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WEAKLY PERTURBED BOUNDARY-VALUE PROBLEMS FOR SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We obtain bifurcation conditions of solutions of boundary-value problems for weakly perturbed systems of linear integro-differential equations from the point $\varepsilon = 0$.

1. Statement of the problem

We consider the boundary-value problems for linear systems of integro-differential equations with a small parameter ε

$$\dot{x}(t) - \Phi(t)\int_a^b \big[A(s)x(s) + B(s)\dot{x}(s)\big]\,ds = f(t) + \varepsilon\int_a^b \big[K(t,s)x(s) + K_1(t,s)\dot{x}(s)\big]\,ds, \tag{1}$$

$$lx(\cdot,\varepsilon) = \alpha \in \mathbb{R}^p \tag{2}$$

and the structure of the set of its solutions in the space $D_2[a,b]$ of n-dimensional absolutely continuously differentiable vector functions

$$x = x(t,\varepsilon) : x(\cdot,\varepsilon) \in D_2[a,b], \quad \dot{x}(\cdot,\varepsilon) \in L_2[a,b], \quad x(t,\cdot) \in C(0,\varepsilon_0].$$

Here, $A(t), B(t), \ \Phi(t), \ K(t,s), K_1(t,s), \ f(t)$ are $(m \times n), (m \times n), (n \times m)$, $(n \times n), (n \times n)$ and $(n \times \ 1)$ matrices, respectively, with entries from the space $L_2[a,b]$, $\alpha = \operatorname{col}(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p) \in R^p$, l is a bounded linear vector functional defined in the space $D_2[a;b]$, $l = \operatorname{col}(l_1, l_2, l_3, \dots, l_p) \colon D_2[a;b] \to R^p$, $l_i \colon D_2[a;b] \to R$. The columns of the matrix $\Phi(t)$ are assumed to be linear independent on $[a,b]$.

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We first consider that the problem (1), (2) belongs to the Fredholm case ($n \neq p$) and obtain bifurcation conditions of solution of this problem from the point $\varepsilon = 0$.

Parallel with the weakly perturbed boundary-value problem (1), (2), we consider the following generating boundary-value problem ($\varepsilon = 0$):

$$\dot{x}(t) - \Phi(t)\int_a^b \big[A(s)x(s) + B(s)\dot{x}(s)\big]\,ds = f(t), \tag{3}$$

$$lx(\cdot) = \alpha \,\epsilon\, \mathbb{R}^p. \tag{4}$$

Assume that the boundary-value problem (3), (4) does not have solutions for arbitrary inhomogeneities $f(t)\epsilon L_2[a,b]$ and $\alpha\epsilon R^p$. Then according to [1], we can formulate the following criterion for the solvability of boundary-value problem (3), (4).

Theorem 1. *Let $rankQ = n_2 \leq min(p, r_1)$. The homogeneous boundary-value problem (3), (4) ($f(t) = 0, \alpha = 0$) possesses exactly r_2 ($r_2 = r_1 - n_2$) linearly independent solutions of the form:*

$$x(t, c_{r_2}) = \Psi_0(t)P_{D_{r_1}}P_{Q_{r_2}}c_{r_2}, c_{r_2}\epsilon\mathbb{R}^{r_2},$$

$$r_1 = m + n - rankD, \quad r_2 = m + n - rankD - rankQ.$$

The inhomogeneous problem (3), (4) is solvable if and only if $f(t)\epsilon L_2[a,b]$ and $\alpha\epsilon R^p$ satisfy conditions

$$P_{D^*_{d_1}}\tilde{b} = 0, \quad P_{Q^*_{d_2}}\Big(\alpha - l\big(F(\cdot)\big)\Big) = 0, \tag{5}$$

$$d_1 = m - rankD, \quad d_2 = p - rankQ.$$

In this case, the problem (3), (4) possesses an r_2-parameter family of solutions

$$x(t) = \Psi_0(t)P_{D_{r_1}}P_{Q_{r_2}}c_{r_2} + \Psi_0(t)P_{D_{r_1}}Q^+\Big(\alpha - l\big(F(\cdot)\big)\Big) + F(t),$$

where $Q = lX_{r_1}(\cdot)$ is an $p \times r_1$ matrix, the matrix Q^+ is pseudoinverse (in the Moore–Penrose sense) [2] to the matrix Q,

$$F(t) = \tilde{f}(t) + \Psi_0(t)D^+\tilde{b}, \quad X_{r_1}(t) = \Psi_0(t)P_{D_{r_1}}$$

is an $n \times r_1$ matrix,

$$D = \left[I_m - \int_a^b \big[A(s)\Psi(s) + B(s)\Phi(s)\big]\,ds, -\int_a^b A(s)\,ds \right]$$

is an $m \times (m+n)$ matrix.

