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ON SOME DIFFERENCE EQUATIONS OF FIRST ORDER

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ABSTRACT. One considers two boundary value problems for the Laplacian and biharmonic operator in a plane sector with boundary conditions of the Dirichlet and the Neumann type on the angle sides. The system of difference equations of first order to which this problem is reduced, is explicitly written out. Certain cases of solvability for such difference equations of first order are described; it will be useful for studying similar equations.

1. Introduction

In 90th the author has suggested the wave factorization method for studying solvability of pseudo differential equations in non-smooth domains. It has permitted to obtain a solvability picture for model pseudo differential equations in canonical non-smooth domains of cone type. Besides, for roughly speaking, positive index of wave factorization starting from form of general solution one can describe certain correct statements of boundary value problems for pseudo differential equations and to obtain for them the analogue of algebraic Shapiro--Lopatinskii condition [1], [4]. But using the reduction to the boundary for a lot of cases we obtain the system of linear difference equations with variable coefficients of arbitrary order n instead of the system of linear algebraic equations [7]. In some simple cases there were the difference equations of first order, which can be solved by the special methods [5], [6]. We will give here some calculations based on the wave factorization related to boundary value problems for the Laplacian, and give the obtained difference equations. In conclusion we give some results devoted to solvability of simplest difference equations of first order. Here we use also the factorization idea.

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The difference equations given in Sections 2, 3 look very hard, but the author hopes, it is possible their further simplification and studying solvability. Some calculations were done by my postgraduate student M. I. Khodotova.

2. Difference equations in the oblique derivative problem for the Laplacian in a plane sector

We consider the following problem finding the function u_+, which is defined in $C_+^a = \left\{x \in \mathbf{R}^2 : x_2 > a|x_1|\right\}$ (for simplicity we take $a = 1$) and satisfy the Laplace equation

$$(\Delta u_+)(x) = 0, \qquad x \in C_+^a, \tag{1}$$

and boundary Neumann condition on one angle side ∂C_+^a

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{x_2=x_1,\, x_1>0} = 0 \tag{2}$$

and boundary condition

$$\left. a\frac{\partial u}{\partial x_2} + b\frac{\partial u}{\partial x_1} + cu\right|_{x_2=-x_1,\, x_1<0} = g, \tag{3}$$

on the other one, where $\mathbf{n}$ is the normal vector to the straight line $x_2 = x_1$.

Using wave factorization method [4] and changing variables in the problem (1)–(3) $x_1' = x_1+x_2$, $x_2' = x_1-x_2$, going to Fourier images, we have the following general form for the solution of (1):

$$\widetilde{U}_+(t_1, t_2) = a_{\neq}^{-1}(t_1, t_2)\left(\widetilde{c}(t_1) + \widetilde{d}(t_2)\right). \tag{4}$$

The formula (4) is a special case of the formula for general solution of elliptic pseudodifferential equation with symbol $A(\xi)$, which admits the homogeneous wave factorization in the form (see [4] for details)

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi).$$

We use the following notations: $\widetilde{U}_+(t_1, t_2) = \widetilde{u}_+\big((t_2+t_1)/2,\, (t_2-t_1)/2\big)$,

$$a_{\neq}(t_1, t_2) = A_{\neq}\big((t_2+t_1)/2,\ (t_2-t_1)/2\big), \quad \widetilde{c}(t_1) = \widetilde{c_0}(t_1), \quad \widetilde{d}(t_2) = \widetilde{d_0}(t_2),$$

where the unknown functions $\widetilde{c_0}$, $\widetilde{d_0}$ satisfy the conditions:

$$\int_{-\infty}^{+\infty} t_1\widetilde{U_+}(t_1,\, t_2)\, dt_1 = 0,$$

$$a\int_{-\infty}^{+\infty} t_1\widetilde{U_+}\left(t_1,t_2\right)\, dt_2 + b\int_{-\infty}^{+\infty} t_2\widetilde{U_+}\left(t_1,t_2\right)\, dt_2 + c\int_{-\infty}^{+\infty} \widetilde{U_+}\left(t_1,t_2\right)\, dt_2 \; = \widetilde{g}'\left(t_1\right). \quad (5)$$

Substituting (4) into (5) we obtain the following system of linear integral equations with respect to $\widetilde{c}$, $\widetilde{d}$:

$$\int_{-\infty}^{+\infty} t_1 a_{\neq}^{-1}\left(t_1,t_2\right)\widetilde{c}\left(t_1\right)dt_1 + \widetilde{d}\left(t_2\right)\int_{-\infty}^{+\infty} t_1 a_{\neq}^{-1}\left(t_1,t_2\right)dt_1 = 0, \quad (6)$$

$$\widetilde{c}\left(t_1\right)\int_{-\infty}^{+\infty}\left(at_1+bt_2+c\right)a_{\neq}^{-1}\left(t_1,t_2\right)dt_2$$
$$+\int_{-\infty}^{+\infty}\left(at_1+bt_2+c\right)a_{\neq}^{-1}\left(t_1,t_2\right)\widetilde{d}\left(t_2\right)dt_2 = \widetilde{g}'\left(t_1\right).$$

To simplify this system we need to calculate the following integrals:

$$b\left(t_1\right)\overset{\text{def}}{=}\int_{-\infty}^{+\infty} a_{\neq}^{-1}\left(t_1,t_2\right)dt_2, \qquad b_1\left(t_1\right)\overset{\text{def}}{=}\int_{-\infty}^{+\infty} t_2 a_{\neq}^{-1}\left(t_1,t_2\right)dt_2,$$

$$b_2\left(t_2\right)\overset{\text{def}}{=}\int_{-\infty}^{+\infty} t_1 a_{\neq}^{-1}\left(t_1,t_2\right)dt_1.$$

We remind [4]

$$a_{\neq}\left(t_1,t_2\right)=\begin{cases}\left(t_2-t_1\right)/\sqrt{2}+\sqrt{-t_1t_2}\,sgn\left(t_2-t_1\right), & t_1t_2<0,\ t_1\neq t_2,\\ \left(t_2-t_1\right)/\sqrt{2}+i\sqrt{t_1t_2}, & t_1t_2>0.\end{cases}$$

One can see the function $b\left(t_1\right)$ is homogeneous of order zero, and hence it is sufficient to calculate $b\left(\pm 1\right)$.

