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INVERSE PROBLEM FOR AN ULTRAPARABOLIC EQUATION

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ABSTRACT. This article is devoted to the solvability of the inverse problem for linear ultraparabolic equation. The problem contains the unknown function in the boundary condition. The existence and the uniqueness of the solution for the mixed problem for linear ultraparabolic equation with the nonhomogeneous boundary conditions on the space variables are also obtained.

1. Introduction

In the theory of partial differential equations a problem in which the solution of the equation and some of the coefficients of the equation, or its right-hand side, or the initial data are unknown, is called an inverse problem. This problem contains the same conditions as a direct problem, and conditions related to the presence of additional unknown functions. These conditions are called overdetermination conditions and they can take an integral or final form.

The inverse problems for the equation of parabolic type are introduced in the work of A. Tikhonov in 1935 [14], where the problem of recovering the initial data of the Cauchy problem for the heat conduction equation with the final overdetermination is considered. In 1962 [4] using methods of the theory of integral equations and Schauder fixed point theorems for the first time the unknown coefficient that depends on time in parabolic equation was determined. Later, various methods of inverse problems for parabolic equations were developed in the works [3], [5], [9], [11] and others. In the work [10] the problem of recovering a kernel of the integrodifferential equation with the two time variables is studied and an application to an ultraparabolic integrodifferential equation is given.

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In this paper the inverse problem for a linear ultraparabolic equation is considered. The problem determines the unknown solution of the equation and unknown boundary condition with respect to the part of spatial variables. In order to find the unknown function in the boundary condition, the integral overdetermination condition is posed. Moreover, some classes of linear ultraparabolic equations for which the mixed problem with known nonhomogeneous conditions has a unique solution are found. For the wave equation an inverse problem of recovering of the boundary conditions is considered in [1]. To establish the solvability of the mixed boundary value problem for ultraparabolic equation we applied the Galerkin method. Note that in the works [6], [7], [12], [13] using this method the solvability of mixed problems for ultraparabolic equations in bounded or unbounded domains is studied.

2. Formulation of problems

Let $\Omega \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^l$ be bounded domains with boundaries $\partial\Omega$ and ∂D correspondingly; $T > 0$, $x \in \Omega$, $y \in D$, $t \in (0,T)$, $Q_\tau = \Omega \times D \times (0,\tau)$, $\tau \in (0,T]$, $G = \Omega \times D$, $\Pi_T = D \times (0,T)$.

Denote $\Sigma_T = \partial\Omega \times D \times (0,T)$, $S_T = \Omega \times \partial D \times (0,T)$, ν_1, ν_2—the outward unit normal vectors to the surfaces S_T and Σ_T correspondingly,

$$S_T^1 = \left\{ (x,y,t) \in S_T : \sum_{i=1}^{l} \lambda_i(x,y,t)\cos(\nu_1, y_i) < 0 \right\},$$

$$S_T^2 = \left\{ (x,y,t) \in S_T : \sum_{i=1}^{l} \lambda_i(x,y,t)\cos(\nu_1, y_i) \geqslant 0 \right\}.$$

In the domain Q_T we consider the problem

$$u_t + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} - \sum_{i,j=1}^{n} \big(a_{ij}(x,y,t) u_{x_i}\big)_{x_j} + c(x,y,t)u = f(x,y,t); \tag{1}$$

$$\frac{\partial u}{\partial \nu}\Big|_{\Sigma_T} = p(t)h(x,y,t), \quad (x,y,t) \in \Sigma_T; \qquad u|_{S_T^1} = 0; \tag{2}$$

$$u(x,y,0) = u_0(x,y), \qquad (x,y) \in G; \tag{3}$$

$$\int_G K(x,y)u(x,y,t)\,dx\,dy = E(t), \qquad t \in [0,T], \tag{4}$$

where

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x,y,t) u_{x_i} \cos(\nu_2, x_j).$$

Our task is to find a pair of functions (u,p) that satisfies the equation (1), the boundary conditions (2), the initial condition (3) and the overdetermination condition (4).

Let us introduce the following spaces:

- $L^\infty(Q_T) :=$
 $\{w : Q_T \to \mathbb{R}|\ w$ is measurable and there is a constant C such that $|w(x,y,t)| \leqslant C$ a.e. on $Q_T\}$ with $\|w; L^\infty(Q_T)\| = \inf\{C : |w(x,y,t)| \leqslant C$ a.e. on $Q_T\}$;
- $L^2(G) :=$
 $\{w\colon G \to \mathbb{R}| \int_G w^2\,dx\,dy < \infty\}$ with $\|w; L^2(G)\| = \left(\int_G |w(x,y)|^2\,dx\,dy\right)^{\frac{1}{2}}$;
- $L^2(Q_T) :=$
 $\{w : Q_T \to \mathbb{R}| \int_{Q_T} w^2\,dx\,dy\,dt < \infty\}$ with $\|w; L^2(Q_T)\| =$
 $= \left(\int_{Q_T} |w(x,y,t)|^2\,dx\,dy\,dt\right)^{\frac{1}{2}}$;
- $W^{1,2}(Q_T) :=$
 $\{w : Q_T \to \mathbb{R}|\ w, w_{x_i}, w_{y_j}, w_t \in L^2(Q_T), i = 1,\dots,n, j = 1,\dots,l\}$ with $\|w; W^{1,2}(Q_T)\| =$
 $= \|w; L^2(Q_T)\| + \sum_{i=1}^{l} \|w_{y_i}; L^2(Q_T)\| + \sum_{i=1}^{n} \|w_{x_i}; L^2(Q_T)\| + \|w_t; L^2(Q_T)\|$;
- $C^k(O)$—the space of all k-times continuously differentiable functions on O;
- $V\big(0,T; W(G)\big) :=$
 $\{w : [0,T] \to W(G)|\ \|w(\cdot,\cdot,t); W(G)\| \in V(0,T)\}$ (where V, W are Banach spaces);
- $V\big(D; W(\Omega)\big) :=$
 $\{w : D \to W(\Omega)|\ \|w(\cdot,y); W(\Omega)\| \in V(D)\}$;
- $V\big(\Pi_T; W(\Omega)\big) :=$
 $\{w : \Pi_T \to W(\Omega)|\ \|w(\cdot,y,t); W(\Omega)\| \in V(\Pi_T)\}$;
- $V_1(Q_T) :=$
 $\{w : Q_T \to \mathbb{R}|\ w, w_{x_i}, w_t \in L^2(Q_T),\ i = 1,\dots,n\}$ with $\|w; V_1(Q_T)\| =$
 $= \|w; L^2(Q_T)\| + \sum_{i=1}^{n} \|w_{x_i}; L^2(Q_T)\| + \|w_t; L^2(Q_T)\|$;
- $V_2(Q_T) :=$
 $\{w : Q_T \to \mathbb{R}|\ w,\ w_{x_i} \in L^2(Q_T), i = 1,\dots,n\}$ with $\|w; V_2(Q_T)\| =$
 $= \|w; L^2(Q_T)\| + \sum_{i=1}^{n} \|w_{x_i}; L^2(Q_T)\|$.

