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# THE STURM–LIOUVILLE PROBLEM WITH SINGULAR POTENTIAL AND THE EXTREMA OF THE FIRST EIGENVALUE

Elena S. Karulina—Anton A. Vladimirov

ABSTRACT. We get the infima and suprema of the first eigenvalue of the problem

$$-y'' + qy = \lambda y,$$
$$\begin{cases} y'(0) - k_0^2 y(0) = 0, \\ y'(1) + k_1^2 y(1) = 0, \end{cases}$$

where $q$ belongs to the set of constant-sign summable functions on $[0, 1]$ such that

$$\int_0^1 q\,dx = 1 \quad \text{or} \quad \int_0^1 q\,dx = -1.$$

## 1. Introduction

**1.1.** Consider the Sturm–Liouville problem

$$-y'' + (q - \lambda)y = 0, \tag{1}$$

$$\begin{cases} y'(0) - k_0^2 y(0) = 0, \\ y'(1) + k_1^2 y(1) = 0, \end{cases} \tag{2}$$

where the real coefficients $k_0 \geq 0$ and $k_1 \geq k_0$ are fixed, the solution $y$ belongs to the space $W_1^2[0, 1]$, the equality (1) is considered as holding almost everywhere at $[0, 1]$, and the potential $q \in L_1[0, 1]$ is a constant-sign function such that one


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of the integral conditions holds:

$$\int_0^1 q\,dx = 1 \quad \text{or} \quad \int_0^1 q\,dx = -1. \tag{3}$$

The aim of this paper is to get the infima and suprema of the first eigenvalue of the problem (1)–(3).

**1.2.** The problem (1)–(3) is a partial case of the problem (1), (2) with $q \in A_\gamma$ or $-q \in A_\gamma$, where $\gamma \in \mathbb{R} \setminus \{0\}$ and

$$A_\gamma \rightleftharpoons \left\{ q \in L_1[0,1] : q(x) \geq 0 \ \text{a.e.} \ \text{and} \int_0^1 q^\gamma\,dx = 1 \right\}. \tag{4}$$

Denote by $\lambda_1(q)$ the minimal eigenvalue of the problem (1) or

$$-y'' - \lambda q y = 0 \tag{5}$$

with some self-adjoint boundary conditions. Consider for each $\gamma \in \mathbb{R} \setminus \{0\}$ four values $m_\gamma^\pm \rightleftharpoons \inf_{q \in A_\gamma} \lambda_1(\pm q)$ and $M_\gamma^\pm \rightleftharpoons \sup_{q \in A_\gamma} \lambda_1(\pm q)$. The estimates of $m_\gamma^+$ and $M_\gamma^+$ for the equation (5) with the Dirichlet boundary conditions were obtained in [1]. The analogous results about the Dirichlet problem for the equation (1) were obtained in [2], [3]. In [4] the problem (5), (2) was studied.

The values $m_\gamma^+$ and $M_\gamma^+$ for the problem (1), (2) with $q \in A_\gamma$ were considered by one of the authors in [5] for all $\gamma \neq 0$. The most detailed and precise results were obtained for the case $\gamma \neq 1$.

The case $\gamma = 1$ is in some kind special. In [3] and [5], for (1) with various boundary conditions, the precise results for $M_1^+$ were obtained by the method quite different from used for $\gamma \neq 1$. In [5] for $m_1^+$ only inequality $m_1^+ \geq 1/4$ was obtained. In [3] for $m_1^-$ it was proved that this infimum is attained at the non-summable potential $q^* = -\boldsymbol{\delta}_{1/2}$.

In this paper we extend the class of considered potentials from $L_1[0,1]$ to the space $W_2^{-1}[0,1]$ (see [6] and 2.1 later). The space $W_2^{-1}[0,1]$, in particular, contains a Dirac delta function $\boldsymbol{\delta}_\zeta$ with support located at an arbitrary point $\zeta \in [0,1]$. This generalization of the problem lets us to get the precise description of $M_1^-$ and $m_1^\pm$ and to prove that they are attained at the potentials from the extended class.

**1.3.** The main results of the paper are the following four theorems:

**1.3.1. Theorem.** *By definition, put*

$$\alpha_\mu \rightleftharpoons \frac{1}{\sqrt{\mu}} \arctan \frac{k_0^2}{\sqrt{\mu}}, \quad \beta_\mu \rightleftharpoons \frac{1}{\sqrt{\mu}} \arctan \frac{k_1^2}{\sqrt{\mu}}. \tag{6}$$

*Then* $M_1^+$ *is a unique solution to the equation*

$$1 - \alpha_\mu - \beta_\mu = \mu^{-1} \tag{7}$$

*and is attained at the potential* $q^* \in L_1[0,1]$ *such that*

$$q^*(x) = \begin{cases} M_1^+ & \text{for } x \in \big[\alpha_{M_1^+}, 1 - \beta_{M_1^+}\big], \\ 0 & \text{otherwise.} \end{cases}$$

**1.3.2. Theorem.** *If* $k_0^2 + k_1^2 \le 1$, *then* $M_1^- = k_0^2 + k_1^2 - 1$ *and is attained at the potential*

$$q^* \rightleftharpoons -k_0^2 \boldsymbol{\delta}_0 - k_1^2 \boldsymbol{\delta}_1 - \big(1 - k_0^2 - k_1^2\big).$$