Let us find the value $b\left(1\right)$. Taking into account the definition for the function we will represent $a_{\neq}\left(1,t\right)=\left(t-1\right)/\sqrt{2}-\sqrt{-t}$ for all $t\in \mathrm{R}$ taking such branch $\sqrt{z}$ of the square root for which its image is in lower half-plane. We have

$$\int_{-\infty}^{+\infty}\frac{e^{-ixt}dt}{a_{\neq}\left(1,\,t\right)} = -\sqrt{2}\int_{-\infty}^{+\infty}\frac{e^{ixt}dt}{t+\sqrt{2t}+1}.$$

Let us expand the fraction $\left(t+\sqrt{2t}+1\right)^{-1}$ into the simplest ones and obtain

$$\int\limits_{-\infty}^{+\infty}\frac{e^{-ixt}dt}{a_{\neq}(1,t)}=i\left(\int\limits_{-\infty}^{+\infty}\frac{e^{ixt}dt}{\sqrt{t}+(1-i)/\sqrt{2}}-\int\limits_{-\infty}^{+\infty}\frac{e^{ixt}dt}{\sqrt{t}+(1+i)/\sqrt{2}}\right).$$

The first integral is vanishing, because the function under integral sign is holomorphic in upper and lower half-planes. The second integral for $x<0$ is vanishing in view of residue theorem, but for $x>0$

$$-i\int\limits_{-\infty}^{+\infty}\frac{e^{ixt}dt}{\sqrt{t}+(1+i)/\sqrt{2}}=-i\left(2\pi i\operatorname*{Res}_{z=i}\left[\frac{e^{ixz}}{\sqrt{z}+(1+i)/\sqrt{2}}\right]\right)=-4\pi e^{i\pi/4-x},$$

from which $b(1)=-4\pi e^{i\pi/4}$. The same calculations give $b(-1)=4\pi e^{-i\pi/4}$. The second integral $b_1(t_1)$ is a function homogeneous of order 1, and for its calculation we do the following: first we calculate $b_1(\pm 1)$, and then $b_1(t_1)=b_1(sgnt_1)\,t_1 sgnt_1$. At the end we obtain that

$$b_1(1)=-4\pi i e^{i\pi/4}=4\pi e^{-i\pi/4},\qquad b_1(-1)=4\pi i e^{-i\pi/4}=4\pi e^{i\pi/4}.$$

The function represented by the integral $b_2(t_2)$ is homogeneous of order 1 also, and for its calculation it is sufficient to find $b_2(\pm 1)$, and therefore,

$$b_2(t_2)=b_2(sgnt_2)\,t_2 sgnt_2.$$

We have $b_2(1)=4\pi i e^{-i\pi/4}=4\pi e^{i\pi/4}$, $b_2(-1)=-4\pi i e^{i\pi/4}=4\pi e^{-i\pi/4}$.

The system (6) can be rewritten as

$$\int\limits_{-\infty}^{+\infty}K(t_1,t_2)\,\widetilde{c}(t_1)\,dt_1+\widetilde{d}(t_2)=0, \tag{7}$$

$$\begin{aligned}a\widetilde{c}(t_1)&+\big(bb_1(t_1)\,b^{-1}(t_1)+c\big)t_1^{-1}\widetilde{c}(t_1)\\&+\left(a+ct_1^{-1}\right)\int\limits_{-\infty}^{+\infty}L(t_1,t_2)\,\widetilde{d}(t_2)\,dt_2\\&+\int\limits_{-\infty}^{+\infty}M(t_1,t_2)\,\widetilde{d}(t_2)\,dt_2=\widetilde{g_1}(t_1),\end{aligned}$$

where we use the following notations:

$$\begin{aligned}K(t_1,t_2)&=a_{\neq}^{-1}(t_1,t_2)\,b_2^{-1}(t_2)\,t_1,\\L(t_1,t_2)&=a_{\neq}^{-1}(t_1,t_2)\,b^{-1}(t_1),\\M(t_1,t_2)&=a_{\neq}^{-1}(t_1,t_2)\,b^{-1}(t_1)\,t_2t_1^{-1},\\\widetilde{g_1}(t_1)&=\widetilde{g'}(t_1)\,b^{-1}(t_1)\,t_1^{-1}.\end{aligned}$$

For all $t_1 > 0$, $t_2 > 0$ we denote

$$K_{11}(t_1,t_2) = K(t_1,t_2), \qquad K_{12}(t_1,t_2) = K(-t_1,t_2),$$
$$K_{21}(t_1,t_2) = K(-t_1,-t_2), \qquad K_{22}(t_1,t_2) = K(t_1,-t_2).$$

Analogously, we define the kernels $M_{ij}(t_1,t_2)$, $L_{ij}(t_1,t_2)$, $i,j = 1,2$.

Let us note all kernels are the functions homogeneous of order -1.

Denote $c_0(t_1)$ the restriction $\widetilde{c}(t_1)$ on $(0;+\infty)$, $c_1(t_1)$ the restriction $\widetilde{c}(-t_1)$ on $(0;+\infty)$; the same we define the functions $d_0(t_2)$, $d_1(t_2)$. Finally, we denote $g_{10}(t_1)$ the restriction $\widetilde{g_1}(t_1)$ on $(0;+\infty)$, $g_{11}(t_1)$ the restriction $\widetilde{g_1}(-t_1)$ on $(0;+\infty)$. As a result instead of the system of two linear integral equation (7) with respect to two unknowns $\widetilde{c}(t_1)$, $\widetilde{d}(t_2)$ we obtain the system of four linear integral equations with respect to four unknown functions $c_0(t_1)$, $c_1(t_1)$, $d_0(t_2)$, $d_1(t_2)$ on the positive half-axis, for which their kernels are homogeneous of the order -1. It permits to apply the Mellin transform [3] for studying solvability of this system.

We remind the Mellin transform is defined by the formula

$$\hat{f}(s) = \int_0^\infty f(x)x^{s-1}\,dx, \qquad s = \sigma + i\tau,$$

at least for functions $f(x) \in C_0^\infty(\mathbf{R}_+)$. The integral converges for all complex s and it is an entire analytic function. If we change variable $x = e^t$, then the Mellin transform passes into the Fourier transform of function $f(e^t)$

$$\hat{f}(s) = \int_{-\infty}^\infty e^{ts} f(e^t)\,dt, \qquad s = \sigma + i\tau.$$

Thus, all properties of the Mellin transform can be obtained from corresponding properties of the Fourier transform. Particularly, the inversion formula of the Mellin transform for $f(x) \in C_0^\infty(\mathbf{R})$ has the following form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(s)t^{-s}d\tau, \quad s = \sigma + i\tau.$$

