' (Riemann as quoted in Ehm 2010, 145). Throughout the *Habilitationsvortrag*, we see that the investigative hypothesis lying at the basis of geometry, and which was Riemann's main concern, was infinitesimals. Developing this approach enabled Riemann to investigate the links between different laws of nature—knowledge of which is based on the exactness of our description of phenomena in infinitesimal regions. Gaining knowledge of the external world from the behaviour of infinitesimal parts constitutes the backbone of Riemann's research program. In fact, *Habilitationsvortrag* of is a summary of Riemann's metric approach, which aimed to find the concept of an n-dimensional manifold equipped with the notion of distance between infinitely close points.

2.7. Kant and Riemann on Pure Intuition

Despite all the reasons for thinking that Riemann's philosophy of geometry and Kant's spatial intuition do not go together, I would like to argue that they are not necessarily inconsistent. In *Habilatitionsvortrag*, Riemann maintains that the main principles that lie at the foundation of geometry are hypotheses, and that their value is determined within 'the bounds of observation'. Here I want to underline the phrase 'the bounds of observation'. Riemann stresses that there exists some form of limit, which may well be the same as the perceptual capacity given in Kant's spatial intuition. For Kant, space and time, which are *the forms of pure intuition*, are not *concepts*. That is, one should be able to use concepts as *predicates* of subjects; but with 'space' and 'time', such predication is not possible. We can talk of the spatiality and the temporality of a thing, but while doing so, what we talk about are not space and time, but the parts that are derived from space and time. We don't conceive the external world as islands of space-time; instead we conceive it within the integrity of space-time—if we think of space and time as *complete* and *unique* then the suggestion that space and time are concepts can be ruled out. In addition, with the help of the idea of 'incongruent counterparts', we can understand why 'space' and 'time' are the pure forms of intuition but not concepts. Since we cannot define our right and left hands, we cannot name them on a conceptual level. Thus, for Kant space and time function as the conditions of all possible experience. A close reading of Kant suggests that he did not say space had to have the properties described in Euclidean geometry; rather, and at most, that we necessarily perceive space *as if it were Euclidean*. Kant's point was that as humans, or perhaps as living beings, we perceive space in some geometric system, simply by virtue of being human. The Euclidean system introduces geometrical constraints. It is true that Riemann introduces his concept of manifold in a rather quasi-philosophical way. However, according to Michael Spivak, Riemann was clear that manifolds are, locally, similar to n-dimensional Euclidean space:

> However, it is quite obvious that the notion was thoroughly clear in his own mind and that he recognized that manifolds were characterized by the fact that they are locally like n-dimensional Euclidean space. (1975, 155)

Riemann thought that geometry must start from infinitesimals. The metric given by the standard Euclidean distance $ds^2 = \sum_{ij} \delta_{ij} dx^i dx^j$ in n-dimensional Euclidean space $\mathbb{E}^n$ is the same distance relation as $\mathbb{R}^n$.[14] In Riemann's characterization of n-dimensional curvature a region of manifolds

[14] Here it is important to note that we do not say that the Pythagorean Theorem holds in every Riemannian manifold; rather, what we try to say is that by means of the notion of manifold we can transport some known theorems of Euclidean operations to n-dimensions. For example, the Pythagorean Theorem is valid in both $\mathbb{R}^2$ and $\mathbb{R}^n$. In these different geometric systems of n-dimensions, relations are no different from those in classical geometry. To give an example, in the classical Euclidean system in two dimensions, we employ the Pythagorean Theorem $a^2 + b^2 = c^2$, while

counts as flat if the distance between any pair of points in it satisfies Euclidean metric. Kant's claim about space being an *a priori* form of pure intuition, and Riemann's point about intuitive space in this sense do not necessarily rule each other out. Here it is crucial to distinguish between the foundational topological structure of a Euclidean space and the space itself, which, according to definition, constitutes a metrical space. This differentiation holds significant philosophical importance in grasping the core structure of Riemann's dissertation. He establishes the metric as a differential quadratic form based on certain *a priori* assumptions at the outset of the second part of his dissertation. It's essential to avoid conflating these with the empirical determinations of metric coefficients in the third part. Specifically, empirical evidence would not be capable of discerning between a quadratic form and a more intricate metrical function, such as one derived from a fourth-degree homogeneous form. I think that for Riemann topological structure is unique and necessary but metrical structure is subject to empirical investigation. Riemann aimed to establish the foundational essence of the metric as a differential quadratic form, particularly evident in the early part of his *Habilatitionsvortrag*'s second section. Specifically, concerning Riemannian manifolds, the ability to adjust metric coefficients for various physical applications does not imply that the fundamental concept itself—the notion of a differential quadratic form—is subject to fluctuation. Indeed, Hermann Weyl clearly distinguishes within a Riemannian manifold between two aspects: 1) the fundamental concept of a differential quadratic form, which Riemann establishes *a priori* in the construction outlined in the second part of his dissertation—Weyl termed this "the essence of space/the metric" (1923, 102-103); and 2) the contingent variation of the metric coefficients, which is subject to change and can be linked to empirical investigations. Similarly, Ernst Cassirer (1923) recognized the clear division between the concept of the Riemannian metric itself and the variation of its coefficients. He proposed that the overarching idea of a Riemannian manifold, when considered in its entirety with variable coefficients, could be regarded as the fixed *a priori* space essential for relativistic physics. Hence, Riemann's line of reasoning both adheres to and extends beyond Kant's. For all these reasons, we can

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in three dimensions it takes the form of $a^2 + b^2 + c^2 = d^2$, and in n-dimensional case it will become $x_1^2 + x_2^2 + \cdots + x_n^2 = z^2$.

perhaps call Riemann's philosophy of geometry a neo-Kantian philosophy of geometry.