Let us assume that such condition **(S)** holds; there exists $\Gamma_1 \subset \partial D \subset \mathbb{R}^{l-1}$ such that mes $\Gamma_1 > 0$ and the surface $S_T^1 = \Omega \times \Gamma_1 \times (0,T)$. We also consider functions $a_{ij},\ c,\ \lambda_i,\ f, K, u_0, h, E$ that satisfy the following hypotheses:

(A): $a_{ij} \in C(\overline{Q}_T)$, $a_{ij} = a_{ji}$, $\sum_{i,j=1}^{n} a_{ij}(x,y,t)\xi_i\xi_j \geqslant a_0|\xi|^2$
for all $(x,y,t) \in Q_T$, $a_0 > 0$;

(C): $c \in L^\infty(Q_T)$, $c(x,y,t) \geqslant c_0$, a.e. on Q_T, c_0 is constant;

(F): $f \in L^2(Q_T)$;

(U): $u_0 \in L^2(G)$;

(L): $\lambda_i \in L^\infty\big(0,T;C(\overline{G})\big)$, $\lambda_{iy_i} \in L^\infty(Q_T)$ for all $i = 1,\ldots,l$;

(K): $K \in C^1\big(D;C^2(\overline{\Omega})\big)$, $\frac{\partial K}{\partial \nu}\Big|_{\partial\Omega\times D} = 0$, $K|_{\Omega\times\Gamma_2} = 0$;

(H): $h \in L^\infty\big(0,T;L^2(D;W^{1,2}(\Omega))\big)$;

(E): $E \in C^2\big([0,T]\big)$.

We differentiate the overdetermination condition (4) on t and substitute therein u_t from the equation (1)

$$\int_G K(x,y)\Big(f(x,y,t) - \sum_{i=1}^{l} \lambda_i(x,y,t)u_{y_i} + \\ + \sum_{i,j=1}^{n} (a_{ij}(x,y,t)u_{x_i})_{x_j} - c(x,y,t)u\Big)\,dx\,dy = E'(t), \quad t \in [0,T]. \quad (5)$$

Then we integrate by parts in the second and the third term of (5) and express $p(t)$ from the result

$$p(t) = \left[\int_D \int_{\partial\Omega} K(x,y)h(x,y,t)\,d\sigma\,dy\right]^{-1} \\ \times \Bigg(E'(t) - \int_G \Big(\Big\{\sum_{i=1}^{l}\big(\lambda_i(x,y,t)K(x,y)\big)_{y_i} + \sum_{i,j=1}^{n}\big(K_{x_j}(x,y)a_{ij}(x,y,t)\big)_{x_i} - \\ - K(x,y)c(x,y,t)\Big\}u + K(x,y)f(x,y,t)\Big)\,dx\,dy\Bigg), \qquad t \in [0,T]. \quad (6)$$

Denote $G_\eta = \big\{(x,y,t)|(x,y) \in G, t = \eta\big\}$, $\eta \in [0,T]$.

Definition 1. A pair of functions (u,p) is a weak solution to the problem (1)–(4) if $u \in L^2\big(0,T;W^{1,2}(G)\big) \cap C\big([0,T];L^2(G)\big)$, $u|_{S_T^1} = 0$, $p \in L^2(0,T)$, $p(0) = 0$ and (u,p) satisfies the equality

$$\int_{G_T} uv\,dx\,dy + \int_{Q_T}\Big[-uv_t + \sum_{i=1}^{l}\lambda_i(x,y,t)u_{y_i}v + \\ + \sum_{i,j=1}^{n} a_{ij}(x,y,t)u_{x_i}v_{x_j} + c(x,y,t)uv - f(x,y,t)v\Big]\,dx\,dy\,dt \\ = \int_{\Sigma_T} p(t)h(x,y,t)v\,d\sigma\,dy\,dt + \int_{G_0} u_0(x,y)v\,dx\,dy \quad (7)$$

for all functions $v \in V_1(Q_T)$ and the formula (6) holds.

Given a function $p^* \in W^{1,2}(0,T)$ such that $p^*(0) = 0$, consider the mixed problem for the equation (1) with the initial condition (3) and the boundary conditions

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Sigma_T} = p^*(t)h(x,y,t), \quad (x,y,t) \in \Sigma_T; \quad u|_{S_T^1} = 0. \tag{8}$$

Definition 2. A function $u^* \in W^{1,2}(Q_T)$, $u^*|_{S_T^1} = 0$ is a solution to the problem (1), (3), (8) if it satisfies the equality

$$\int\limits_{Q_T} \left[u_t^* v + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i}^* v + \sum_{i,j=1}^{n} a_{ij}(x,y,t) u_{x_i}^* v_{x_j} + c(x,y,t) u^* v \right] dx\, dy\, dt$$
$$= \int\limits_{Q_T} f(x,y,t) v\, dx\, dy\, dt + \int\limits_{\Sigma_T} p^*(t) h(x,y,t) v\, d\sigma\, dy\, dt \tag{9}$$

for all functions $v \in V_2(Q_T)$ and the initial condition (3) holds.

3. Existence and uniqueness of solution for direct problem

Let us prove the existence and the uniqueness of the solution to the problem (1), (3), (8).

Theorem 1. *Suppose that the hypotheses* **(A)**, **(C)**, **(L)**, **(U)**, **(H)**, **(S)**, **(F)** *and the conditions* a_{ijt}, a_{ijy_s}, $c_{y_s} \in L^\infty(Q_T)$, $f_{y_s} \in L^2(Q_T)$, $f|_{S_T^1} = 0$, h_{y_s}, $h_t \in L^2\big(\Pi_T; W^{1,2}(\Omega)\big)$, $i,j = 1,\dots,n$, $s = 1,\dots,l$, $u_0 \in W^{1,2}(G)$, $\frac{\partial u_0}{\partial \nu}|_{\partial\Omega\times D} = 0$, $u_0|_{\Omega\times\Gamma_1} = 0$, $h|_{\partial\Omega\times\Gamma_1\times(0,T)} = 0$, $p^* \in W^{1,2}(0,T)$, $p^*(0) = 0$ *are fulfilled. Then a solution to the problem* (1), (3), (8) *exists.*