The Parseval equality for the Mellin transform

$$\int_0^{+\infty} t^{2\sigma-1}|f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{f}(s)|^2\,d\tau, \quad s = \sigma + i\tau,$$

particularly, for $\sigma = 1/2$ we have

$$\int\limits_0^{+\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} |\hat f(s)|^2\,d\tau, \qquad s = 1/2 + i\tau,$$

or, in other words,

$$\int\limits_0^{+\infty} |f(t)|^2\,dt = \frac{1}{2\pi i}\int\limits_{1/2-i\infty}^{1/2+i\infty} |\hat f(s)|^2\,ds,$$

meaning the right integral as

$$\lim_{y\to\infty}\int\limits_{1/2-iy}^{1/2+iy} |\hat f(s)|^2\,ds.$$

The last transform reduces the (4×4)-system mentioned to a system of linear difference equations with respect to unknowns $\widehat{c_0}\,(\lambda)$, $\widehat{c_1}\,(\lambda)$, $\widehat{d_0}\,(\lambda)$, $\widehat{d_1}\,(\lambda)$:

$$\widehat{K_{11}}\,(\lambda)\,\widehat{c_0}\,(\lambda) + \widehat{K_{12}}\,(\lambda)\,\widehat{c_1}\,(\lambda) + \widehat{d_0}\,(\lambda) = 0,$$

$$\widehat{K_{21}}\,(\lambda)\,\widehat{c_0}\,(\lambda) + \widehat{K_{22}}\,(\lambda)\,\widehat{c_1}\,(\lambda) + \widehat{d_1}\,(\lambda) = 0,$$

$$\begin{aligned}
&a\widehat{c_0}\,(\lambda) + \left(bb_1\,(1)\,b^{-1}\,(1) + c\right)\widehat{c_0}\,(\lambda-1) + \left(a\widehat{L_{11}}\,(\lambda) + b\widehat{M_{11}}\,(\lambda)\right)\widehat{d_0}\,(\lambda)\\
&+\left(a\widehat{L_{12}}\,(\lambda) + b\widehat{M_{12}}\,(\lambda)\right)\widehat{d_1}\,(\lambda) + c\widehat{L_{11}}\,(\lambda)\,\widehat{d_0}\,(\lambda-1) + c\widehat{L_{12}}\,(\lambda)\,\widehat{d_1}\,(\lambda-1)\\
&\quad= \widehat{g_{10}}\,(\lambda),\\
&a\widehat{c_1}\,(\lambda) - \left(bb_1\,(-1)\,b^{-1}\,(-1) + c\right)\widehat{c_1}\,(\lambda-1) + \left(a\widehat{L_{21}}\,(\lambda) + b\widehat{M_{21}}\,(\lambda)\right)\widehat{d_0}\,(\lambda)\\
&+\left(a\widehat{L_{22}}\,(\lambda) + b\widehat{M_{22}}\,(\lambda)\right)\widehat{d_1}\,(\lambda) - c\widehat{L_{21}}\,(\lambda)\,\widehat{d_0}\,(\lambda-1) - c\widehat{L_{22}}\,(\lambda)\,\widehat{d_1}\,(\lambda-1)\\
&\quad= \widehat{g_{20}}\,(\lambda),
\end{aligned} \tag{8}$$

where $\widehat{\ }$ denotes the Mellin transform for corresponding functions, $\widehat{K_{ij}}(\lambda)$, $\widehat{M_{ij}}(\lambda)$, $\widehat{L_{ij}}\,(\lambda)$ are the Mellin transforms for the functions $K_{ij}\,(1,t)$, $M_{ij}\,(t,1)$, $L_{ij}\,(t,1)$, $i,j = 1,2$, respectively. So, it is necessary to calculate the Mellin transforms for the functions $K_{ij}\,(1,t)$, $M_{ij}\,(t,1)$, $L_{ij}\,(t,1)$:

$$K_{11}\,(1,t) = a_{\neq}^{-1}\,(1,t)\,b_1^{-1}\,(t) = (4\pi)^{-1}\,e^{-i\pi/4}t^{-1}\left((t-1)/\sqrt{2} + i\sqrt{t}\right)^{-1},$$

$$K_{12}\,(1,t) = -a_{\neq}^{-1}\,(-1,t)\,b_1^{-1}\,(t) = -\,(4\pi)^{-1}\,e^{-i\pi/4}t^{-1}\left((t+1)/\sqrt{2} + \sqrt{t}\right)^{-1},$$

$$K_{21}(1,t)=-a_{\neq}^{-1}(-1,-t)\,b_1^{-1}(-t)=-(4\pi)^{-1}e^{i\pi/4}t^{-1}\left((1-t)/\sqrt{2}+i\sqrt{t}\right)^{-1},$$
$$K_{22}(1,t)=\ a_{\neq}^{-1}(1,-t)\,b_1^{-1}(-t)=-(4\pi)^{-1}e^{i\pi/4}t^{-1}\left((t+1)/\sqrt{2}+\sqrt{t}\right).$$

By analogous calculations we find $M_{ij}(t,1)$, $L_{ij}(t,1)$. Further we have:

$$L_{11}(t,1)=-(4\pi)^{-1}e^{-i\pi/4}\left((1-t)/\sqrt{2}+i\sqrt{t}\right)^{-1}=-itK_{21}(1,t),$$
$$L_{12}(t,1)=\ (4\pi)^{-1}e^{i\pi/4}\left((t+1)/\sqrt{2}+\sqrt{t}\right)^{-1}=-tK_{22}(1,t),$$
$$L_{21}(t,1)=\ (4\pi)^{-1}e^{i\pi/4}\left((t-1)/\sqrt{2}+i\sqrt{t}\right)^{-1}=itK_{11}(1,t),$$
$$L_{22}(1,t)=(4\pi)^{-1}e^{-i\pi/4}\left((t+1)/\sqrt{2}+\sqrt{t}\right)^{-1}=-tK_{12}(1,t);$$

$$M_{11}(t,1)=\ t^{-1}L_{11}(t,1),$$
$$M_{12}(t,1)=\ t^{-1}L_{12}(t,1),$$
$$M_{21}(t,1)=-t^{-1}L_{21}(t,1),$$
$$M_{22}(t,1)=-t^{-1}L_{22}(t,1).$$

From computational point of view it is more convenient to find the Mellin transform of the functions $L_{ij}(t,1)$, $i,j=1,2$, and then, taking into account property of the Mellin transform, $\widehat{t^{-1}f(t)}=\widehat{f}(\lambda-1)$, and then to obtain the Mellin transform of the functions $M_{ij}(t,1)$, $K_{ij}(1,t)$.