As Luciano Boi puts it, Kant suggests that one role of the intuition of space in external sensibility is to lend a rational and structural coherence to various empirical phenomena. However, this coherence appears to rely less on the internal structure of the phenomena themselves and more on a faculty inherent to subjectivity. In essence, intuition functions as a frame of reference or an organizing principle (2019, 3). Indeed, Boi quotes Riemann in order to show Riemann agrees on and even more precise concerning this function of spatial intuition:

> The hypothesis that space is an infinite and three-dimensional manifold is a hypothesis which applies to our whole perception of the external world, and which allows us, at every instant, to complete the realm of our actual perceptions and construct the possible places of objects; in fact, this hypothesis is constantly confirmed in all of these applications. (...) But the infinitude of space is by no means a necessary consequence of what precedes. (Riemann, 1990)

According to Kant, in order to represent to oneself various kinds of spaces, all of which are logically possible, one needs first to possess the concept of space. Riemann's concept of manifold can actually be thought as this concept of space that Kant thought was necessary for representing various kinds of spaces to ourselves. Riemannian manifolds can represent non-Euclidean spaces, each of which is dependent on the chosen metrical system employed on the same manifold. Thus, on this view of Riemann's philosophy of geometry, spatial intuition is not being dismissed. What is more, Riemann—rather than being in opposition to Kant—shows that there are valuable conceptual resources to be found when applying geometry to physics in the very large and the very small.

In *Habilitationsvortrag*, Riemann also makes a distinction between 'unlimitedness' and 'infiniteness'. This distinction can be understood in light of a distinction between the qualitative features of space, i.e; 'the extent relations', and features relating to distance, i.e., 'measure relations'. In more modern terms, 'relations of extension' correspond to 'topological relations', while 'measure relations' correspond to 'metrical relations'. We can see this distinction when considering the surface of a sphere: it is not infinite in extent but unbounded. For Riemann, properties such as unboundedness and three dimensionality of space are known with an empirical certainty greater than that of any experience of the external world:

> That space is an unlimited, triply extended manifold is an assumption which is employed for every apprehension of the external world; by it at every moment the domain of real perceptions is supplemented and the possible locations of an object that is sought for are constructed, and in these applications the assumption is continually being verified. (1929/1959, 423)

Riemann's stress on unboundedness is followed by a question about whether our certainty about unboundedness is compatible with our certainty about the infinitude of space. I think it is not inappropriate to claim that, for Riemann, when we say that 'space is a three-dimensional manifold', has the same empirical certainty as the statement 'it is unbounded'. The above quote shows that, for Riemann, our perception of the external world is limited to three-dimensional Euclidean geometry—but he makes no reference to Kantian spatial intuition in this context. Kant argues for the intuitive nature of space at B40 in *Critique of Pure Reason* by appealing to its unboundedness—where the unboundedness of space is supposed to be guaranteed by our prior recognition. The idea that on a single topology many metric relations are possible can be used to interpret Kant's understanding of space. In the *metaphysical exposition* there are four main propositions about space: Space is not an empirical concept which has been derived from outer experience. (1781/2007, B38, 68)

The basic idea here is that if the representation of space is presupposed then relational aspect of things is possible: 'Space is a necessary a priori representation which underlies all other intuitions'. (1781/2007, A24/B39, 68) Space is the sole requirement of the possibility of external appearances; therefore, it must be an *a priori intuition*: 'Space is not a discursive or, as we say, general concept of relations of things in general, but a pure intuition'. (1781/2007, A25, 69) Here Kant argues that we can represent to ourselves only one space, and we can only consider parts of this unique space. Parts cannot precede this whole space; therefore, they can only exist *in* this space; and therefore, space is necessarily one and is an *a priori intuition:*

'Space is represented as an infinite given magnitude'. (1781/2007, B40, 69) Kant's idea here is that a concept can have infinitely many different representatives as instances of it, but the concept itself cannot be represented in infinitely many different ways. Every concept contains infinitely many representations under itself, but not within itself. Therefore, space can only be thought of in this latter way. As such, the original representation of space is not a concept, but an *a priori* intuition. In this metaphysical exposition, I do not think that Kant is giving a topology of space any different to the concepts of being unbounded and of continuous intuition. Torretti (1984, 33) suggests that:

> Since Kant conceived the 'manifold of a priori intuition' called space, not as a mere point-set, but as a (presumably three-dimensional) continuum, we must suppose that he would expected 'the mere form of intuition' to constrain the understanding to bestow a definite topological structure on the object of geometry. But, apart from this, the understanding may freely determine it, subject to no other laws than its own. Since the propositions of classical geometry are not logically necessary, nothing can prevent the understanding from developing a variety of alternative geometries (compatible with the prescribed topology), and using them in physics.