Here,

$$\Psi(t) = \int_a^t \Phi(s)\, ds, \quad \Psi_0(t) = \Big[\Psi(t), I_n\Big]$$

is $n \times (n+m)$ matrix,

$$\tilde{b} = \int_a^b \big[A(s)\tilde{f}(s) + B(s)f(s)\big] ds.$$

P_Q, P_{Q^*} are $r_1 \times r_1$, $p \times p$ matrices, orthoprojectors acting from R^{r_1}, R^p into the kernel and cokernel of the matrix Q, respectively. The matrix $P_{Q_{r_2}}$ $(P_{Q^*_{d_2}})$ composed of a complete system of r_2 (d_2) linearly independent columns (rows) of the matrix $P_Q(P_{Q^*})$. P_D, P_{D^*} are $(m+n) \times (m+n)$, $m \times m$ matrices, orthoprojectors acting from R^{m+n}, R^m into the kernel and cokernel of the matrix D, respectively. The matrix $P_{D_{r_1}}(P_{D^*_{d_1}})$ composed of a complete system of r_1 (d_1) linearly independent columns (rows) of the matrix $P_D(P_{D^*})$.

We obtain bifurcation conditions of solution of weakly perturbed systems (1) from the point $\varepsilon = 0$ [3]. Conditions of solution of weakly perturbed systems of linear integro-differential equations from the point $\varepsilon = 0$. We propose an iterative procedure converges finding solutions in the part of Laurent series.

Consider the case when one of the conditions (5) is not fulfilled. Then the boundary-value problem (3), (4) does not have solutions.

It is of interest to analyze whether it is possible to make the problem (3), (4) solvable by introducing linear perturbation and (in the case of positive answer to this question) determine the perturbation $K(t,s)$ and $K_1(t,s)$ required to make the boundary-value problem (1), (2) everywhere solvable.

2. Main result

Our aim is to establish conditions for the existence and develop an algorithm for the construction of the solution $x = x(t,\varepsilon)$ of the boundary-value problem (1), (2).

The general method used for the analysis of the posed problem is based on the generalized inverse operators [1], [2], [4] and the Vishik-Lyusternik method [5], [6].

Apply the Vishyk-Lyusternik method [5], which allows to find effective coefficients of the problem (1), (2) in the form of convergent series

$$x(t,\varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k x_k(t, c_k) = \frac{x_{-1}(t, c_{-1})}{\varepsilon} + x_0(t, c_0) + \varepsilon x_1(t, c_1) + \cdots \qquad (6)$$

Substituting the series (6) in the boundary-value problem (1), (2) and equate the coefficients of the same powers of ε.

Iterative process. The problem of determination of the coefficient $x_{-1}(t, c_{-1})$ of the term with ε^{-1} in the series (6) is reduced to the problem of finding solutions of the homogeneous boundary-value problem:

$$\dot{x}_{-1}(t, c_{-1}) - \Phi(t)\int_a^b \big[A(s)x_{-1}(s, c_{-1}) + B(s)\dot{x}_{-1}(s, c_{-1})\big]\,ds = 0,$$

$$lx_{-1}(\cdot, c_{-1}) = 0.$$

By virtue of Theorem 1, the homogeneous problem is always solvable and possesses an r_2-parameter ($r_2 = r_1 - n_2$) family of solutions $x_{-1}(t, c_{-1}) = X_{r_2}(t)c_{-1}$, where the r_2-dimensional vector column $c_{-1}\epsilon\mathbb{R}^{r_2}$ is determined from the condition of solvability of the problem used for determining the coefficient $x_0(t, c_0)$ of the series (6),

$$X_{r_2}(t) = \Psi_0(t)P_{D_{r_1}}P_{Q_{r_2}}$$

is an $n \times r_2$ matrix.