We have:
$$\int_0^{+\infty}\frac{t^{\lambda-1}}{(t-1)/\sqrt{2}+i\sqrt{t}}=2\sqrt{2}\int_0^{+\infty}\frac{y^{2\lambda-1}}{y^2+\sqrt{2}iy-1}.$$

Let us note, the following representation is valid
$$\left(y^2+\sqrt{2}iy-1\right)^{-1}=-\frac{i+1}{2}\left(1+\frac{\sqrt{2}}{i-1}y\right)^{-1}-\frac{i-1}{2}\left(1+\frac{\sqrt{2}}{i+1}y\right)^{-1}.$$

Hence, we obtain
$$\begin{aligned}\int_0^{+\infty}\frac{t^{\lambda-1}dt}{a_{\neq}(1,t)}&=2\sqrt{2}\Bigg(\frac{i-1}{2}\frac{\pi}{\left(\sqrt{2}/(i+1)\right)^{2\lambda}}\cos ec\,(2\pi\lambda)\\&\qquad-\frac{i+1}{2}\frac{\pi}{\left(\sqrt{2}/(i-1)\right)^{2\lambda}}\cos ec\,(2\pi\lambda)\Bigg)\\&=-2\pi\cos ec\,(2\pi\lambda)\left(e^{i(\pi/4)(2\lambda-1)}-e^{i(3\pi/4)(2\lambda-1)}\right),\end{aligned}$$

which implies

$$\widehat{L_{21}}\,(\lambda) = -\,(1/2)\,e^{i\pi/4}\cos ec\,(2\pi\lambda)\left(e^{i(\pi/4)(2\lambda-1)} - e^{i(3\pi/4)(2\lambda-1)}\right),$$

and then

$$\widehat{K_{11}} = -i\widehat{L_{21}}\,(\lambda-1)\,,$$

$$\widehat{K_{11}}\,(\lambda) = (i/2)\,e^{i\pi/4}\cos ec\big(2\pi\,(\lambda-1)\big)\left(e^{i(\pi/4)(2\lambda-3)} - e^{i(3\pi/4)(2\lambda-3)}\right)$$

$$= (1/2)\,e^{i\pi/4}\cos ec\,(2\pi\lambda)\left(e^{i(\pi/4)(2\lambda-1)} - e^{i(3\pi/4)(2\lambda-1)}\right).$$

$$M_{21}\,(t,1) = -\widehat{L_{21}}\,(\lambda-1)\,,$$

hence,

$$\widehat{M_{21}}\,(\lambda) = -\,(1/2)\,e^{3i\pi/4}\cos ec\,(2\pi\lambda)\left(e^{i(\pi/4)(2\lambda-1)} - e^{i(3\pi/4)(2\lambda-1)}\right).$$

Analogously,

$$\widehat{L_{22}}\,(\lambda) = -e^{-i\pi/4}\cos ec\,(2\pi\lambda)\sin\big((\pi/4)\,(2\lambda-1)\big),$$

$$\widehat{K_{12}}\,(\lambda) = \widehat{M_{22}}\,(\lambda) = -\widehat{L_{22}}\,(\lambda-1) = -e^{-i\pi/4}\cos ec\,(2\pi\lambda)\cos\big((\pi/4)\,(2\lambda-1)\big),$$

$$\widehat{L_{12}}\,(\lambda) = -e^{i\pi/4}\cos ec\,(2\pi\lambda)\sin\big((\pi/4)\,(2\lambda-1)\big),$$

$$\widehat{K_{22}}\,(\lambda) = \widehat{M_{12}}\,(\lambda) = -\widehat{L_{12}}\,(\lambda-1) = -e^{i\pi/4}\cos ec\,(2\pi\lambda)\cos\big((\pi/4)\,(2\lambda-1)\big),$$

$$\widehat{L_{11}}\,(\lambda) = (1/2)\,e^{-i\pi/4}\cos ec\,(2\pi\lambda)\left(e^{-i(3\pi/4)(2\lambda-1)} - e^{-i(\pi/4)(2\lambda-1)}\right),$$

$$\widehat{K_{21}}\,(\lambda) = i\widehat{L_{11}}\,(\lambda-1)$$

$$= (1/2)\,e^{-i\pi/4}\cos ec\,(2\pi\lambda)\left(e^{-i(3\pi/4)(2\lambda-1)} + e^{-i(\pi/4)(2\lambda-1)}\right),$$

$$\widehat{M_{21}}\,(\lambda) = -\widehat{L_{11}}\,(\lambda-1)$$

$$= -\,(1/2)\,e^{i\pi/4}\cos ec\,(2\pi\lambda)\left(e^{-i(3\pi/4)(2\lambda-1)} + e^{-i(\pi/4)(2\lambda-1)}\right).$$

It is not hard work to reduce the system (8) by elementary transformations to the system of two difference equations with two unknowns $\widehat{c_0}\,(\lambda)$, $\widehat{c_1}\,(\lambda)$.

So, the solving the problem (1) with boundary conditions (2), (3) reduces to solving the following system:

$$\begin{cases} \widehat{A}\,(\lambda)\,\widehat{c_0}\,(\lambda) + \widehat{B}\,(\lambda)\,\widehat{c_1}\,(\lambda) + \widehat{P}\,(\lambda)\,\widehat{c_0}\,(\lambda-1) + \widehat{R}\,(\lambda)\,\widehat{c_1}\,(\lambda-1) = \widehat{g_{10}}\,(\lambda)\,, \\ \widehat{A_1}\,(\lambda)\,\widehat{c_0}\,(\lambda) + \widehat{B_1}\,(\lambda)\,\widehat{c_1}\,(\lambda) + \widehat{P_1}\,(\lambda)\,\widehat{c_0}\,(\lambda-1) + \widehat{R_1}\,(\lambda)\,\widehat{c_1}\,(\lambda-1) = \widehat{g_{20}}\,(\lambda)\,, \end{cases}$$

where

$$\widehat{A}(\lambda) = a\left(1 - \widehat{K_{11}}(\lambda)\widehat{L_{11}}(\lambda) - \widehat{K_{21}}(\lambda)\widehat{L_{12}}(\lambda)\right) - b\left(\widehat{K_{11}}(\lambda)\widehat{M_{11}}(\lambda) + \widehat{K_{21}}(\lambda)\widehat{M_{12}}(\lambda)\right),$$

$$\widehat{B}(\lambda) = -a\left(\widehat{K_{12}}(\lambda)\widehat{L_{11}}(\lambda) + \widehat{K_{22}}(\lambda)\widehat{L_{12}}(\lambda)\right) - b\left(\widehat{K_{12}}(\lambda)\widehat{M_{11}}(\lambda) + \widehat{K_{22}}(\lambda)\widehat{M_{12}}(\lambda)\right),$$

$$\widehat{P}(\lambda) = bb_1(1)\,b^{-1}(1) + c\left(1 - \widehat{K_{11}}(\lambda)\widehat{L_{11}}(\lambda) - \widehat{K_{12}}(\lambda)\widehat{L_{12}}(\lambda)\right),$$