P r o o f.
We shall use the Galerkin method. Let $\{\varphi^k\}_{k=1}^\infty$ be an arbitrary orthogonal fundamental system of the space $W^{1,2}(\Omega)$, which is orthonormal in the space $L^2(\Omega)$, $\{\psi^s\}_{s=1}^\infty$ be the orthogonal basis of the space $\{v : v \in H^1(D),\, v|_{\Gamma_1} = 0\}$, orthonormal in the space $L^2(D)$, where ψ^s ($s \geqslant 1$) are eigenfunctions of the problem

$$\Delta_y u = \mu u, \quad u|_{\Gamma_1} = 0, \quad \left.\frac{\partial u}{\partial \nu_1}\right|_{\Gamma_2} = 0, \tag{10}$$

that correspond to the eigenvalues μ_s. Here

$$\Delta_y = \sum_{j=1}^{l} \frac{\partial^2}{\partial y_j^2}, \qquad \Gamma_2 = \partial D \backslash \Gamma_1.$$

Let us consider the functions

$$u^{*,N}(x,y,t)=\sum_{k,s=1}^{N} c^N_{k,s}(t)\varphi^k(x)\psi^s(y), \qquad N\in\mathbb{N},$$

where $c^N_{k,s}(t)$, $k,s=1,\dots,N$, are the solutions to the problem

$$\begin{aligned}&\int_G L\big(u^{*,N},\varphi^k\psi^s\big)\,dx\,dy\\ &:=\int_G \Big[u^{*,N}_t\varphi^k(x)\psi^s(y)+\sum_{i=1}^{l}\lambda_i(x,y,t)u^{*,N}_{y_i}\varphi^k(x)\psi^s(y)+\\ &\quad+\sum_{i,j=1}^{n}a_{ij}(x,y,t)u^{*,N}_{x_i}\varphi^k_{x_j}(x)\psi^s(y)+c(x,y,t)u^{*,N}\varphi^k(x)\psi^s(y)-\\ &\qquad\qquad -f(x,y,t)\varphi^k(x)\psi^s(y)\Big]dx\,dy\\ &=\int_D\int_{\partial\Omega}p^*(t)h(x,y,t)\varphi^k(x)\psi^s(y)\,d\sigma\,dy,\end{aligned}\tag{11}$$

$$c^N_{k,s}(0)=u^N_{0,k,s};\quad u^N_0(x,y)=\sum_{k,s=1}^{N}u^N_{0,k,s}\varphi^k(x)\psi^s(y),$$
$$\lim_{N\to\infty}\|u^N_0-u_0\|_{W^{1,2}(G)}=0.\tag{12}$$

Note that (11), (12) is the Cauchy problem for the system of ordinary differential equations of the first order on the vector of functions $\vec{c}_N(t):=\big(c^N_{1,1}(t), c^N_{2,1}(t),\ \dots,\ c^N_{N,1}(t), c^N_{1,2}(t), c^N_{2,2}(t),\dots,\ c^N_{N,2}(t),\dots,c^N_{1,N}(t),c^N_{2,N}(t),\dots,c^N_{N,N}(t)\big)$. Due to the conditions of Theorem 1 on the coefficients of the system (11) and to the Carathéodory theorem [4, p. 54] there exists a unique absolutely continuous solution on $[0,\tau_0]$, $\tau_0<T$ to the problem (11), (12). It is easy to prove that

$$\tau_0=T.$$

After multiplying (11) by $c^N_{k,s}(t)e^{-\alpha t}$, where the constant $\alpha>0$, summing up with respect to s and k, integrating with respect to t over the interval $[0,\tau]$, we obtain

$$\int_{Q_\tau}L\big(u^{*,N},u^{*,N}\big)e^{-\alpha t}\,dx\,dy\,dt=\int_{\Sigma_\tau}p^*(t)h(x,y,t)u^{*,N}e^{-\alpha t}\,d\sigma\,dy\,dt.\tag{13}$$

We estimate the right-hand side in (13) using the estimate [8, p. 77, formula 6.24]

$$\int_{\partial\Omega}v^2\,d\sigma\leqslant\int_\Omega\big[\varepsilon|\nabla v|^2+C(\varepsilon)v^2\big]\,dx\tag{14}$$

for an arbitrary $\varepsilon > 0$, and the constant $C(\varepsilon)$ depending on ε and not depending on v

$$\mathcal{I}_1 = \int\limits_{\Sigma_\tau} p^*(t)h(x,y,t)u^{*,N} e^{-\alpha t}\, d\sigma\, dy\, dt \leqslant \frac{1}{2}\int\limits_{\Sigma_\tau} \big(p^*(t)h(x,y,t)\big)^2 e^{-\alpha t}\, d\sigma\, dy\, dt$$
$$+ \frac{1}{2}\int\limits_{Q_\tau}\left[\varepsilon\sum_{i=1}^{n}\big(u^{*,N}_{x_i}\big)^2 + C(\varepsilon)\big(u^{*,N}\big)^2\right] e^{-\alpha t}\, dx\, dy\, dt.$$

Integrating by parts in the first and second term of (13) and taking into account $\mathcal{I}_1$ we obtain the estimate

$$\int\limits_{G_\tau}\big(u^{*,N}\big)^2 e^{-\alpha\tau}\, dx\, dy$$
$$+\int\limits_{Q_\tau}\left[\big(\alpha - \lambda^1 l + 2c_0 - 1 - C(\varepsilon)\big)\big(u^{*,N}\big)^2 + (2a_0 - \varepsilon)\sum_{i=1}^{n}\big(u^{*,N}_{x_i}\big)^2\right] e^{-\alpha t}\, dx\, dy\, dt$$
$$+\int\limits_{S^2_\tau}\sum_{i=1}^{l}\lambda_i\big(u^{*,N}\big)^2\cos(\nu_1, y_i) e^{-\alpha t}\, dx\, d\sigma_1\, dt$$
$$\leqslant \int\limits_{G_0}\big(u_0^N\big)^2\, dx\, dy + \int\limits_{Q_T}(f)^2 e^{-\alpha t}\, dx\, dy\, dt + \int\limits_{\Sigma_\tau}\big(p^*(t)h(x,y,t)\big)^2 e^{-\alpha t}\, d\sigma\, dy\, dt, \quad (15)$$

where

$$S^2_\tau = \Omega\times\Gamma_2\times(0,\tau), \qquad \lambda^1 = \max_i \operatorname*{ess\,sup}_{Q_T}|\lambda_{y_i}(x,y,t)|.$$