The problem used for determining the coefficient $x_0(t, c_0)$ of the term with ε^0 in the series (6) reduces to the problem of finding solutions of the following boundary-value problem

$$\dot{x}_0(t, c_0) - \Phi(t)\int_a^b \big[A(s)x_0(s, c_0) + B(s)\dot{x}_0(s, c_0)\big]\,ds = f_0(t), \tag{7}$$

$$lx_0(\cdot, c_0) = \alpha, \tag{8}$$

where inhomogeneity of the problem (7), (8) is established by the formula

$$f_0(t) = f(t) + \int_a^b \big[K(t, s)x_{-1}(s, c_{-1}) + K_1(t, s)\dot{x}_{-1}(s, c_{-1})\big]\,ds.$$

By virtue of Theorem 1, the problem (7), (8) is solvable if and only if $f_0(t)\epsilon L_2[a; b]$ and $\alpha\epsilon\mathbb{R}^p$ satisfy the following conditions

$$P_{D^*_{d_1}}\tilde{b}_0 = 0, \quad P_{Q^*_{d_2}}\Big(\alpha - l\big(F_0(\cdot)\big)\Big) = 0, \tag{9}$$

where

$$F_0(t) = \tilde{f}_0(t) + \Psi_0(t) D^+ \tilde{b}_0,$$

$$\tilde{b}_0 = \int_a^b \big[A(s)\tilde{f}_0(s) + B(s) f_0(s)\big] ds,$$

$$\tilde{f}_0(t) = \int_a^t f_0(s)\, ds.$$

Substituting $x_{-1}(t, c_{-1}) = X_{r_2}(t) c_{-1}$ in equations (9), we arrive at the following systems of algebraic equations for $c_{-1} \epsilon \mathbb{R}^{r_2}$

$$B_1 c_{-1} = -P_{D^*_{d_1}} \tilde{b}, \quad B_2 c_{-1} = P_{Q^*_{d_2}} \big(\alpha - lF(\cdot)\big) \tag{10}$$

where

$$B_1 := P_{D^*_{d_1}} \int_a^b \big[A(s)\tilde{L}(s) + B(s) L(s)\big] ds, \tag{11}$$

$$B_2 := P_{Q^*_{d_2}} l \left(\tilde{L}(t) + \Psi_0(t) D^+ \int_a^b \big[A(s)\tilde{L}(s) + B(s) L(s)\big] ds \right). \tag{12}$$

$$L(t) = \int_a^b \big[K(t,s) X_{r_2}(s) + K_1(t,s) \dot{X}_{r_2}(s)\big] ds, \qquad \tilde{L}(t) = \int_a^t L(s)\, ds,$$

here $L(t), B_1, B_2$ are $n \times r_2$, $d_1 \times r_2$, $d_2 \times r_2$ matrices with components belonging to the space $L_2[a;b]$. The systems (10) are solvable if and only if the following conditions

$$P_{B_1^*} P_{D^*_{d_1}} = 0, \quad P_{B_2^*} P_{Q^*_{d_2}} = 0 \tag{13}$$

are satisfied. Here $P_{B_1^*}$, $P_{B_2^*}$ are $d_1 \times d_1$, $d_2 \times d_2$ matrices, orthoprojectors acting from R^{d_1}, R^{d_2} into the cokernel of the matrices B_1 and B_2, respectively.

Thus, in the case, where conditions (13) are satisfied, we get the algebraic equation

$$B_0 c_{-1} = g_{-1} \tag{14}$$

and find a constant $c_{-1} \epsilon R^{r_2}$:

$$c_{-1} = B_0^+ g_{-1},$$

where

$$g_{-1} := \begin{bmatrix} -P_{D^*_{d_1}} \tilde{b} \\ P_{Q^*_{d_2}} \big(\alpha - lF(\cdot)\big) \end{bmatrix}, \quad B_0 := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

B_0 is an $(d_1+d_2)\times r_2$ matrix, the matrix B_0^+ is pseudoinverse (in the Moore--Penrose sense) [2] to the matrix B_0. If the conditions (13) are satisfied, then the problem (7), (8) possesses an r_2-parameter family of solutions

$$x_0(t,c_0)=X_{r_2}(t)c_0+\Psi_0(t)P_{D_{r_1}}Q^+\big(\alpha-lF_0(\cdot)\big)+F_0(t),$$

where c_0 is an r_2-dimensional constant vector determined in the next stage of the process from the condition of solvability of the problem of determination of the coefficient $x_1(t,c_1)$ of the series (6).