*If* $k_0^2 + k_1^2 \ge 1$ *and* $k_1^2 - k_0^2 \le 1$, *then* $M_1^-$ *is the minimal eigenvalue of the problem*

$$-y'' = \lambda y, \tag{8}$$

$$2y'(0) - \big(k_0^2 + k_1^2 - 1\big)y(0) = 2y'(1) + \big(k_0^2 + k_1^2 - 1\big)y(1) = 0 \tag{9}$$

*and is attained at the potential*

$$q^* \rightleftharpoons -\big(1 + k_0^2 - k_1^2\big)\boldsymbol{\delta}_0/2 - \big(1 - k_0^2 + k_1^2\big)\boldsymbol{\delta}_1/2.$$

*If* $k_1^2 - k_0^2 \ge 1$, *then* $M_1^-$ *is the minimal eigenvalue of the problem* (8) *with*

$$y'(0) - k_0^2 y(0) = y'(1) + \big(k_1^2 - 1\big)y(1) = 0 \tag{10}$$

*and is attained at the potential* $q^* \rightleftharpoons -\boldsymbol{\delta}_1$.

**1.3.3. Theorem.** $m_1^+$ *is the minimal eigenvalue of the problem* (8) *with*

$$y'(0) - k_0^2 y(0) = y'(1) + \big(k_1^2 + 1\big)y(1) = 0 \tag{11}$$

*and is attained at the potential* $q^* \rightleftharpoons \boldsymbol{\delta}_1$.

**1.3.4. Theorem.** *If for some* $\mu \ge -k_0^4$ *and some* $\zeta \in (0,1)$ *the problem*

$$-y'' = \mu y \qquad \text{at } (0,\zeta) \cup (\zeta,1), \tag{12}$$

$$\begin{aligned} y'(0) - k_0^2 y(0) &= 2y'(\zeta - 0) - y(\zeta) \\ &= 2y'(\zeta + 0) + y(\zeta) = y'(1) + k_1^2 y(1) = 0 \end{aligned} \tag{13}$$

*has a continuous positive solution, then* $m_1^- = \mu$ *and* $m_1^-$ *is attained at the potential* $q^* \rightleftharpoons -\boldsymbol{\delta}_\zeta$. *Otherwise* $m_1^-$ *is the minimal eigenvalue of the problem* (8) *with*

$$y'(0) - (k_0^2 - 1)y(0) = y'(1) + k_1^2 y(1) = 0$$

*and is attained at the potential* $q^* \rightleftharpoons -\boldsymbol{\delta}_0$.

Some additional remarks on solvability of the boundary problem (12), (13) will be given in the subsection 3.6.

**1.4.** Let us give some examples that illustrate the theorems from the previous subsection. In the case $k_0 = k_1 = 0$ we get $m_1^+ = \lambda_1(\boldsymbol{\delta}_1) = 0.740174(\pm 10^{-6})$. In the case $k_0^2 = k_1^2 > 1/2$ we get $m_1^- = \lambda_1(-\boldsymbol{\delta}_{1/2})$. In the case $k_0^2 = k_1^2 = 1/2$ we have $m_1^- = \lambda_1(-\boldsymbol{\delta}_\zeta) = -1/4$ for any $\zeta \in [0,1]$. In the case $k_0^2 = k_1^2 < 1/2$ we have $m_1^- = \lambda_1(-\boldsymbol{\delta}_0)$.

## 2. The set $\Gamma_1$ and related topics

**2.1.** We suppose that all considered functional spaces are real.

By $W_2^{-1}[0,1]$ denote the Hilbert space that is a completion of $L_2[0,1]$ in the norm

$$\|y\|_{W_2^{-1}[0,1]} \rightleftharpoons \sup_{\|z\|_{W_2^1[0,1]}=1} \int_0^1 yz\,dx.$$

When $y \in W_2^{-1}[0,1]$, by $\int_0^1 yz\,dx$ we sometimes denote the result

$$\langle y, z\rangle \rightleftharpoons \lim_{n\to\infty} \int_0^1 y_n z\,dx, \qquad \text{where} \quad y = \lim_{n\to\infty} y_n,\ y_n \in L_2[0,1],$$

of applying the linear functional $y$ to the function $z \in W_2^1[0,1]$.

For any fixed $q \in L_1[0,1]$ and $\lambda \in \mathbb{R}$ the map taking each $y \in W_1^2[0,1]$ satisfying (2) to

$$-y'' + (q-\lambda)y \in L_1[0,1]$$

can be extended by continuity to the bounded operator $T_q(\lambda) : W_2^1[0,1] \to W_2^{-1}[0,1]$. Using integration by part, we get

$$(\forall y, z \in W_2^1[0,1]) \quad \bigl\langle T_q(\lambda)y, z\bigr\rangle = \int_0^1 \bigl[y'z' + (q-\lambda)\,yz\bigr]\,dx + k_0^2\,y(0)z(0) + k_1^2\,y(1)z(1). \quad (14)$$

Consider the linear operator pencil[1] $T_q\colon \mathbb{R} \to \mathcal{B}\bigl(W_2^1[0,1], W_2^{-1}[0,1]\bigr)$ that takes any $\lambda \in \mathbb{R}$ to the operator $T_q(\lambda)$ described by (14). The spectral problem for $T_q$ may be considered as a reformulation (or as a generalization in case when $q \in W_2^{-1}[0,1]$ is not summable) of the boundary value problem (1), (2). We can do this due to the following two facts.