$$\widehat{R}(\lambda) = -c\left(\widehat{K_{12}}(\lambda)\widehat{L_{11}}(\lambda) + \widehat{K_{22}}(\lambda)\widehat{L_{12}}(\lambda)\right),$$

$$\widehat{A}_1(\lambda) = a\left(\widehat{K_{21}}(\lambda)\widehat{L_{22}}(\lambda) - \widehat{K_{11}}(\lambda)\widehat{L_{21}}(\lambda)\right) + b\left(\widehat{K_{21}}(\lambda)\widehat{M_{22}}(\lambda) - \widehat{K_{11}}(\lambda)\widehat{M_{21}}(\lambda)\right),$$

$$\widehat{B_1}(\lambda) = a\left(1 - \widehat{K_{12}}(\lambda)\widehat{L_{21}}(\lambda) - \widehat{K_{22}}(\lambda)\widehat{L_{22}}(\lambda)\right) - b\left(\widehat{K_{12}}(\lambda)\widehat{M_{21}}(\lambda) + \widehat{K_{22}}(\lambda)\widehat{M_{22}}(\lambda)\right),$$

$$\widehat{P_1}(\lambda) = c\left(\widehat{K_{11}}(\lambda)\widehat{L_{21}}(\lambda) + \widehat{K_{21}}(\lambda)\widehat{L_{22}}(\lambda)\right),$$

$$\widehat{R_1}(\lambda) = c\left(\widehat{K_{22}}(\lambda)\widehat{L_{22}}(\lambda) + \widehat{K_{12}}(\lambda)\widehat{L_{21}}(\lambda) - 1\right) - bb_1(-1)\,b^{-1}(-1).$$

3. Difference equations in the oblique derivative problem for the biharmonic equation in a plane sector

Here we consider the following problem: finding the function u_+, which is defined in $C_+^1 = \left\{x \in \mathbf{R}^2 : x_2 > |x_1|\right\}$ and satisfy the biharmonic equation

$$\Delta^2 u_+(x) = 0, \qquad x \in C_+^1, \tag{9}$$

and boundary conditions on angle sides

$$\begin{cases} \left.\frac{\partial u_+}{\partial \mathbf{n}}\right|_{x_2=x_1, x_1>0} = g_1, \\ u_+|_{x_2=x_1, x_1>0} = g_2 \end{cases} \tag{10}$$

on one angle side, and

$$\begin{cases} \left.\frac{\partial u_+}{\partial \mathbf{n}}\right|_{x_2=-x_1, x_1<0} = g_3, \\ u_+|_{x_2=-x_1, x_1<0} = g_4, \end{cases} \tag{11}$$

on the other one, where $\mathbf{n}$ is unit normal vector for the straight line $x_2 = x_1$.

Using the wave factorization we have the formula for general solution of the equation (9) and changing variables

$$x_1' = x_1 + x_2, \qquad x_2' = x_1 - x_2$$

in the problem (9)–(11) and going to Fourier images, we have the equation

$$\widetilde{U}_+(t_1,t_2) = a_{\neq}^{-2}(t_1,t_2)\left(\tilde{c}_0(t_1) + \tilde{c}_1(t_1)\,t_2 + \tilde{d}_0(t_2) + \tilde{d}_1(t_2)\,t_1\right) \tag{12}$$

and four conditions for determining the unknown functions $\widetilde{c_0}, \widetilde{d_0}, \widetilde{c_1}, \widetilde{d_1}$:

$$\begin{cases} \int\limits_{-\infty}^{+\infty} t_1\widetilde{U_+}(t_1,t_2)\,dt_1 = \tilde{g}_1'(t_2), \\ \int\limits_{-\infty}^{+\infty} \widetilde{U_+}(t_1,t_2)\,dt_1 = \tilde{g}_2'(t_2), \end{cases} \quad \begin{cases} \int\limits_{-\infty}^{+\infty} t_2\widetilde{U_+}(t_1,t_2)\,dt_2 = \tilde{g}_3'(t_1), \\ \int\limits_{-\infty}^{+\infty} \widetilde{U_+}(t_1,t_2)\,dt_2 = \tilde{g}_4'(t_1), \end{cases} \tag{13}$$

for which we use the following notations:

$$\begin{aligned} \widetilde{U}_+(t_1,t_2) &= \widetilde{u}_+\big((t_2+t_1)/2,\ (t_2-t_1)/2\big), \\ a_{\neq}(t_1,t_2) &= A_{\neq}\big((t_2+t_1)/2,\ (t_2-t_1)/2\big), \\ \widetilde{c}(t_1) &= \widetilde{c_0}(t_1), \\ \widetilde{d}(t_2) &= \widetilde{d_0}(t_2). \end{aligned}$$

Substituting (12) into (13) we obtain the following system of linear integral equations with respect to $\widetilde{c}$, $\widetilde{d}$:

$$\begin{cases} \int\limits_{-\infty}^{+\infty} t_1 a_{\neq}^{-2}(t_1,t_2)\,\tilde{c}_0(t_1)\,dt_1 + t_2\int\limits_{-\infty}^{+\infty} t_1 a_{\neq}^{-2}(t_1,t_2)\,\tilde{c}_1(t_1)\,dt_1 \\ \quad +\tilde{d}_0(t_2)\int\limits_{-\infty}^{+\infty} t_1 a_{\neq}^{-2}(t_1,t_2)\,dt_1 \\ \quad +\tilde{d}_1(t_2)\int\limits_{-\infty}^{+\infty} t_1^2 a_{\neq}^{-2}(t_1,t_2)\,dt_1 = \tilde{g}_1(t_2), \\ \int\limits_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1,t_2)\,\tilde{c}_0(t_1)\,dt_1 + t_2\int\limits_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1,t_2)\,\tilde{c}_1(t_1)\,dt_1 \\ \quad +\tilde{d}_0(t_2)\int\limits_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1,t_2)\,dt_1 \\ \quad +\tilde{d}_1(t_2)\int\limits_{-\infty}^{+\infty} t_1 a_{\neq}^{-2}(t_1,t_2)\,dt_1 = \tilde{g}_2(t_2), \end{cases}$$

$$\begin{cases} \tilde{c}_0(t_1) \int\limits_{-\infty}^{+\infty} t_2 a_{\neq}^{-2}(t_1,t_2)\,dt_2 + \tilde{c}_1(t_1) \int\limits_{-\infty}^{+\infty} t_2^2 a_{\neq}^{-2}(t_1,t_2)\,dt_2 \\ \quad + \int\limits_{-\infty}^{+\infty} t_2 a_{\neq}^{-2}(t_1,t_2)\,\tilde{d}_0(t_2)\,dt_2 \\ \quad + t_1 \int\limits_{-\infty}^{+\infty} t_2 a_{\neq}^{-2}(t_1,t_2)\,\tilde{d}_1(t_2)\,dt_2 = \tilde{g}_3(t_1), \\ \tilde{c}_0(t_1) \int\limits_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1,t_2)\,dt_2 + \tilde{c}_1(t_1) \int\limits_{-\infty}^{+\infty} t_2 a_{\neq}^{-2}(t_1,t_2)\,dt_2 \\ \quad + \int\limits_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1,t_2)\,\tilde{d}_0(t_2)\,dt_2 \\ \quad + t_1 \int\limits_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1,t_2)\,\tilde{d}_1(t_2)\,dt_2 = \tilde{g}_4(t_1). \end{cases}$$