Hence, based on the idea that Kant does not give a unique determination of space, it is possible to argue that any possible space would have a geometrical structure that is not graspable by human understanding. Yet topological properties, such as continuity, three dimensionality, and unboundedness count as constraints directly imposed by the mere form of intuition. Riemannian manifolds are compatible with constraints imposed by Kantian spatial intuition in a topological sense.

Kant's pure intuition (*reine Anschauung*) is one with which we represent ourselves in physical space. Since we can think of empty space, but not the absence of the space, the concept is *a priori*. Second, for Kant space as pure intuition is the same as the physical space of ordinary experience—that is, empirically real and transcendentally ideal. According to some commentators (Wiredu 1970; Friedman 1992 and 1999), it is also possible to talk about logically possible space, for Kant. So, what is logically possible space for Kant? Kant distinguishes between the logical possibility of a concept and the objective reality of a concept:

> Thus there is no contradiction in the concept of a figure which is enclosed within two straight lines since the concepts of two straight lines and their coming together contain no negation of a figure. The impossibility arises not from the concept itself, but in connection with its construction in space, that is, from the conditions of space and its determination.[15] (1781/2007, A221/B268, 240)

Kant argues that such a concept is not self-contradictory or logically possible and is in fact objectively real. He defines the possible as that which is objectively real; and he refers these concepts as 'fictitious', denying that they tell us anything about space. Thus, Kant equates intuited space with physical space, and for this reason he thinks that logically possible spaces are not really informative about space. Riemann argues along the same lines. After pointing out that the simplest case for space is determined by $ds^2 = \sum_{ij} g_{ij} dx^i dx^j$, he says that:

> The next case in order of the simplicity would probably contain the manifolds in which the line-element can be expressed by the fourth root of a differential expression of the fourth degree. Investigation of this more general class indeed would require no essentially different principles, but would consume considerable time and throw relatively little new light upon the theory of space, particularly since the results cannot be expressed geometrically. (1929/1959, 417)

[15] This quotation combines Kant's logical criterion of possibility with Friedman's assertion that Kant's notion of real possibility can be replaced with our notion of physical possibility. Friedman's point is that in Kant's distinction between conditions of thought and conditions of cognition, the former does not correspond to our notion of logical possibility—rather, logical possibility as given by the conditions of thought plus intuition corresponds to pure mathematics. On the other hand, real possibility as given by the conditions of thought plus empirical intuition corresponds to the '(pure part of) mathematical physics' (Friedman 1992, 94).

Riemann's strategy is to identify intuited space as just one logically possible space, and not necessarily as a true description of physical space (Nowak 1989, 20). What's new here is an altered definition of space as manifold, which eliminated the necessity for a definition of three-dimensional Euclidean space and, by implication, the necessity for propositions using concepts in Euclidean geometry (propositions that were formed out of the concepts). In doing so, Riemann shows the possibility of getting rid of the 'necessity' of concepts in a system of geometry (concepts being necessary for a system). For Riemann, the concept of manifold (n-dimensional topological space) is the most general structure common to this infinite multiplicity of spaces. In this sense, he is able to identify logically possible geometries, just as Kant had suggested. As such, we can say that for Riemann this structure represents the general condition for the perception of the matters of fact and, therefore, that it is the *a priori* form of spatial intuition. Thus, for Riemann there is an *a priori* intuition of space, which is not metrical but topological. We can therefore say that what Riemann's philosophy of geometry denies is Kant's claim about the equality of intuited space with physical space; and not Kant's point about the *a priori* necessity of a general concept of space for any theory of geometry.

3. Conclusion

Riemann's ideas seem to have strong philosophical implications, yet I don't think that they were developed against Kant's philosophy of geometry. Rather, these concerns guided him to find a satisfactory basis for studying nature from more general point of view. Revolutionizing mathematics and physics was not what Riemann intended. He wanted to deal with a problem that had been around for a while—namely, is there something besides Euclid? How can we be sure about Euclid's axioms? Riemann suggests a procedure for being sure; he says that we can view any space as an n-fold extended manifold, where n is the number of arbitrary directions in which we can go. Thus, the problem for Riemann is different; that is, Riemann had no real interest in the problem of the foundations of geometry as such, such that the problem of parallels belongs to the foundations of elementary geometry—yet as his 1851 dissertation shows, he wanted to develop and use geometric ideas on *n*-dimensional manifolds as an aid to mathematics and physics. Riemann was fundamentally interested, not in synthetic *a priori* propositions, but in the geometry of physical space. In this sense, we can only say that Riemann had no particular commitment to the truth of Euclidean geometry; yet this does not mean that he wanted to cast doubt upon the Euclidean axioms. Rather, he had the goal of developing geometry so that it would become accessible to science and empirical verification.

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