[1]A linear operator pencil $L$ is an operator-valued function such that $L(\lambda) = A + \lambda B$, where $\lambda \in \mathbb{R}$, $A$ and $B$ are some operators not depending on $\lambda$.

**2.1.1.** *For all $q \in L_1[0,1]$ and $\lambda \in \mathbb{R}$ the function $y \in W_2^1[0,1]$ belongs to the kernel of the operator $T_q(\lambda)$, if and only if $y \in W_1^2[0,1]$ and $y$ is a solution of the problem* (1), (2).

P r o o f. It directly follows from the definition of the operator $T_q(\lambda)$ that for any solution $y \in W_1^2[0,1]$ of the problem (1), (2) the equality $T_q(\lambda)y = 0$ holds.

Let us prove the converse. Consider some $y \in \ker T_q(\lambda)$, and put

$$w(x) \rightleftharpoons y'(x) - \int_0^x (q-\lambda)y\,dt. \tag{15}$$

For any $z \in \overset{\circ}{W}{}_2^1[0,1]$, using (14), we have

$$0 = \langle T_q(\lambda)y, z\rangle = \int_0^1 wz'\,dx. \tag{16}$$

Since the set of the derivatives of all functions $z \in \overset{\circ}{W}{}_2^1[0,1]$ is an orthogonal complement in $L_2[0,1]$ of the set of all constants, from (16) it follows that the function $w \in L_2[0,1]$ is constant. Combining this with (15), we get that the function $y'$ is absolutely continuous and its generalized derivative equals $(q-\lambda)y$. Now, using (14), we see that for any $z \in W_2^1[0,1]$ we get

$$0 = \langle T_q(\lambda)y, z\rangle = \left[-y'(0) + k_0^2 y(0)\right] z(0) + \left[y'(1) + k_1^2 y(1)\right] z(1),$$

so $y$ satisfies the conditions (2). □

**2.1.2.** *For any $q \in W_2^{-1}[0,1]$ the spectrum of the linear operator pencil $T_q$ is purely discrete, simple and bounded from below.*

P r o o f. Note that for any $y \in W_2^1[0,1]$ we have

$$\|y^2\|_{W_2^1[0,1]} \le \sup_{x\in[0,1]} |y(x)| \cdot \sqrt{\int_0^1 \left[y^2 + 4(y')^2\right] dx} \le 2\|y\|_{C[0,1]} \cdot \|y\|_{W_2^1[0,1]},$$

then, by the embedding theorem, we get

$$\|y^2\|_{W_2^1[0,1]} \le C\,\|y\|^2_{W_2^1[0,1]}, \tag{17}$$

where $C$ is some constant.

Since $C[0,1]$ is densely embedded in $W_2^{-1}[0,1]$, for any $\varepsilon \in (0,1)$ there exists a function $\tilde{q} \in C[0,1]$ such that

$$\|\tilde{q} - q\|_{W_2^{-1}[0,1]} \le \varepsilon/C.$$

Using this and the inequality (17), for any $y \in W_2^1[0,1]$ we get

$$\left| \int_0^1 (\tilde{q} - q)\, y^2\, dx \right| \leq \|\tilde{q} - q\|_{W_2^{-1}[0,1]} \cdot \|y^2\|_{W_2^1[0,1]} \leq \varepsilon\, \|y\|^2_{W_2^1[0,1]}. \tag{18}$$

Further, for any $\kappa > \|\tilde{q}\|_{C[0,1]} + 1$ we have $\int_0^1 \tilde{q} y^2\, dx \geq (1-\kappa) \int_0^1 y^2\, dx$. Combining this with (14) and (18), we obtain

$$IT_q(-\kappa) \geq 1 - \varepsilon, \tag{19}$$

where by $I\colon W_2^{-1}[0,1] \to W_2^1[0,1]$ we denote an isometry that satisfies

$$(\forall y \in W_2^{-1}[0,1])\, (\forall z \in W_2^1[0,1]) \qquad \langle Iy, z\rangle_{W_2^1[0,1]} = \langle y, z\rangle.$$

The existence and uniqueness of this isometry follows from the Riesz theorem about the representation of a functional in a Hilbert space [7, § 30, § 99].

From the estimate (19) it follows [7, § 104] that the operator $S \rightleftharpoons IT_q(-\kappa)$ is boundedly invertible. Taking into account (14), we have $IT_q(\lambda) \equiv S - (\lambda+\kappa) J^* J$, where $J\colon W_2^1[0,1] \to L_2[0,1]$ is the embedding operator. So for any $\lambda \in \mathbb{R}$ the existence of a bounded inverse of the operator $T_q(\lambda)$ is equivalent to the existence of a bounded inverse of the operator $1 - (\lambda + \kappa) S^{-1/2} J^* J S^{-1/2}$. Since $J$ is compact, it follows that the spectrum of $T_q$ is purely discrete, semi-simple and bounded from below.