Choosing ε and α from the inequalities $2a_0 > \varepsilon$ and $\alpha > \lambda^1 l - 2c_0 + 1 + C(\varepsilon)$, from (15) we derive

$$\int\limits_{G_\tau}\big(u^{*,N}\big)^2\, dx\, dy + \int\limits_{Q_\tau}\left[\big(u^{*,N}\big)^2 + \sum_{i=1}^{n}\big(u^{*,N}_{x_i}\big)^2\right] dx\, dy\, dt$$
$$\leqslant M_1\left(\int\limits_{G_0}\big(u_0^N\big)^2\, dx\, dy + \int\limits_{Q_T}(f)^2\, dx\, dy\, dt + \int\limits_{\Sigma_T}\big(p^*(t)h(x,y,t)\big)^2\, d\sigma\, dy\, dt\right), \quad (16)$$

where the constant M_1 does not depend on N. Multiply (11) by the eigenvalue of the problem (10) μ_s and by $c^N_{k,s}(t)e^{-\alpha t}$, sum it up with respect to k and s and replace the term $\mu_s u^{*,N}$ on $\Delta_y u^{*,N}$, we have

$$\int\limits_{Q_\tau} L\left(u^{*,N}, \sum_{k=1}^{l} u^{*,N}_{y_k y_k}\right) e^{-\alpha t} dx\, dy\, dt = \int\limits_{\Sigma_\tau} p^*(t)h(x,y,t)\sum_{k=1}^{l} u^{*,N}_{y_k y_k} e^{-\alpha t}\, d\sigma\, dy\, dt. \quad (17)$$

Since $h|_{\partial\Omega\times\Gamma_1\times(0,T)} = 0$ and $u^{*,N}_{y_k}|_{S^2_T} = 0$, we obtain

$$\begin{aligned}
\mathcal{I}_2 &= \int\limits_{\Sigma_\tau} p^*(t)h(x,y,t)\sum_{k=1}^{l} u^{*,N}_{y_k y_k} e^{-\alpha t}\, d\sigma\, dy\, dt \\
&= -\int\limits_{\Sigma_\tau} p^*(t)\sum_{k=1}^{l} h_{y_k}(x,y,t) u^{*,N}_{y_k} e^{-\alpha t}\, d\sigma\, dy\, dt \\
&\leqslant \frac{1}{2}\int\limits_{\Sigma_\tau}\sum_{k=1}^{l}\big(p^*(t)h_{y_k}(x,y,t)\big)^2 e^{-\alpha t}\, d\sigma\, dy\, dt \\
&\quad + \frac{1}{2}\int\limits_{Q_\tau}\left[\varepsilon\sum_{k=1}^{l}\sum_{i=1}^{n}\big(u^{*,N}_{y_k x_i}\big)^2 + C(\varepsilon)\sum_{k=1}^{l}\big(u^{*,N}_{y_k}\big)^2\right] e^{-\alpha t}\, dx\, dy\, dt.
\end{aligned}$$

We estimate the left-hand side from (17) similarly to [6]. Then using $\mathcal{I}_2$, from (17) we obtain the estimate

$$\begin{aligned}
&\int\limits_{G_\tau}\sum_{k=1}^{l}\big(u^{*,N}_{y_k}\big)^2 e^{-\alpha\tau}\, dx\, dy \\
&\quad + \int\limits_{Q_\tau}\Bigg[\sum_{k=1}^{l}\big(\alpha-\lambda^1 l + 2c_0 - 2 - C(\varepsilon)\big)\big(u^{*,N}_{y_k}\big)^2 + \\
&\qquad\qquad + \sum_{k=1}^{l}(2a_0-\delta-\varepsilon)\sum_{i=1}^{n}\big|u^{*,N}_{y_k x_i}\big|^2\Bigg] e^{-\alpha t}\, dx\, dy\, dt \\
&\quad - \int\limits_{S^1_\tau}\sum_{i=1}^{l}\lambda_i\big(u^{*,N}_{y_i}\big)^2\cos(\nu_1, y_i) e^{-\alpha t}\, dx\, d\sigma_1\, dt \\
&\leqslant \int\limits_{G_0}\sum_{k=1}^{l}\big(u^N_{0y_k}\big)^2\, dx\, dy \\
&\quad + \int\limits_{Q_\tau}\left[\sum_{k=1}^{l}(f_{y_k})^2 + (c_1)^2\big(u^{*,N}\big)^2 + \frac{l(a_1)^2}{\delta}\sum_{i=1}^{n}\big(u^{*,N}_{x_i}\big)^2\right] e^{-\alpha t}\, dx\, dy\, dt \\
&\quad + \int\limits_{\Sigma_\tau}\sum_{k=1}^{l}\big(p^*(t)h_{y_k}(x,y,t)\big)^2 e^{-\alpha t}\, d\sigma\, dy\, dt,
\end{aligned} \tag{18}$$

where

$$a_1 = \max_{i,j,k}\operatorname*{ess\,sup}_{Q_T}|a_{ijy_k}|, \quad c_1 = \max_{k}\operatorname*{ess\,sup}_{Q_T}|c_{y_k}|, \quad \delta > 0\,.$$

Choose $\varepsilon, \delta, \alpha$ from the inequalities

$$2a_0 > \delta + \varepsilon, \qquad \alpha > \lambda^1 l - 2c_0 + 2 + C(\varepsilon).$$

Using (16), we derive the estimate

$$\begin{aligned}
&\int\limits_{G_\tau} \sum_{k=1}^{l} \big(u^{*,N}_{y_k}\big)^2 \, dx\, dy + \int\limits_{Q_\tau} \left[\sum_{k=1}^{l} \big(u^{*,N}_{y_k}\big)^2 + \sum_{k=1}^{l} \sum_{i=1}^{n} \big|u^{*,N}_{y_k x_i}\big|^2 \right] dx\, dy\, dt \\
&\leqslant M_2 \left[\int\limits_{G_0} \left(\sum_{k=1}^{l} \big(u^N_{0y_k}\big)^2 + \big(u^N_0\big)^2 \right) dx\, dy + \int\limits_{Q_T} \left(\sum_{k=1}^{l} (f_{y_k})^2 + (f)^2 \right) dx\, dy\, dt + \right. \\
&\qquad \left. + \int\limits_{\Sigma_T} \sum_{k=1}^{l} \big(p^*(t) h_{y_k}\big(x,y,t\big)\big)^2 \, d\sigma\, dy\, dt + \int\limits_{\Sigma_T} \big(p^*(t) h\big(x,y,t\big)\big)^2 \, d\sigma\, dy\, dt \right], \qquad (19)
\end{aligned}$$

where the constant M_2 does not depend on N. Multiplying (11) by $c^N_{k,st}(t)$, summing up with respect to s and k and integrating with respect to t from 0 to τ we have the equality

$$\int\limits_{Q_\tau} L\big(u^{*,N}, u^{*,N}_t\big) \, dx\, dy\, dt = \int\limits_{\Sigma_\tau} p^*(t) h(x,y,t) u^{*,N}_t \, d\sigma\, dy\, dt. \qquad (20)$$