As in the previous case we write the expression

$$a_{\neq}(t_1,t_2) = \begin{cases} \left((t_2-t_1)/\sqrt{2} + \sqrt{-t_1 t_2}\, sgn\,(t_2-t_1)\right)^2, & t_1 t_2 < 0,\ t_1 \neq t_2, \\ (t_2-t_1)/\sqrt{2} + i\sqrt{t_1 t_2}, & t_1 t_2 > 0. \end{cases}$$

For simplifying the system above and taking into account the concrete form of the symbol $a_{\neq}(t_1,t_2)$, we will calculate the following integrals:

$$b(t_1) \stackrel{\text{def}}{=} \int\limits_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1,t_2)\,dt_2, \qquad b_1(t_1) \stackrel{\text{def}}{=} \int\limits_{-\infty}^{+\infty} t_2 a_{\neq}^{-2}(t_1,t_2)\,dt_2,$$

$$b_2(t_1) \stackrel{\text{def}}{=} \int\limits_{-\infty}^{+\infty} t_2^2 a_{\neq}^{-2}(t_1,t_2)\,dt_2, \qquad b_3(t_2) \stackrel{\text{def}}{=} \int\limits_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1,t_2)\,dt_1,$$

$$b_4(t_2) \stackrel{\text{def}}{=} \int\limits_{-\infty}^{+\infty} t_1 a_{\neq}^{-2}(t_1,t_2)\,dt_1, \qquad b_5(t_2) \stackrel{\text{def}}{=} \int\limits_{-\infty}^{+\infty} t_1^2 a_{\neq}^{-2}(t_1,t_2)\,dt_1.$$

These integrals can be calculated by the same way as above for the Laplacian. Let us calculate $b(t_1) = \int_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1,t_2)\,dt_2$. It is sufficient to calculate $b(\pm 1)$. Let us find the value $b(1)$. We have

$$\int\limits_{-\infty}^{+\infty} \frac{e^{-ixt}\,dt}{a_{\neq}(1,t)} = -2\int\limits_{-\infty}^{+\infty} \frac{e^{ixt}\,dt}{\left(t+\sqrt{2t}+1\right)^2}.$$

If we expand the fraction $\left(t+\sqrt{2t}+1\right)^{-2}$ into the simplest ones, we will obtain:

$$\int\limits_{-\infty}^{+\infty}\frac{e^{-ixt}dt}{a_{\neq}(1,t)}=\int\limits_{-\infty}^{+\infty}\frac{e^{ixt}dt}{\left(\sqrt{t}+(1-i)/\sqrt{2}\right)^2}-\int\limits_{-\infty}^{+\infty}\frac{e^{ixt}dt}{\left(\sqrt{t}+(1+i)/\sqrt{2}\right)^2}$$

$$+\sqrt{2}i\int\limits_{-\infty}^{+\infty}\frac{e^{ixt}dt}{\sqrt{t}+(1-i)/\sqrt{2}}-\sqrt{2}i\int\limits_{-\infty}^{+\infty}\frac{e^{ixt}dt}{\sqrt{t}+(1+i)/\sqrt{2}},$$

$$-\sqrt{2}i\int\limits_{-\infty}^{+\infty}\frac{e^{ixt}dt}{\sqrt{t}+(1+i)/\sqrt{2}}=-4\sqrt{2}\pi e^{\frac{i\pi}{4}-x},$$

$$\int\limits_{-\infty}^{+\infty}\frac{e^{ixt}dt}{\left(\sqrt{t}+(1+i)/\sqrt{2}\right)^2}=2\pi i\operatorname*{Res}_{z=i}\left[\frac{e^{ixz}}{\left(\sqrt{z}+(1+i)/\sqrt{2}\right)^2}\right]$$

$$=\frac{2\sqrt{2}\pi ie^{\frac{i\pi}{4}-x}}{2e^{\frac{i\pi}{4}}+\sqrt{2}+\sqrt{2}e^{\frac{i\pi}{2}}},$$

from which it implies that

$$b(1)=-2\sqrt{2}\pi e^{i\pi/4}\left(2-\frac{i}{\sqrt{2}e^{i\pi/4}+1+e^{i\pi/2}}\right).$$

By the similar way we find that

$$b(-1)=2\sqrt{2}\pi e^{-i\pi/4}\left(2-\frac{i}{\sqrt{2}e^{i\pi/4}+1+e^{i\pi/2}}\right).$$

Let us calculate the second integral

$$b_1(t_1)=\int\limits_{-\infty}^{+\infty}t_2a_{\neq}^{-2}(t_1,t_2)\,dt_2:$$

$$b_1(1)=-2\sqrt{2}\pi e^{i\pi/4}\left(2i+\frac{1}{\sqrt{2}e^{i\pi/4}+1+e^{i\pi/2}}\right),$$

$$b_1(-1)=2\sqrt{2}\pi e^{-i\pi/4}\left(2i+\frac{1}{\sqrt{2}e^{i\pi/4}+1+e^{i\pi/2}}\right).$$

$$b_2(t_1)=\int\limits_{-\infty}^{+\infty}t_2^2a_{\neq}^{-2}(t_1,t_2)\,dt_2:$$

$$b_2(1) = -2\sqrt{2}\pi e^{i\pi/4}\left(2 + \frac{i}{\sqrt{2}e^{i\pi/4} + 1 + e^{i\pi/2}}\right),$$
$$b_2(-1) = 2\sqrt{2}\pi e^{-i\pi/4}\left(2 - \frac{i}{\sqrt{2}e^{i\pi/4} + 1 + e^{i\pi/2}}\right).$$

$$b_3(t_2) = \int_{-\infty}^{+\infty} a_{\neq}^{-2}(t_1, t_2)\, dt_1 :$$

$$b_3(1) = 2\sqrt{2}\pi e^{i\pi/4}\left(-2i + \frac{1}{\sqrt{2}e^{i\pi/4} + 1 - e^{i\pi/2}}\right),$$
$$b_3(-1) = 2\sqrt{2}\pi e^{-i\pi/4}\left(-2i + \frac{1}{\sqrt{2}e^{i\pi/4} + 1 - e^{i\pi/2}}\right).$$