The spectrum of the pencil $T_q$ is simple since (see [6], [8, Propositions 2, 10]) for any $\lambda \in \mathbb{R}$ the kernel of the operator $T_q(\lambda)$ is formed by the first components $Y_1$ of the solutions to the boundary value problem

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -u^2 & -u \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \tag{20}$$

$$Y_2(0) - k_0^2 Y_1(0) = Y_2(1) + \left[k_1^2 + \omega\right] Y_1(1) = 0. \tag{21}$$

Here $u \in L_2[0,1]$ and $\omega \in \mathbb{R}$ are taken from the representation

$$(\forall y \in W_2^1[0,1]) \qquad \int_0^1 (q - \lambda) y\, dx = -\int_0^1 u y'\, dx + \omega\, y(1) \tag{22}$$

of the potential $q \in W_2^{-1}[0,1]$. □

**2.2.** For the eigenvalues

$$\lambda_1(q) < \lambda_2(q) < \cdots < \lambda_n(q) < \cdots$$

of the pencil $T_q$ we have the following propositions.

**2.2.1.** (See [8, Proposition 10].) *For any* $n \geq 1$, $q \in W_2^{-1}[0,1]$ *and* $\lambda \in \mathbb{R}$ *the inequality* $\lambda > \lambda_n(q)$ *is equivalent to the existence of* $n$*-dimentional subspace* $\mathfrak{N} \subset W_2^1[0,1]$ *that satisfies*

$$(\forall y \in \mathfrak{N} \setminus \{0\}) \qquad \big\langle T_q(\lambda)y, y \big\rangle < 0.$$

**2.2.2.** *For any* $n \geq 1$ *the function* $\lambda_n \colon W_2^{-1}[0,1] \to \mathbb{R}$ *is continuous.*

P r o o f. Consider some $q \in W_2^{-1}[0,1]$ and $\varepsilon \in (0, 1/2)$. For any $y \in W_2^1[0,1]$, $\lambda \in \mathbb{R}$ and $\tilde{q} \in W_2^{-1}[0,1]$ such that $\|\tilde{q} - q\|_{W_2^{-1}[0,1]} < \varepsilon/C$, where $C$ is the same as in (17), we get

$$\begin{aligned}
\big\langle T_{\tilde{q}}(\lambda)y, y \big\rangle &\geq \big\langle T_q(\lambda)y, y \big\rangle - \varepsilon\, \|y\|^2_{W_2^1[0,1]} \\
&\geq \big\langle T_q(\lambda)y, y \big\rangle - \varepsilon\, \|y\|^2_{W_2^1[0,1]} \\
&\quad - \varepsilon \cdot \Big\langle T_{2q}\big(\lambda_1(2q)\big)y, y \Big\rangle - \varepsilon k_0^2 y^2(0) - \varepsilon k_1^2 y^2(1) \\
&= (1 - 2\varepsilon) \cdot \left\langle T_q \left( \frac{\lambda + \varepsilon \cdot [1 - \lambda_1(2q)]}{1 - 2\varepsilon} \right) y, y \right\rangle .
\end{aligned}$$

Consequently, from the variational principle 2.2.1 it follows that any $\lambda > \lambda_n(\tilde{q})$ satisfies

$$\frac{\lambda + \varepsilon \cdot [1 - \lambda_1(2q)]}{1 - 2\varepsilon} > \lambda_n(q).$$

Since we can choose $\lambda$ arbitrarily close to $\lambda_n(\tilde{q})$, we have

$$\lambda_n(\tilde{q}) \geq (1 - 2\varepsilon)\, \lambda_n(q) - \varepsilon \cdot \big[1 - \lambda_1(2q)\big].$$

By the same method we get

$$\lambda_n(\tilde{q}) \leq (1 + 2\varepsilon)\, \lambda_n(q) + \varepsilon \cdot \big[1 - \lambda_1(2q)\big].$$

□

**2.3.** Let $\Gamma_1$ be the closure in $W_2^{-1}[0,1]$ of the set $A_1$ defined by (4). Put by definition

$$\Lambda(X) \rightleftharpoons \big\{ \lambda \in \mathbb{R} : (\exists q \in X) \quad \lambda = \lambda_1(q) \big\},$$

where $X \subseteq W_2^{-1}[0,1]$ is some set of generalized functions. The set $\Lambda(X)$ is formed by all the possible values of $\lambda_1(q)$ for all $q \in X$. By $-X$ we, as usually, denote the set

$$\big\{ q \in W_2^{-1}[0,1] : (\exists r \in X) \quad q = -r \big\}.$$

**2.3.1.** *Suppose* $X$ *is a dense subset of* $\Gamma_1$, *then the closures of* $\Lambda(\pm X)$ *and* $\Lambda(\pm\Gamma_1)$ *coincide.*

**2.3.2.** *The extrema* $m_1^\pm \rightleftharpoons \inf \Lambda(\pm A_1)$ *and* $M_1^\pm \rightleftharpoons \sup \Lambda(\pm A_1)$, *defined in 1.2, satisfy the equalities* $m_1^\pm = \inf \Lambda(\pm \Gamma_1)$ *and* $M_1^\pm = \sup \Lambda(\pm \Gamma_1)$.

The proposition 2.3.1 immediately follows from 2.2.2. The proposition 2.3.2 immediately follows from 2.3.1.

**2.3.3.** *The set* $\Gamma_1$ *consists of all non-negative*[2] *distributions* $q \in W_2^{-1}[0,1]$ *such that* $\int_0^1 q\,dx = 1$.