We estimate the right-hand side in (20) by using the estimate (14):

$$\begin{aligned}
\mathcal{I}_3 &= \int\limits_{\Sigma_\tau} p^*(t) h(x,y,t) u^{*,N}_t \, d\sigma\, dy\, dt \\
&\leqslant \int\limits_{\partial\Omega \times D} \big(p^*(\tau)\big)^2 \big(h(x,y,\tau)\big)^2 d\sigma\, dy \\
&\quad + \int\limits_{\Sigma_\tau} \Big(\big(p^*(t) h(x,y,t)\big)_t\Big)^2 d\sigma\, dy\, dt \\
&\quad + \int\limits_{G_\tau} \left[C(\varepsilon) \big(u^{*,N}\big)^2 + \varepsilon \sum_{i=1}^{n} \big(u^{*,N}_{x_i}\big)^2 \right] dx\, dy \\
&\quad + \int\limits_{Q_\tau} \left[C(\varepsilon) \big(u^{*,N}\big)^2 + \varepsilon \sum_{i=1}^{n} \big(u^{*,N}_{x_i}\big)^2 \right] dx\, dy\, dt.
\end{aligned}$$

According to $\mathcal{I}_3$, from (20) we get

$$\int\limits_{Q_\tau}\left(u_t^{*,N}\right)^2 dx\,dy\,dt \;+\; (a_0-2\varepsilon)\int\limits_{G_\tau}\sum_{i=1}^{n}\left|u_{x_i}^{*,N}\right|^2 dx\,dy \;\leqslant\; 3\int\limits_{Q_\tau} f^2\,dx\,dy\,dt$$
$$+\int\limits_{Q_\tau}\left[\left((a_3)^2+2\varepsilon\right)\sum_{i=1}^{n}\left|u_{x_i}^{*,N}\right|^2+3\lambda^1 l\sum_{i=1}^{l}\left(u_{y_i}^{*,N}\right)^2+\left(3c_2+2C(\varepsilon)\right)\left(u^{*,N}\right)^2\right]dx\,dy\,dt$$
$$+\,2\int\limits_{\partial\Omega\times D}\left(p^*(\tau)\right)^2\left(h(x,y,\tau)\right)^2 d\sigma\,dy+2\int\limits_{\Sigma_\tau}\left(\left(p^*(t)h(x,y,t)\right)_t\right)^2 d\sigma dy\,dt$$
$$+\,2\int\limits_{G_\tau}C(\varepsilon)\left(u^{*,N}\right)^2 d\sigma\,dy+a_2\int\limits_{G_0}\sum_{i=1}^{n}\left(u_{0x_i}^{*,N}\right)^2 d\sigma\,dy, \tag{21}$$

where

$$a_2=\max_{ij}\operatorname*{ess\,sup}_{G_0}|a_{ij}|,\quad c_2=\operatorname*{ess\,sup}_{Q_T}|c|,\quad a_3=\max_{ij}\operatorname*{ess\,sup}_{Q_T}|a_{ijt}|.$$

From (21) there follows the estimate

$$\int\limits_{Q_\tau}\left(u_t^{*,N}\right)^2 dx\,dy\,dt+\int\limits_{G_\tau}\sum_{i=1}^{n}\left|u_{x_i}^{*,N}\right|^2 dx\,dy\leqslant M_3, \tag{22}$$

where the constant M_3 does not depend on N. From estimates (16), (19), (22) there follows the existence of some subsequence $\left\{u^{*,N_k}\right\}_{N_k=1}^{\infty}$ of the sequence $\left\{u^{*,N}\right\}_{N=1}^{\infty}$ such that $u^{*,N_k}\to u^*$ weakly in $W^{1,2}(Q_T)$ as $N_k\to\infty$. Then similarly to [8] we prove that u^* is the solution to the problem (1), (3), (8). $\square$

Theorem 2. *Under the hypotheses* **(A)**, **(C)**, **(L)** *the problem* (1), (3), (8) *cannot have more than one solution.*

P r o o f. Assume that there exist two solutions $u^{*,1}$, $u^{*,2}$ to the problem (1), (3), (8). Their difference $u^{*,1,2}=u^{*,1}-u^{*,2}$ satisfies the equality

$$\int\limits_{Q_T}\left[u_t^{*,1,2}v+\sum_{i=1}^{l}\lambda_i(x,y,t)u_{y_i}^{*,1,2}v\;+\right.$$
$$\left.+\sum_{i,j=1}^{n}a_{ij}(x,y,t)u_{x_i}^{*,1,2}v_{x_j}+c(x,y,t)u^{*,1,2}v\right]dx\,dy\,dt=0$$

for all functions $v\in V_2(Q_T)$ and $u^{*,1,2}(x,y,0)=0$.

Choose here $v = u^{*,1,2}e^{-\alpha t}, \alpha = \lambda^1 l - 2c_0 + 1$. Then similarly to (15) we find

$$\int\limits_{G_T} \big(u^{*,1,2}\big)^2 e^{-\alpha\tau}\,dx\,dy + \int\limits_{Q_T} \left[\big(\alpha - \lambda^1 l + 2c_0\big)\big(u^{*,1,2}\big)^2 + 2a_0 \sum_{i=1}^{n} \big|u^{*,1,2}_{x_i}\big|^2 \right] e^{-\alpha t}\,dx\,dy\,dt + \int\limits_{S_T^2} \sum_{i=1}^{l} \lambda_i \big(u^{*,1,2}\big)^2 \cos(\nu_1, y_i) e^{-\alpha t}\,dx\,d\sigma_1\,dt \leqslant 0.$$

From here we conclude $u^{*,1,2} := 0$, $u^{*,1} = u^{*,2}$. □

Remark 1. If u^* is the solution to the problem (1), (3), (8), then

$$\sum_{i,j=1}^{n} \big(a_{ij}(x,y,t)u^*_{x_i}\big)_{x_j} \in L^2(Q_T).$$

Remark 2. If u^* is the solution to the problem (1), (3), (8), then

$$\int\limits_{G_\tau} u^* v\,dx\,dy + \int\limits_{Q_\tau} \left[-u^* v_t + \sum_{i=1}^{l} \lambda_i(x,y,t)u^*_{y_i} v + \sum_{i,j=1}^{n} a_{ij}(x,y,t)u^*_{x_i} v_{x_j} + c(x,y,t)u^* v - f(x,y,t)v \right] dx\,dy\,dt = \int\limits_{\Sigma_\tau} p^*(t)h(x,y,t)v\,d\sigma dy\,dt + \int\limits_{G_0} u_0(x,y)v\,dx\,dy$$