The problem of determination of the coefficient $x_1(t,c_1)$ of the term with ε^1 in the series (6) reduces to the problem of determination of solutions of the following boundary-value problem:

$$\dot{x}_1(t,c_1)-\Phi(t)\int_a^b\big[A(s)x_1(s,c_1)+B(s)\dot{x}_1(s,c_1)\big]ds=f_1(t), \tag{15}$$

$$lx_1(\cdot,c_1)=0, \tag{16}$$

here inhomogeneity of this problem is established by the formula

$$f_1(t)=\int_a^b\big[K(t,s)x_0(s,c_0)+K_1(t,s)\dot{x}_0(s,c_0)\big]ds.$$

The conditions of solvability of the problem (15), (16)

$$P_{D^*_{d_1}}\tilde{b}_1=0,\quad P_{Q^*_{d_2}}l\big(F_1(\cdot)\big)=0. \tag{17}$$

Accordingly

$$F_1(t)=\tilde{f}_1(t)+\Psi_0(t)D^+\tilde{b}_1,$$

$$\tilde{b}_1=\int_a^b\big[A(s)\tilde{f}_1(s)+B(s)f_1(s)\big]ds,$$

$$\tilde{f}_1(t)=\int_a^t f_1(s)\,ds.$$

We substitute $x_0(t,c_0)=X_{r_2}(t)c_0+F_0(t)+\Psi_0(t)P_{D_{r_1}}Q^+\big(\alpha-lF_0(\cdot)\big)$ in (17) and obtain similarly to (10) algebraic equations for $c_0\epsilon\mathbb{R}^{r_2}$

$$B_1c_0=-P_{D^*_{d_1}}\int_a^b\big[A(s)\tilde{M}_0(s)+B(s)M_0(s)\big]ds, \tag{18}$$

$$B_2c_0 = -P_{Q^*_{d_2}}\, l\left(\tilde{M}_0(\cdot) + \Psi_0(\cdot)D^+\int_a^b \big[A(s)\tilde{M}_0(s) + B(s)M_0(s)\big]\,ds\right). \tag{19}$$

The equations (18), (19) are solvable if and only if the conditions (13) are true, where

$$M_0(t) = \int_a^b \Big[K(t,s)\Big(F_0(s) + \Psi_0(s)P_{D_{r_1}}Q^+\big(\alpha - lF_0(\cdot)\big)\Big)$$
$$+ K_1(t,s)\Big(F_0(s) + \Psi_0(s)P_{D_{r_1}}Q^+\big(\alpha - lF_0(\cdot)\big)\Big)'\Big]\,ds,$$

$$\tilde{M}_0(t) = \int_a^t M_0(s)\,ds.$$

Thus, in the case where the conditions (13) are satisfied, we get the algebraic equation

$$B_0c_0 = g_0 \tag{20}$$

and find a constant $c_0 \epsilon \mathbb{R}^{r_2}$,

$$c_0 = B_0^+ g_0, \tag{21}$$

where

$$g_0 := \begin{bmatrix} -P_{D^*_{d_1}} \int_a^b \big[A(s)\tilde{M}_0(s) + B(s)M_0(s)\big]\,ds \\ -P_{Q^*_{d_2}}\, l\Big(\tilde{M}_0(\cdot) + \Psi_0(\cdot)D^+\int_a^b \big[A(s)\tilde{M}_0(s) + B(s)M_0(s)\big]\,ds\Big) \end{bmatrix}.$$

Thus, if the conditions (13) are satisfied, then the problem (15), (16) possesses an r_2-parameter family of solutions

$$x_1(t,c_1) = X_{r_2}(t)c_1 + F_1(t) - \Psi_0(t)P_{D_{r_1}}Q^+ l\big(F_1(\cdot)\big),$$

where c_1 is an r_2-dimensional constant vector determined in the next stage of the process from the condition of solvability of the problem of determination of the coefficient $x_2(t,c_2)$ of the series (6).