P r o o f. Since for any $q \in \Gamma_1$ there exists a sequence of functions from $A_1$ such that its limit equals $q$, it follows that the generalized function $q$ is non-negative and satisfies $\int_0^1 q\,dx = 1$.

Let us prove the converse. Suppose $q \in W_2^{-1}[0,1]$ is a non-negative generalized function and satisfies $\int_0^1 q\,dx = 1$. Then (see [6], [8, § 2.3]) there exists a function $u \in L_2[0,1]$ such that

$$(\forall y \in W_2^1[0,1]) \qquad \int_0^1 qy\,dx = -\int_0^1 uy'\,dx + y(1). \tag{23}$$

Put by definition

$$\Pi_{\gamma,\eta,\theta}(x) \rightleftharpoons \begin{cases} \frac{x-\gamma}{\eta-\gamma} & \text{for } x \in [\gamma,\eta], \\ \frac{\theta-x}{\theta-\eta} & \text{for } x \in [\eta,\theta], \\ 0 & \text{otherwise} \end{cases}$$

for any reals $\gamma < \eta < \theta$. Suppose $0 < a < b < c < d < 1$. Substituting the functions $\Pi_{-1,0,a} + \Pi_{0,a,b}$, $\Pi_{a,b,c} + \Pi_{b,c,d}$ and $\Pi_{c,d,1} + \Pi_{d,1,2}$ for $y$ in (23), we get

$$0 \le \frac{1}{b-a}\int_a^b u\,dx \le \frac{1}{d-c}\int_c^d u\,dx \le 1.$$

From these inequalities it follows that the function $u \in L_2[0,1]$ is non-decreasing and satisfies $\operatorname{vrai\,inf}_{x\in[0,1]} u(x) \ge 0$ and $\operatorname{vrai\,sup}_{x\in[0,1]} u(x) \le 1$.

Since there exists a sequence $\{u_n\}_{n=0}^\infty$ of non-decreasing piecewise linear functions such that $u_n(0) = 0$, $u_n(1) = 1$ and $u = \lim_{n\to\infty} u_n$, it follows that $q = \lim_{n\to\infty} u_n'$, where $u_n' \in A_1$. □

**2.4.** Consider the function $F$ implicitely defined by the equation

$$\lambda_1\big(F(\mu,\zeta)\boldsymbol{\delta}_\zeta\big) = \mu, \tag{24}$$

where $\mu \in \mathbb{R}$ and $\zeta \in [0,1]$. The following three propositions give us some information about this function.

[2]The generalized function $q \in W_2^{-1}[0,1]$ is called non-negative if for any non-negative function $y \in W_2^1[0,1]$ the inequality $\langle q, y\rangle \ge 0$ holds.

**2.4.1.** *For any $\zeta \in [0,1]$ the function $F(\cdot,\zeta)$ is single-valued, strictly increasing, and its domain is the interval $(-\infty, f^+)$ with some $f^+ > 0$.*

P r o o f. For any $a \in \mathbb{R}$ there exists [8, Proposition 11] a positive eigenfunction $y \in \ker T_{a\boldsymbol{\delta}_\zeta}(\mu)$ corresponding to the eigenvalue $\mu \rightleftharpoons \lambda_1(a\boldsymbol{\delta}_\zeta)$, so for any $b < a$ we have

$$\langle T_{b\boldsymbol{\delta}_\zeta}(\mu)y, y\rangle = \langle T_{a\boldsymbol{\delta}_\zeta}(\mu)y, y\rangle + (b-a)\cdot y^2(\zeta) < 0.$$

Using 2.2.1, we now get $\lambda_1(b\boldsymbol{\delta}_\zeta) < \mu$. So the function $F(\cdot,\zeta)$ is the inverse of the strictly increasing and, according to 2.2.2, continuous map $a \mapsto \lambda_1(a\boldsymbol{\delta}_\zeta)$. Therefore, the function $F(\cdot,\zeta)$ is single-valued and strictly increasing.

Further, for any $a \in \mathbb{R}$ from the equality

$$\Big\langle T_{a\boldsymbol{\delta}_\zeta}\big(a+k_0^2+k_1^2\big)1, 1\Big\rangle = a - \big(a+k_0^2+k_1^2\big) + k_0^2 + k_1^2 = 0$$

and the proposition 2.2.1 it follows that $\lambda_1(a\boldsymbol{\delta}_\zeta) \le a + k_0^2 + k_1^2$. Therefore, the domain of $F(\cdot,\zeta)$ is unbounded from below. Also for any $a > 0$ we have $\lambda_1(a\boldsymbol{\delta}_\zeta) > 0$, so the right bound of $\operatorname{dom} F(\cdot,\zeta)$ is positive. □

**2.4.2.** *The function $F$ is continuous.*

P r o o f. Consider an arbitrary point $(\mu_0,\zeta_0) \in \operatorname{dom} F$ and suppose $a^\pm$ satisfy $a^- < F(\mu_0,\zeta_0) < a^+$. For any point $(\mu,\zeta) \in \mathbb{R}\times[0,1]$ sufficiently close to $(\mu_0,\zeta_0)$ from 2.4.1 and 2.2.2 we obtain the inequalities $\lambda_1(a^-\boldsymbol{\delta}_\zeta) < \mu < \lambda_1(a^+\boldsymbol{\delta}_\zeta)$. Hence there exists $a \in (a^-, a^+)$ such that $\mu = \lambda_1(a\boldsymbol{\delta}_\zeta)$, so for the point $(\mu,\zeta)$ the equation (24) has a solution $F(\mu,\zeta) = a$. □