$$\text{for all } v \in V_1(Q_T) \quad \text{and} \quad \tau \in (0;T].$$

4. Existence and uniqueness of solution for the inverse problem

Theorem 3. *Let the conditions* **(A)**, **(C)**, **(L)**, **(U)**, **(H)**, **(E)**, **(K)**, **(F)**, **(S)**, $a_{ij} \in W^{1,\infty}(Q_T)$, $a_{ijx_it}, c_{y_s}, c_t \in L^\infty(Q_T)$, $f_{y_s}, f_t \in L^2(Q_T)$, $f|_{S_T^1} = 0$, λ_{st}, $\lambda_{sy_st} \in L^2(Q_T)$, $h_{y_s}, h_t \in L^2\big(\Pi_T; W^{1,2}(\Omega)\big)$, $i,j = 1,\dots,n$, $s = 1,\dots,l$, $u_0 \in W^{1,2}(G)$, $\frac{\partial u_0}{\partial \nu}|_{\partial\Omega\times D} = 0$, $u_0|_{\Omega\times\Gamma_1} = 0$, $h|_{\partial\Omega\times\Gamma_1\times(0,T)} = 0$ *hold and, additionally,* $\int_D \int_{\partial\Omega} K(x,y)h(x,y,t)\,d\sigma\,dy \neq 0$,

$$E'(0) = \int_G \Bigg(K(x,y)f(x,y,0) + \Bigg(\sum_{i=1}^{l} \big(\lambda_i(x,y,0)K(x,y)\big)_{y_i} + \\ + \sum_{i,j=1}^{n} \big(K_{x_j}(x,y)a_{ij}(x,y,0)\big)_{x_i} - K(x,y)c(x,y,0)\Bigg) u_0(x,y)\Bigg)\, dx\, dy. \quad (23)$$

Then a weak solution to the problem (1)–(4) *exists.*

P r o o f. Let $m \in \mathbb{N}$. Now we construct the approximation of the solution to the problem (1)–(4) (u^m, p^m) (similarly to [1]) in such way: $p^1(t) := 0$, u^m satisfies the equality

$$\int_{G_\tau} u^m v\, dx\, dy \\ + \int_{Q_\tau} \Bigg[-u^m v_t + \sum_{i=1}^{l} \lambda_i(x,y,t) u^m_{y_i} v + \sum_{i,j=1}^{n} a_{ij}(x,y,t) u^m_{x_i} v_{x_j} + \\ + \; c(x,y,t)u^m v - f(x,y,t)v \Bigg]\, dx\, dy\, dt \\ = \int_{\Sigma_\tau} p^m(t)h(x,y,t)v\, d\sigma\, dy\, dt + \int_{G_0} u_0(x,y)v\, dx\, dy, \quad (24)$$

for all $v \in V_1(Q_T)$, $\tau \in (0;T]$, $m \geqslant 1$,

$$u^m(x,y,0) = u_0(x,y), \qquad (x,y) \in G; \quad (25)$$

$$p^m(t) = \Bigg[\int_D \int_{\partial\Omega} K(x,y)h(x,y,t)\, d\sigma\, dy \Bigg]^{-1} \\ \times \Bigg(E'(t) + \int_G \Bigg(-K(x,y)f(x,y,t) - \sum_{i=1}^{l} \big(\lambda_i(x,y,t)K(x,y)\big)_{y_i} u^{m-1} - \\ - \sum_{i,j=1}^{n} \big(K_{x_j}(x,y)a_{ij}(x,y,t)\big)_{x_i} u^{m-1} + K(x,y)c(x,y,t)u^{m-1} \Bigg) dx\, dy \Bigg), \\ t \in [0,T], \quad m \geqslant 2. \quad (26)$$

Under the conditions on the coefficients of the equation (1) of Theorem 3 $p^m \in W^{1,2}(0,T)$. Moreover, because of equality (23) $p^m(0) = 0$, $m \geqslant 1$.

Note that for each $m \in \mathbb{N}$ there exists a unique function $u^m \in W^{1,2}(Q_T)$ which satisfies (24) with the initial condition (25) if p^m is known (see Theorem 1,2). The sequence (u^m, p^m), $m \in \mathbb{N}$, is constructed. Now we show that this sequence converges to the solution of the problem (1)–(4). Consider the differences for $m \geqslant 2$,

$$z^m(x,y,t) = u^m(x,y,t) - u^{m-1}(x,y,t), \qquad r^m(t) = p^m(t) - p^{m-1}(t).$$

Using (24), we find

$$\int_{G_\tau} z^m v\,dx\,dy + \int_{Q_\tau} \Bigg[-z^m v_t + \sum_{i=1}^{l} \lambda_i(x,y,t) z^m_{y_i} v + \\ + \sum_{i,j=1}^{n} a_{ij}(x,y,t) z^m_{x_i} v_{x_j} + c(x,y,t) z^m v \Bigg] dx\,dy\,dt \\ = \int_{\Sigma_\tau} r^m(t) h(x,y,t) v\,d\sigma\,dy\,dt, \qquad m \geqslant 2 \quad (27)$$

for all functions

$$v \in V_1(Q_T) \quad \text{and} \quad \tau \in (0,T] \quad \text{and, besides,} \quad z^m(x,y,0) = 0.$$

Setting in (27)

$$v(x,y,t) = z^m(x,y,t) e^{-\alpha t}, \qquad \alpha \geqslant 0,$$

we obtain the equality

$$\int_{G_\tau} (z^m)^2 e^{-\alpha\tau}\,dx\,dy + \int_{S^2_\tau} \sum_{i=1}^{l} \lambda_i |z^m|^2 \cos(\nu_1, y_i) e^{-\alpha t}\,dx\,d\sigma_1\,dt \\ + \int_{Q_\tau} \Bigg[\alpha (z^m)^2 - \sum_{i=1}^{l} \lambda_{iy_i}(x,y,t)(z^m)^2 + \\ + 2\sum_{i,j=1}^{n} a_{ij}(x,y,t) z^m_{x_i} z^m_{x_j} + 2c(x,y,t)(z^m)^2 \Bigg] e^{-\alpha t}\,dx\,dy\,dt \\ = 2\int_{\Sigma_\tau} r^m(t) h(x,y,t) z^m e^{-\alpha t}\,d\sigma\,dy\,dt, \qquad m \geqslant 2. \quad (28)$$

Note that the right-hand side of equality (28) satisfies

$$2\int\limits_{\Sigma_\tau} r^m(t)h(x,y,t)z^m e^{-\alpha t}\,d\sigma\,dy\,dt \leqslant (h^0)^2\cdot \mathrm{mes}\,D\cdot \mathrm{mes}\,\partial\,\Omega\cdot\delta$$

$$\times\int\limits_0^\tau \big(r^m(t)\big)^2 e^{-\alpha t}\,dt + \frac{1}{\delta}\int\limits_{Q_\tau}\left[\varepsilon\sum_{i=1}^n \big(z^m_{x_i}\big)^2 + C(\varepsilon)(z^m)^2\right]e^{-\alpha t}\,dx\,dy\,dt,$$

where $\varepsilon>0$, $\delta>0$, $h^0 = \mathrm{ess}\sup\limits_{\Sigma_T}|h(x,y,t)|$.