The problem of determination of the coefficient $x_2(t,c_2)$ of the term with ε^2 in the series (6) reduces to the problem of determination of solutions of the following boundary-value problem

$$\dot{x}_2(t,c_2) - \Phi(t)\int_a^b \big[A(s)x_2(s,c_2) + B(s)\dot{x}_2(s,c_2)\big]\,ds = f_2(t), \tag{22}$$

$$lx_2(\cdot,c_2) = 0, \tag{23}$$

here inhomogeneity of this problem is established by the formula

$$f_2(t) = \int_a^b \big[K(t,s)x_1(s,c_1) + K_1(t,s)\dot{x}_1(s,c_1)\big]\,ds.$$

The conditions of solvability of the problem (22), (23) are

$$P_{D^*_{d_1}}\tilde{b}_2 = 0, \quad P_{Q^*_{d_2}}\, l\big(F_2(\cdot)\big) = 0, \tag{24}$$

where, define

$$\begin{aligned} F_2(t) &= \tilde{f}_2(t) + \Psi_0(t)D^+\tilde{b}_2, \\ \tilde{b}_2 &= \int_a^b \big[A(s)\tilde{f}_2(s) + B(s)f_2(s)\big]\,ds, \\ \tilde{f}_2(t) &= \int_a^t f_2(s)\,ds. \end{aligned}$$

We substitute $x_1(t,c_1) = X_{r_2}(t)c_1 + F_1(t) - \Psi_0(t)P_{D_{r_1}}Q^+ l\big(F_1(\cdot)\big)$ in (24) and get similarly to (10) the algebraic equations

$$B_1 c_1 = -P_{D^*_{d_1}} \int_a^b \Big[A(s)\tilde{M}_1(s) + B(s)M_1(s)\Big]\,ds, \tag{25}$$

$$B_2 c_1 = -P_{Q^*_{d_2}}\, l\left(\tilde{M}_1(\cdot) + \Psi_0(\cdot)D^+ \int_a^b \big[A(s)\tilde{M}_1(s) + B(s)M_1(s)\big]\,ds\right). \tag{26}$$

The equations (25), (26) are solvable if and only if the conditions (13) are true, where

$$\begin{aligned} M_1(t) &= \int_a^b \Big[K(t,s)\big(F_1(s) - \Psi_0(s)P_{D_{r_1}}Q^+ lF_1(\cdot)\big) \\ &\qquad + K_1(t,s)\big(F_1(s) - \Psi_0(s)P_{D_{r_1}}Q^+ lF_1(\cdot)\big)'\Big]\,ds, \\ \tilde{M}_1(t) &= \int_a^t M_1(s)\,ds. \end{aligned}$$

Thus, in the case where conditions (13) are satisfied, we get the algebraic equation

$$B_0 c_1 = g_1 \tag{27}$$

and find a constant $c_1 \epsilon \mathbb{R}^{r_2}$,

$$c_1 = B_0^+ g_1, \tag{28}$$

where

$$g_1 := \begin{bmatrix} -P_{D_{d_1}^*} \int\limits_a^b \big[A(s)\tilde{M}_1(s) + B(s)M_1(s)\big] ds \\ -P_{Q_{d_2}^*} l\Big(\tilde{M}_1(\cdot) + \Psi_0(\cdot)D^+ \int\limits_a^b \big[A(s)\tilde{M}_1(s) + B(s)M_1(s)\big] ds\Big) \end{bmatrix}.$$

Thus, if the conditions (13) are satisfied, then the problem (22), (23) possesses an r_2-parameter family of solutions

$$x_2(t, c_2) = X_{r_2}(t)c_2 + F_2(t) - \Psi_0(t)P_{D_{r_1}} Q^+ l F_2(\cdot),$$

where c_2 is an r_2-dimensional constant vector determined in the next stage of the process from the condition of solvability of the problem of determination of the coefficient $x_3(t, c_3)$ of the series (6).