**2.4.3.** *A point $(\mu,\zeta) \in (0,+\infty)\times[0,1]$ belongs to domain of the function $F$ if and only if the following conditions hold:*

$$\sqrt{\mu}\cdot(\zeta-\alpha_\mu) \in (-\pi/2,\pi/2), \qquad \sqrt{\mu}\cdot(1-\beta_\mu-\zeta) \in (-\pi/2,\pi/2), \tag{25}$$

*where $\alpha_\mu$ and $\beta_\mu$ are defined by (6). In this case the equality*

$$F(\mu,\zeta) = \sqrt{\mu}\cdot\Big\{\tan\big[\sqrt{\mu}\cdot(\zeta-\alpha_\mu)\big] + \tan\big[\sqrt{\mu}\cdot(1-\beta_\mu-\zeta)\big]\Big\} \tag{26}$$

*holds.*

*For any $\zeta \in [0,1]$ the equality*

$$F(0,\zeta) = -\frac{k_0^2}{1+k_0^2\zeta} - \frac{k_1^2}{1+k_1^2(1-\zeta)} \tag{27}$$

*holds.*

*For any $\mu < 0$ and $\zeta \in [0,1]$ the equality*

$$F(\mu,\zeta) = -\sqrt{|\mu|}\cdot\Big\{G\left(\sqrt{|\mu|}, k_0^2, \zeta\right) + G\left(\sqrt{|\mu|}, k_1^2, 1-\zeta\right)\Big\}, \tag{28}$$

*where*

$$G(\nu,\kappa,x) \rightleftharpoons \begin{cases} \tanh\left(\nu x + \ln\sqrt{\dfrac{\nu+\kappa}{\nu-\kappa}}\right) & \textit{for } \nu > \kappa, \\ 1 & \textit{for } \nu = \kappa, \\ \coth\left(\nu x + \ln\sqrt{\dfrac{\kappa+\nu}{\kappa-\nu}}\right) & \textit{for } \nu < \kappa, \end{cases}$$

*holds.*

P r o o f. Consider $\mu \in \mathbb{R}$ and $\zeta \in (0,1)$ such that $(\mu,\zeta) \in \operatorname{dom} F$. According to (20)–(22), the equality $T_q(\mu)y = 0$, where $q \rightleftharpoons F(\mu,\zeta)\boldsymbol{\delta}_\zeta$, is equivalent to the boundary problem

$$-y'' = \mu y \qquad \text{at } (0,\zeta) \cup (\zeta,1), \tag{29}$$

$$y'(\zeta+0) - y'(\zeta-0) = F(\mu,\zeta)y(\zeta), \tag{30}$$

$$y'(0) - k_0^2 y(0) = y'(1) + k_1^2 y(1) = 0. \tag{31}$$

From [8, Proposition 11] and (24) it follows that any non-trivial solution to the problem (29)–(31) is constant-sign.

In the case $\mu > 0$ any solution to the problem (29), (31) has the form

$$y(x) = \begin{cases} A \cdot \cos\bigl[\sqrt{\mu}\cdot(1-\beta_\mu-\zeta)\bigr]\cdot\cos\bigl[\sqrt{\mu}\cdot(x-\alpha_\mu)\bigr] & \text{for } x < \zeta, \\ A \cdot \cos\bigl[\sqrt{\mu}\cdot(1-\beta_\mu-x)\bigr]\cdot\cos\bigl[\sqrt{\mu}\cdot(\zeta-\alpha_\mu)\bigr] & \text{for } x > \zeta, \end{cases} \tag{32}$$

where $A$ is some constant. This function is constant-sign if and only if the conditions (25) hold. Using (30), we now get (26). The values $\zeta \in \{0,1\}$ are finally included in the consideration using the propositions 2.4.2 and 2.2.2.

The cases $\mu = 0$ and $\mu < 0$ are considered on the base of (29)–(31) by analogous way using the solution

$$y(x) = \begin{cases} A \cdot \bigl[1 + k_1^2(1-\zeta)\bigr]\cdot\bigl[1 + k_0^2 x\bigr] & \text{for } x < \zeta, \\ A \cdot \bigl[1 + k_1^2(1-x)\bigr]\cdot\bigl[1 + k_0^2 \zeta\bigr] & \text{for } x > \zeta \end{cases} \tag{33}$$

in the case $\mu = 0$, and the solution

$$y(x) = \begin{cases} A \cdot g\left(\sqrt{|\mu|}, k_1^2, 1-\zeta\right)\cdot g\left(\sqrt{|\mu|}, k_0^2, x\right) & \text{for } x < \zeta, \\ A \cdot g\left(\sqrt{|\mu|}, k_1^2, 1-x\right)\cdot g\left(\sqrt{|\mu|}, k_0^2, \zeta\right) & \text{for } x > \zeta, \end{cases} \tag{34}$$

where

$$g(\nu,\kappa,x) \rightleftharpoons \begin{cases} \cosh\left(\nu x + \ln\sqrt{\dfrac{\nu+\kappa}{\nu-\kappa}}\right) & \text{for } \nu > \kappa, \\ e^{\nu x} & \text{for } \nu = \kappa, \\ \sinh\left(\nu x + \ln\sqrt{\dfrac{\kappa+\nu}{\kappa-\nu}}\right) & \text{for } \nu < \kappa, \end{cases}$$