So, from (28) there follows the estimate

$$\int\limits_{G_\tau}(z^m)^2 e^{-\alpha\tau}\,dx\,dy + \int\limits_{S^2_\tau}\sum_{i=1}^l \lambda_i|z^m|^2\cos(\nu_1,y_i)e^{-\alpha t}\,dx\,d\sigma_1 dt$$

$$+\int\limits_{Q_\tau}\left[\left(\alpha - l\lambda^1 + 2c_0 - \frac{C(\varepsilon)}{\delta}\right)(z^m)^2 + \left(2a_0 - \frac{\varepsilon}{\delta}\right)\sum_{i=1}^n |z^m_{x_i}|^2\right]e^{-\alpha t}\,dx\,dy\,dt$$

$$\leqslant (h^0)^2\cdot\mathrm{mes}\,D\cdot\mathrm{mes}\,\partial\Omega\cdot\delta\int\limits_0^\tau\big(r^m(t)\big)^2 e^{-\alpha t}\,dt. \qquad (29)$$

Choose ε and α from the inequalities

$$a_0 \geqslant \frac{\varepsilon}{2\delta}, \qquad \alpha \geqslant l\lambda^1 - 2c_0 + \frac{C(\varepsilon)}{\delta}.$$

Denote

$$M_4 = \frac{(h^0)^2\cdot\mathrm{mes}\,D\cdot\mathrm{mes}\,\partial\Omega\cdot e^{\alpha T}}{\min\left\{\alpha - l\lambda^1 + 2c_0 - \frac{C(\varepsilon)}{\delta}; 2a_0 - \frac{\varepsilon}{\delta}\right\}},$$

$$s^m(t) := \int\limits_\Omega\int\limits_{\Gamma_2}\sum_{i=1}^l \lambda_i|z^m|^2\cos(\nu_1,y_i)\,dx\,d\sigma_1 + \int\limits_G\left[(z^m)^2 + \sum_{i=1}^n |z^m_{x_i}|^2\right]dx\,dy.$$

Then (29) implies the inequality

$$\int\limits_{G_\tau}(z^m)^2\,dx\,dy + \int\limits_0^\tau s^m(t)\,dt \leqslant \delta\cdot M_4\cdot\int\limits_0^\tau\big(r^m(t)\big)^2\,dt, \quad \tau\in(0,T],\ \ m\geqslant 2. \qquad (30)$$

Now let us estimate $r^m(t)$, $m \geqslant 3$, using (26)

$$r^m(t) = \left[\int\limits_D \int\limits_{\partial\Omega} K(x,y)h(x,y,t)\,d\sigma\,dy\right]^{-1}$$
$$\times \int\limits_G \Bigg(-\sum_{i=1}^{l}\big(K(x,y)\lambda_i(x,y,t)\big)_{y_i} - \sum_{i,j=1}^{n}\big(K_{x_j}(x,y)a_{ij}(x,y,t)\big)_{x_i} +$$
$$+ K(x,y)c(x,y,t)\Bigg) z^{m-1}\,dx\,dy, \qquad t \in [0,T]. \tag{31}$$

Denote

$$M_5 = \left[\inf_{[0,T]}\int\limits_D \int\limits_{\partial\Omega} K(x,y)h(x,y,t)\,d\sigma\,dy\right]^{-2}$$
$$\times \sup_{[0,T]}\int\limits_G \Bigg(-\sum_{i=1}^{l}\big(K(x,y)\lambda_i(x,y,t)\big)_{y_i} - \sum_{i,j=1}^{n}\big(K_{x_j}(x,y)a_{ij}(x,y,t)\big)_{x_i} +$$
$$+ K(x,y)c(x,y,t)\Bigg)^2\,dx\,dy.$$

Then from (31) we obtain

$$\big(r^m(t)\big)^2 \leqslant M_5\int\limits_G \big(z^{m-1}\big)^2\,dx\,dy \leqslant M_5 s^{m-1}(t), \quad t \in [0,T].$$

Thus,

$$\int\limits_0^\tau \big(r^m(t)\big)^2\,dt \leqslant M_5\left(\int\limits_0^\tau s^{m-1}(t)\,dt + \int\limits_{G_\tau}\big(z^{m-1}\big)^2\,dx\,dy\right),$$
$$\tau \in (0,T],\ m \geqslant 3. \tag{32}$$

After substitution of (32) in (30) we obtain the inequality

$$\int\limits_{G_\tau}(z^m)^2\,dx\,dy + \int\limits_0^\tau s^m(t)\,dt \leqslant \delta\cdot M_6\left(\int\limits_0^\tau s^{m-1}(t)\,dt + \int\limits_{G_\tau}\big(z^{m-1}\big)^2\,dx\,dy\right),$$
$$\tau \in (0,T],\ m \geqslant 3,\ M_6 := M_4\cdot M_5. \tag{33}$$

Moreover,(30) and (32) imply

$$\frac{1}{M_5}\int\limits_0^\tau \big(r^{m+1}(t)\big)^2\,dt \leqslant \int\limits_0^\tau s^m(t)\,dt + \int\limits_{G_\tau}(z^m)^2\,dx\,dy$$
$$\leqslant \delta\cdot M_4\cdot\int\limits_0^\tau \big(r^m(t)\big)^2\,dt, \qquad \tau \in (0,T],\ m \geqslant 2.$$

Thus,

$$\int_0^\tau \bigl(r^m(t)\bigr)^2\,dt \leqslant \delta\cdot M_6 \int_0^\tau \bigl(r^{m-1}(t)\bigr)^2\,dt, \qquad \tau\in(0,T],\ m\geqslant 3. \tag{34}$$

We can choose δ such that $|\delta M_6|<1$. So from (33) and (34) it follows that the sequence $\{p^m\}_{m=1}^\infty$ is fundamental in $L^2(0,T)$, $\{u^m\}_{m=1}^\infty$ is fundamental in $L^2(Q_T)\cap C\bigl([0,T]\,;\ L^2(G)\bigr)$, $\{u^m_{x_i}\}_{m=1}^\infty$ is fundamental in $L^2(Q_T)$. These spaces are complete, so, the sequences converge to u, u_{x_i} strongly in these spaces as $m\to\infty$, and $p^m\to p$ strongly in $L^2(0,T)$.