As above, one can easily show (by induction) that, under the conditions (13), the problem of determination of the coefficients $x_k(t, c_k)$ of the terms with ε^k in the series (6) is reduced to the problem of finding of solutions of the inhomogeneous boundary-value problem

$$\dot{x}_k(t, c_k) - \Phi(t)\int\limits_a^b \big[A(s)x_k(s, c_k) + B(s)\dot{x}_k(s, c_k)\big] ds = f_k(t), \tag{29}$$

$$l x_k(\cdot, c_k) = 0, \tag{30}$$

where inhomogeneity of this problem has the form

$$f_k(t) = \int\limits_a^b \big[K(t, s)x_{k-1}(s, c_{k-1}) + K_1(t, s)\dot{x}_{k-1}(s, c_{k-1})\big] ds.$$

Let

$$\tilde{b}_k = \int\limits_a^b \big[A(s)\tilde{f}_k(s) + B(s)f_k(s)\big] ds, \quad \tilde{f}_k(t) = \int\limits_a^t f_k(s)\, ds,$$

then by virtue of Theorem 1, the problem (29), (30) is solvable if and only if the following conditions

$$P_{Q_{d_1}^*} \tilde{b}_k = 0, \quad P_{D_{d_2}^*} l\big(F_k(\cdot)\big) = 0, \tag{31}$$

are satisfied and possesses an r_2-parameter family of solutions

$$x_k(t,c_k) = X_{r_2}(t)c_k + F_k(t) - \Psi_0(t)P_{D_{r_1}}Q^+ l\big(F_k(\cdot)\big), \qquad c_k \epsilon \mathbb{R}^{r_2}. \tag{32}$$

$$F_k(t) = \tilde{f}_k(t) + \Psi_0(t)B_0^+\tilde{b}_k, \tag{33}$$

where c_k is an r_2-dimensional constant vector determined in the next stage of the process from the condition of solvability of the problem of determination of the coefficient $x_{k+1}(t,c_{k+1})$ of the series (6). We get similarly to (10) the algebraic equations

$$B_1 c_k = -P_{D^*_{d_1}}\int_a^b \big[A(s)\tilde{M}_k(s) + B(s)M_k(s)\big]ds, \tag{34}$$

$$B_2 c_k = -P_{Q^*_{d_2}}\, l\left(\tilde{M}_k(\cdot) + \Psi_0(\cdot)D^+\int_a^b \big[A(s)\tilde{M}_k(s) + B(s)M_k(s)\big]ds\right). \tag{35}$$

The equations (34), (35) are solvable if and only if the conditions (13) are true, where

$$\begin{aligned} M_k(t) =& \int_a^b \Big[K(t,s)\big(F_k(s) - \Psi_0(s)P_{D_{r_1}}Q^+ lF_k(\cdot)\big) \\ &+ K_1(t,s)\big(F_k(s) - \Psi_0(s)P_{D_{r_1}}Q^+ lF_k(\cdot)\big)'\Big]ds, \\ \tilde{M}_k(t) =& \int_a^t M_k(s)\,ds. \end{aligned}$$

Thus, in the case, where the conditions (13) are satisfied, we get the system of algebraic equations for c_k:

$$B_0 c_k = g_k \tag{36}$$

and find a constant $c_k \epsilon \mathbb{R}^{r_2}$,

$$c_k = B_0^+ g_k, \tag{37}$$

where

$$g_k := \begin{bmatrix} -P_{D^*_{d_1}} \int_a^b \big[A(s)\tilde{M}_k(s) + B(s)M_k(s)\big]ds \\ -P_{Q^*_{d_2}}\, l\Big(\tilde{M}_k(\cdot) + \Psi_0(\cdot)D^+\int_a^b \big[A(s)\tilde{M}_k(s) + B(s)M_k(s)\big]ds\Big) \end{bmatrix}.$$

Therefore, the problem (29), (30) is solvable if the conditions (13) are satisfied, then the substituting relation (37) in the expression (32), we obtain the solution of the boundary-value problem (29), (30) in general form.

Thus the following assertion is true

Theorem 2. *Assume that the boundary-value problem* (1), (2) *satisfies the conditions presented above and the generating boundary-value problem* (3), (4) *is unsolvable for arbitrary inhomogeneities* $f(t)\epsilon L_2[a,b]$ *and* $\alpha\epsilon R^p$*. If, in addition, the conditions*

$$P_{B_1^*}P_{D_{d_1}^*}=0, \quad P_{B_2^*}P_{Q_{d_2}^*}=0$$

are satisfied, then, for any $f(t)\epsilon L_2[a,b]$ *and* $\alpha\epsilon R^p$*, the boundary-value problem* (1), (2) *possesses at least one solution in the form of the series* (6) *convergent for sufficiently small fixed* $\varepsilon\epsilon(0;\varepsilon_*]$*, with coefficients in the form* (32), (37).

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