in the case $\mu < 0$. □

## 3. Proofs of the main results

**3.1.** In this section we prove Theorems 1.3.1–1.3.4. We use the notation

$$\Omega^+(y) \rightleftharpoons \Big\{x \in [0,1] : y(x) = \sup\nolimits_{t\in[0,1]} y(t)\Big\},$$
$$\Omega^-(y) \rightleftharpoons \Big\{x \in [0,1] : y(x) = \inf\nolimits_{t\in[0,1]} y(t)\Big\},$$

where $y \in W_2^1[0,1]$ is an arbitrary positive function. Also we take into account proposition 2.3.2.

**3.2. Proof of Theorem 1.3.1.** Consider some potential $q^* \in \Gamma_1$, and some positive eigenfunction $y \in \ker T_{q^*}\big(\lambda_1(q^*)\big)$. Suppose that the support of the generalized function $q^*$ is a subset of $\Omega^+(y)$. Then for any $q \in \Gamma_1$ we, using 2.3.3, have

$$\begin{aligned} 0 &= \Big\langle T_{q^*}\big(\lambda_1(q^*)\big)y, y\Big\rangle \\ &= \int_0^1 \big[(y')^2 - \lambda_1(q^*)\, y^2\big]\, dx + \sup_{x\in[0,1]} y^2(x) + k_0^2\, y^2(0) + k_1^2\, y^2(1) \\ &\geq \int_0^1 \Big[(y')^2 + \big(q - \lambda_1(q^*)\big)\, y^2\Big] dx + k_0^2\, y^2(0) + k_1^2\, y^2(1), \end{aligned}$$

hence
$$\Big\langle T_q\big(\lambda_1(q^*)\big)y, y\Big\rangle \leq 0.$$

It follows that $\lambda_1(q) \leq \lambda_1(q^*)$, therefore $\lambda_1(q^*) = M_1^+$. Thus we have proved that $M_1^+$ is attained at any potential $q^*$ such that $\operatorname{supp} q^* \subseteq \Omega^+(y)$.

Suppose that $\Omega^+(y) = [\tau_0, \tau_1]$, where $\tau_0 \neq \tau_1$. Also suppose that the potential $q^*$ is summable and has the form

$$q^*(x) = \begin{cases} \mu & \text{for } x \in [\tau_0,\, \tau_1], \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu$ is some positive constant. Since $y''(x) = 0$ for all $x \in (\tau_0, \tau_1)$, it follows that $\mu = \lambda_1(q^*)$. Therefore, the eigenfunction $y$ has the form

$$y(x) = \begin{cases} A \cdot \cos\big[\sqrt{\mu} \cdot (x - \alpha_\mu)\big] & \text{for } x < \tau_0, \\ B & \text{for } x \in [\tau_0,\, \tau_1], \\ C \cdot \cos\big[\sqrt{\mu} \cdot (1 - \beta_\mu - x)\big] & \text{for } x > \tau_1, \end{cases}$$

where $A$, $B$ and $C$ are some positive constants, and $\alpha_\mu$, $\beta_\mu$ are defined by (6). From the continuity of $y'$ it follows that $\tau_0 = \alpha_\mu$ and $\tau_1 = 1 - \beta_\mu$, hence $A = B = C$. Finally, from the condition $\int_0^1 q^*\, dx = 1$ we have the equation (7).

To conclude the proof, it remains to note that the equation (7) has a unique solution, because $\alpha_\mu$ and $\beta_\mu$, considered as functions of $\mu > 0$, are non-negative, continuous, non-increasing and tend to zero as $\mu \to +\infty$.

**3.3. Proof of Theorem 1.3.2.** Consider some potential $q^* \in -\Gamma_1$, and some positive eigenfunction $y \in \ker T_{q^*}\big(\lambda_1(q^*)\big)$. Suppose that $\operatorname{supp} q^* \subseteq \Omega^-(y)$. Then for any $q \in -\Gamma_1$ we, using 2.3.3, have

$$\begin{aligned} 0 &= \Big\langle T_{q^*}\big(\lambda_1(q^*)\big)y, y\Big\rangle \\ &= \int_0^1 \big[(y')^2 - \lambda_1(q^*)\, y^2\big]\, dx \; - \inf_{x\in[0,1]} y^2(x) + k_0^2\, y^2(0) + k_1^2\, y^2(1) \\ &\geq \int_0^1 \Big[(y')^2 + \big(q - \lambda_1(q^*)\big)\, y^2\Big] dx + k_0^2\, y^2(0) + k_1^2\, y^2(1), \end{aligned}$$

hence

$$\Big\langle T_q\big(\lambda_1(q^*)\big)y, y\Big\rangle \leq 0.$$

It follows that $\lambda_1(q) \leq \lambda_1(q^*)$, therefore $\lambda_1(q^*) = M_1^-$. Thus we have proved that $M_1^-$ is attained at any potential $q^*$ such that $\operatorname{supp} q^* \subseteq \Omega^-(y)$.