$$b_4(t_2) = \int_{-\infty}^{+\infty} t_1 a_{\neq}^{-2}(t_1, t_2)\, dt_1 :$$

$$b_4(1) = 2\sqrt{2}\pi e^{i\pi/4}\left(2 + \frac{i}{\sqrt{2}e^{-i\pi/4} + 1 - e^{i\pi/2}}\right),$$
$$b_4(-1) = 2\sqrt{2}\pi e^{-i\pi/4}\left(2 + \frac{i}{\sqrt{2}e^{-i\pi/4} + 1 - e^{i\pi/2}}\right).$$

$$b_5(t_2) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} t_1^2 a_{\neq}^{-2}(t_1, t_2)\, dt_1 :$$

$$b_5(1) = 2\sqrt{2}\pi e^{i\pi/4}\left(2i - \frac{1}{\sqrt{2}e^{-i\pi/4} + 1 - e^{i\pi/2}}\right),$$
$$b_5(-1) = 2\sqrt{2}\pi e^{-i\pi/4}\left(2i - \frac{1}{\sqrt{2}e^{-i\pi/4} + 1 - e^{i\pi/2}}\right).$$

The previous system takes the form:

$$\begin{cases} \int\limits_{-\infty}^{+\infty} t_1 a_{\neq}^{-2}(t_1, t_2)\,\tilde{c}_0(t_1)\,dt_1 + t_2 \int\limits_{-\infty}^{+\infty} t_1 a_{\neq}^{-2}(t_1, t_2)\,\tilde{c}_1(t_1)\,dt_1 \\ \qquad + \tilde{d}_0(t_2)\, b_4(t_2) + \tilde{d}_1(t_2)\, b_5^{-1}(t_2) = \tilde{g}_1(t_2), \\ \int\limits_{-\infty}^{+\infty} b_3^{-1}(t_2)\, a_{\neq}^{-2}(t_1, t_2)\,\tilde{c}_0(t_1)\,dt_1 + t_2 \int\limits_{-\infty}^{+\infty} b_3^{-1}(t_2)\, a_{\neq}^{-2}(t_1, t_2)\,\tilde{c}_1(t_1)\,dt_1 \\ \qquad + \tilde{d}_0(t_2) + \tilde{d}_1(t_2)\, b_3^{-1}(t_2)\, b_4(t_2) = \tilde{g}_2(t_2), \end{cases}$$

$$\begin{cases} \tilde{c}_0(t_1)\, b_1(t_1) + \tilde{c}_1(t_1)\, b_2(t_1) + \int\limits_{-\infty}^{+\infty} t_2 a_{\neq}^{-2}(t_1,t_2)\, \tilde{d}_0(t_2)\, dt_2 \\ \quad + t_1 \int\limits_{-\infty}^{+\infty} t_2 a_{\neq}^{-2}(t_1,t_2)\, \tilde{d}_1(t_2)\, dt_2 = \tilde{g}_3(t_1), \\ \tilde{c}_0(t_1) + \tilde{c}_1(t_1)\, b_1(t_1)\, b(t_1) + \int\limits_{-\infty}^{+\infty} b^{-1}(t_1)\, a_{\neq}^{-2}(t_1,t_2)\, \tilde{d}_0(t_2)\, dt_2 \\ \quad + t_1 \int\limits_{-\infty}^{+\infty} b^{-1}(t_1)\, a_{\neq}^{-2}(t_1,t_2)\, \tilde{d}_1(t_2)\, dt_2 = \tilde{g}_4(t_1), \end{cases}$$

and can be reduced as above to a system of (8×8)-system of linear integral equations, and further by Mellin transform to the (8×8)-system of linear difference equations.

4. One solvability case for a system of linear difference equations of first order

Let us consider a system of two linear difference equations of first order with two unknowns $c_1(\lambda)$, $c_2(\lambda)$:

$$\begin{cases} a_{11}(\lambda)\, c_1(\lambda) + a_{12}(\lambda)\, c_2(\lambda) \\ \quad - b_{11}(\lambda)\, c_1(\lambda-1) - b_{12}(\lambda-1)\, c_2(\lambda) = d_1(\lambda), \\ a_{21}(\lambda)\, c_1(\lambda) + a_{22}(\lambda)\, c_2(\lambda) \\ \quad - b_{21}(\lambda)\, c_1(\lambda-1) - b_{22}(\lambda)\, c_2(\lambda-1) = d_2(\lambda), \end{cases} \tag{14}$$

where $d_1(\lambda)$, $d_2(\lambda)$ are given functions.

Our main goal is to find the solution of the system (14).

For this purpose we rewrite the system in a matrix form:

$$A(\lambda)\, C(\lambda) - B(\lambda)\, C(\lambda-1) = D(\lambda), \tag{15}$$

where

$$A(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{pmatrix}, \quad B(\lambda) = \begin{pmatrix} b_{11}(\lambda) & b_{12}(\lambda) \\ b_{21}(\lambda) & b_{22}(\lambda) \end{pmatrix}$$

are matrices of order (2×2), and

$$C(\lambda) = \begin{pmatrix} c_1(\lambda) \\ c_2(\lambda) \end{pmatrix}, \quad D(\lambda) = \begin{pmatrix} d_1(\lambda) \\ d_2(\lambda) \end{pmatrix}$$

are vectors.

Multiply the quantity (15) by $A^{-1}(\lambda)$ from left. We obtain

$$A^{-1}(\lambda)\, A(\lambda)\, C(\lambda) - A^{-1}(\lambda)\, B(\lambda)\, C(\lambda-1) = A^{-1}(\lambda)\, D(\lambda)$$

or

$$C(\lambda) - G(\lambda)\, C(\lambda - 1) = R(\lambda),$$

where $G(\lambda) = A^{-1}(\lambda)\, B(\lambda)$ is matrix of order (2×2), $R(\lambda) = A^{-1}(\lambda)\, D(\lambda)$ is vector.

If we suppose that $G(\lambda)$ admits factorization

$$G(\lambda) = P_1^{-1}(\lambda)\, P_1(\lambda - 1), \tag{16}$$

then substituting (16) into last matrix equation we have:

$$C(\lambda) - P_1^{-1}(\lambda)\, P_1(\lambda - 1)\, C(\lambda - 1) = R(\lambda)$$

or

$$P_1(\lambda)\, C(\lambda) - P_1(\lambda - 1)\, C(\lambda - 1) = P_1(\lambda)\, R(\lambda).$$

It is easily seen the last expression is a system of linear difference equation of first order, and its solution will be in the following form (see [2] for details and notations)

$$P_1(\lambda)\, C(\lambda) = \overset{\lambda}{\underset{c}{\mathbf{S}}}\, P_1(z)\, R(z)\, \Delta z + \varpi,$$

where ϖ is periodic function with period 1.