Moreover, from (19) we have

$$\int_{Q_\tau}\sum_{k=1}^l \bigl(u^{m,N}_{y_k}\bigr)^2\,dx\,dy\,dt \leqslant M_7\Biggl[\int_{G_0}\Biggl(\sum_{k=1}^l \bigl(u^N_{0y_k}\bigr)^2+\bigl(u^N_0\bigr)^2\Biggr)dx\,dy + {} + \int_{Q_T}\Biggl(\sum_{k=1}^l (f_{y_k})^2+(f)^2\Biggr)dx\,dy\,dt + \int_0^T \bigl(p^m(t)\bigr)^2\,dt\Biggr], \tag{35}$$

where the constant M_7 does not depend on N. Note that if $p^m\to p$ strongly in $L^2(0,T)$, then for all $\varepsilon>0$ there exists m_1 such that for all $m>m_1$ the inequality $\bigl(\int_0^T \bigl(p^m(t)-p(t)\bigr)^2\,dt\bigr)^{\frac12}\leqslant\varepsilon$ holds. Let $\varepsilon=1$. Then

$$\int_0^T \bigl(p^m(t)\bigr)^2\,dt \leqslant \Biggl(\Biggl(\int_0^T \bigl(p(t)\bigr)^2\,dt\Biggr)^{\frac12}+1\Biggr)^2.$$

Passing to the limit as $N\to\infty$ in (35) and taking into account the estimate $\|v;L^2(Q_T)\|^2\leqslant\lim_{N\to\infty}\|v^N;L^2(Q_T)\|^2$, we obtain the estimate

$$\int_{Q_\tau}\sum_{k=1}^l \bigl(u^m_{y_k}\bigr)^2\,dx\,dy\,dt \leqslant M_7\Biggl[\int_{G_0}\Biggl(\sum_{k=1}^l (u_{0y_k})^2+(u_0)^2\Biggr)dx\,dy+\int_{Q_T}\Biggl(\sum_{k=1}^l (f_{y_k})^2+(f)^2\Biggr)dx\,dy\,dt + {} + \Biggl(\Biggl(\int_0^T \bigl(p(t)\bigr)^2\,dt\Biggr)^{\frac12}+1\Biggr)^2\Biggr], \qquad \text{for all } m>m_1. \tag{36}$$

The right-hand side of (36) does not depend on m.

Consequently, we can choose a subsequence from $\{u^m\}_{m=1}^{\infty}$ such that $u^m_{y_k} \to u_{y_k}$ in $L^2(Q_T)$ weakly. Letting $m \to \infty$ in (24) and (26) we conclude that (u, p) is a weak solution of the problem (1)–(4). $\square$

Theorem 4. *Let the conditions* **(A)**, **(C)**, **(L)**, **(U)**, **(H)**, **(E)**, **(K)**, **(S)**, $\int_D \int_{\partial\Omega} K(x,y)h(x,y,t)\, d\sigma\, dy \neq 0$ *hold. Then the problem* (1)–(4) *cannot have more than one weak solution.*

P r o o f. Let (u_1, p_1), (u_2, p_2) be two weak solutions to the problem (1)–(4). Then their difference (u, p), where $u = u_1 - u_2, p = p_1 - p_2$, satisfies the equality

$$\int\limits_{G_T} uv\, dx\, dy + \int\limits_{Q_T} \Big[-uv_t + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} v + \sum_{i,j=1}^{n} a_{ij}(x,y,t) u_{x_i} v_{x_j} + c(x,y,t)uv \Big] dx\, dy\, dt = \int\limits_{\Sigma_T} p(t)h(x,y,t)v\, d\sigma\, dy\, dt$$

for all functions $v \in V_1(Q_T)$. Set $v = ue^{-\alpha t}$, $\alpha > 0$. Then as from (13) we find

$$\int\limits_{G_T} u^2 e^{-\alpha T}\, dx\, dy + \int\limits_{Q_T} \Big[\Big(\alpha - \lambda^1 l + 2c_0 - \frac{C(\varepsilon)}{\delta} \Big) u^2 + \Big(2a_0 - \frac{\varepsilon}{\delta} \Big) \sum_{i=1}^{n} (u_{x_i})^2 \Big] e^{-\alpha t} dx\, dy\, dt + \int\limits_{S_T^2} \sum_{i=1}^{l} \lambda_i u^2 \cos(\nu_1, y_i) e^{-\alpha t}\, dx\, d\sigma_1 dt \leqslant \delta (h^0)^2 \mathrm{mes}\, D \cdot \mathrm{mes}\, \partial\Omega \cdot \int\limits_0^T \big(p(t)\big)^2\, dt. \quad (37)$$

Choosing

$$\alpha > \lambda^1 l - 2c_0 + \frac{C(\varepsilon)}{\delta}, \ \varepsilon < 2a_0, \quad \text{and} \quad \delta > 0$$

as an arbitrary small number, we get the estimate

$$\int\limits_{G_T} u^2\, dx\, dy + \int\limits_{Q_T} \Big[u^2 + \sum_{i=1}^{n} |u_{x_i}|^2 \Big] dx\, dy\, dt \leqslant \delta \cdot M_4 \int\limits_0^T \big(p(t)\big)^2\, dt. \quad (38)$$

Moreover, (6) implies the equality

$$p(t) = \left[\int\limits_D \int\limits_{\partial\Omega} K(x,y) h(x,y,t)\, d\sigma\, dy \right]^{-1}$$
$$\times \left(\int\limits_G \left(-\sum_{i=1}^{l} \big(K(x,y)\lambda_i(x,y,t)\big)_{y_i} u - \sum_{i,j=1}^{n} \big(K_{x_j}(x,y) a_{ij}(x,y,t)\big)_{x_i} u + \right.\right.$$
$$\left.\left. + K(x,y) c(x,y,t) u \right) dx\, dy \right), \qquad t \in [0,T].$$

Rising it to the square and integrating with respect to t from 0 to T, as from (31) we get

$$\int\limits_0^T \big(p(t)\big)^2\, dt \leqslant M_5 \int\limits_{Q_T} u^2\, dx\, dy\, dt.$$

Applying here (38), we find

$$\int\limits_0^T \big(p(t)\big)^2\, dt \leqslant M_6 \cdot \delta \cdot \int\limits_0^T \big(p(t)\big)^2\, dt.$$

From here $(1 - M_6 \cdot \delta) \cdot \int_0^T \big(p(t)\big)^2\, dt \leqslant 0$. Choosing $\delta < \frac{1}{M_6}$, we find

$$\int\limits_0^T \big(p(t)\big)^2\, dt \leqslant 0.$$

So, $p := 0$, that is, $p_1 = p_2$. Then (38) implies $u := 0$, so, $u_1 = u_2$. □

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