Hence,

$$C(\lambda) = P_1^{-1}(\lambda) \overset{\lambda}{\underset{c}{\mathbf{S}}}\, P_1(z)\, R(z)\, \Delta z + \varpi = P_1^{-1}(\lambda) \overset{\lambda}{\underset{c}{\mathbf{S}}}\, P_1(z)\, A^{-1}(z)\, D(z)\, \Delta z + \varpi.$$

So, the problem is reduced to finding $P_1(\lambda)$.

Let us consider the equation (16), which we rewrite in the form:

$$\begin{cases} p_{11}(\lambda)\, g_{11}(\lambda) + p_{12}(\lambda)\, g_{21}(\lambda) = p_{11}(\lambda - 1), \\ p_{11}(\lambda)\, g_{12}(\lambda) + p_{12}(\lambda)\, g_{22}(\lambda) = p_{12}(\lambda - 1), \\ p_{21}(\lambda)\, g_{11}(\lambda) + p_{22}(\lambda)\, g_{21}(\lambda) = p_{21}(\lambda - 1), \\ p_{21}(\lambda)\, g_{12}(\lambda) + p_{22}(\lambda)\, g_{22}(\lambda) = p_{22}(\lambda - 1), \end{cases}$$

where $p_{11}(\lambda)$, $p_{12}(\lambda)$, $p_{21}(\lambda)$, $p_{22}(\lambda)$ are the unknown functions.

Such system decomposes into the two systems of linear difference equations of first order with variable coefficients $g_{11}(\lambda)$, $g_{12}(\lambda)$, $g_{21}(\lambda)$, $g_{22}(\lambda)$:

$$\begin{cases} p_{11}(\lambda)\, g_{11}(\lambda) + p_{12}(\lambda)\, g_{21}(\lambda) = p_{11}(\lambda - 1), \\ p_{11}(\lambda)\, g_{12}(\lambda) + p_{12}(\lambda)\, g_{22}(\lambda) = p_{12}(\lambda - 1), \end{cases} \tag{17}$$

$$\begin{cases} p_{21}(\lambda)\, g_{11}(\lambda) + p_{22}(\lambda)\, g_{21}(\lambda) = p_{21}(\lambda - 1), \\ p_{21}(\lambda)\, g_{12}(\lambda) + p_{22}(\lambda)\, g_{22}(\lambda) = p_{22}(\lambda - 1). \end{cases} \tag{18}$$

We will seek the solution of the system (17) in the form $p_{11}(\lambda) = \alpha_1 e^{r\lambda}$, $p_{12}(\lambda) = \alpha_2 e^{r\lambda}$.

Then our system can be rewritten in the form:

$$\begin{cases} g_{11}(\lambda)\,\alpha_1 e^{r\lambda} + g_{21}(\lambda)\,\alpha_2 e^{r\lambda} = \alpha_1 e^{r(\lambda-1)}, \\ g_{12}(\lambda)\,\alpha_1 e^{r\lambda} + g_{22}(\lambda)\,\alpha_2 e^{r\lambda} = \alpha_2 e^{r(\lambda-1)}, \end{cases}$$

or, in the other terms,

$$\begin{cases} \big(g_{11}(\lambda) - e^{-r}\big)\alpha_1 + g_{21}(\lambda)\,\alpha_2 = 0, \\ g_{12}(\lambda)\,\alpha_1 + \big(g_{22}(\lambda) - e^{-r}\big)\alpha_2 = 0. \end{cases}$$

The last system has a non-trivial solution in such case only, if its determinant is equal to zero. Let us compose the determinant and find the non-trivial solutions of the system.

$$\begin{vmatrix} g_{11}(\lambda) - e^{-r} & g_{21}(\lambda) \\ g_{12}(\lambda) & g_{22}(\lambda) - e^{-r} \end{vmatrix} = \det G - e^{-r} tr\, G + e^{-2r} = 0, \qquad (19)$$

$tr\, G$ is trace of the matrix G.

Changing variables $\delta = e^{-r}$, $a = tr\, G$, $b = \det G$, we obtain quadratic equation $\delta^2 - a\delta + b = 0$, which has always two solutions δ_1 and δ_2, and, hence, we have two values r, i.e., r_1 and r_2.

Take $\alpha_1 = 1$. Then the solutions of the systems (17) and (18) can be

$$p_{11}(\lambda) = e^{r_1\lambda}, \quad p_{12}(\lambda) = a e^{r_1\lambda}, \quad p_{21}(\lambda) = e^{r_2\lambda}, \quad p_{22}(\lambda) = b e^{r_2\lambda},$$

where r_1 and r_2 are solutions of our quadratic equation.

Theorem 1. *If the elements of the matrix G such that under certain r the determinant* (19) *is equal zero, then $G(\lambda)$ admits the factorization.*

The method for solving (2×2)-system of linear difference equations of first order mentioned above is valid for a system of three linear difference equations of first order with three unknowns.

Theorem 2. *If the elements of the matrix G, where G is corresponding matrix of order* (3×3) *such that under certain r the following equation*

$$\begin{aligned} &\det G - e^{-2r} tr\, G - e^{-3r} - e^{-r} \\ &\quad \times \big(g_{11}(\lambda)g_{33}(\lambda) + g_{22}(\lambda)g_{33}(\lambda) + g_{11}(\lambda)g_{22}(\lambda) - \\ &\qquad - g_{31}(\lambda)g_{13}(\lambda) - g_{23}(\lambda)g_{32}(\lambda) - g_{21}(\lambda)g_{12}(\lambda)\big) = 0 \qquad (20) \end{aligned}$$

is valid, then $G(\lambda)$ admits the factorization.

Concerning to $(n \times n)$-system of linear difference equations of first order, reasoning by the same way one can obtain an algebraic equation of order n of type (20) and formulate the following result.

Theorem 3. *If the factorization for the matrix G exists, then it is unique up to periodic function.*

The proof of Theorem 3 is easily obtained by taking logarithm of both sides of the equation (16) and reducing to the problem of type

$$R(\lambda) - R(\lambda - 1) = Q(\lambda)$$

with given right-hand side $Q(\lambda)$.

Such problem is solved by the formula [3]

$$R(\lambda) = \overset{\lambda}{\underset{c}{\mathbf{S}}} Q(z)\Delta z + \varpi,$$

where ϖ is periodic function with the period 1.

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