Suppose $k_0^2 + k_1^2 \leq 1$. Consider the generalized function

$$q^* \rightleftharpoons -k_0^2 \boldsymbol{\delta}_0 - k_1^2 \boldsymbol{\delta}_1 - \big(1 - k_0^2 - k_1^2\big),$$

which in this case belongs to $-\Gamma_1$. Using (14), we get that the first eigenfunction of the pencil $T_{q^*}$ is $y \equiv \text{const}$, so $\operatorname{supp} q^* \subseteq \Omega^-(y)$. It follows that $M_1^-$ is attained at the potential $q^*$ and is equal to the corresponding first eigenvalue

$$\lambda_1(q^*) = k_0^2 + k_1^2 - 1.$$

Suppose

$$k_0^2 + k_1^2 \geq 1, \tag{35}$$

$$k_1^2 - k_0^2 \leq 1. \tag{36}$$

Consider the generalized function $q^* \rightleftharpoons -\big(1 + k_0^2 - k_1^2\big)\boldsymbol{\delta}_0/2 - \big(1 - k_0^2 + k_1^2\big)\boldsymbol{\delta}_1/2$, which, due to (36), belongs to $-\Gamma_1$. For such $q^*$ the equation $T_{q^*}(\lambda)y = 0$ is equivalent to the problem (8), (9). The first eigenvalue $\lambda_1(q^*)$, due to (35) and (9), is non-negative and the corresponding eigenfunction is

$$y(x) \equiv \cos\left[\sqrt{\lambda_1(q^*)} \cdot (x - \zeta)\right], \tag{37}$$

where $\zeta = 1/2$. Hence $\operatorname{supp} q^* \subseteq \Omega^-(y)$. It follows that $M_1^-$ is attained at the potential $q^*$ and is equal to the corresponding first eigenvalue $\lambda_1(q^*)$.

Suppose $k_1^2 - k_0^2 \geq 1$. Consider the generalized function $q^* \rightleftharpoons -\boldsymbol{\delta}_1 \in -\Gamma_1$. For such $q^*$ the equation $T_{q^*}(\lambda)y = 0$ is equivalent to the problem (8), (10).

The corresponding first eigenfunction is defined by (37), where $\zeta \in [0, 1/2]$, since $k_1^2 - 1 \geq k_0^2$. Hence $\operatorname{supp} q^* \subseteq \Omega^-(y)$. It follows that $M_1^-$ is attained at the potential $q^*$ and is equal to the corresponding first eigenvalue $\lambda_1(q^*)$.

**3.4. Proof of Theorem 1.3.3.** Consider some potential $q \in \Gamma_1$, and some positive eigenfunction $y \in \ker T_q\big(\lambda_1(q)\big)$. Then for any $\lambda > \lambda_1(q)$, according to 2.3.3, we have

$$0 > \int_0^1 \big[(y')^2 + (q - \lambda)\, y^2\big]\, dx + k_0^2\, y^2(0) + k_1^2\, y^2(1)$$

$$\geq \int_0^1 \big[(y')^2 - \lambda y^2\,\big]\, dx + \inf_{x \in [0,1]} y^2(x) + k_0^2\, y^2(0) + k_1^2\, y^2(1).$$

It follows that there exists $\zeta \in [0, 1]$ such that

$$\int_0^1 \big[(y')^2 + (\boldsymbol{\delta}_\zeta - \lambda)\, y^2\big]\, dx + k_0^2\, y^2(0) + k_1^2\, y^2(1) < 0.$$

So for any $\lambda > m_1^+$ there exists $\zeta \in [0, 1]$ such that $\lambda_1(\boldsymbol{\delta}_\zeta) < \lambda$. Hence, using 2.3.3, we get $m_1^+ = \inf_{x \in [0,1]} \lambda_1(\boldsymbol{\delta}_x)$. This equality is equivalent, according to 2.4.1, to the following fact: $F(m_1^+, x)$ is defined for all $x \in [0, 1]$ and satisfies $\sup_{x \in [0,1]} F(m_1^+, x) = 1$.

Since $m_1^+ > 0$, from 2.4.3 it follows that if $\mu = m_1^+$, then for any $\zeta \in [0, 1]$ the conditions (25) hold. According to (26), (25) and

$$\frac{\partial F(\mu, \zeta)}{\partial \zeta} \equiv \mu \cdot \frac{\cos^2[\sqrt{\mu} \cdot (1 - \beta_\mu - \zeta)] - \cos^2[\sqrt{\mu} \cdot (\zeta - \alpha_\mu)]}{\cos^2[\sqrt{\mu} \cdot (\zeta - \alpha_\mu)] \cdot \cos^2[\sqrt{\mu} \cdot (1 - \beta_\mu - \zeta)]}, \tag{38}$$

it follows that the function $F(\mu, \cdot)$ can have at some point $\zeta \in (0, 1)$ a local extremum satisfying $F(\mu, \zeta) > 0$ only if $\zeta = (1 - \beta_\mu + \alpha_\mu)/2$, $\zeta > \alpha_\mu$ and $\zeta < 1 - \beta_\mu$. But this conditions imply, according to (38), that such $\zeta$ must be a point of strict local minimum of the function $F(\mu, \cdot)$. Therefore, $F(\mu, \cdot)$ cannot have a supremum in $(0, 1)$, so we get $m_1^+ = \inf\big\{\lambda_1(\boldsymbol{\delta}_0), \lambda_1(\boldsymbol{\delta}_1)\big\}$. Note that for the potential $q^* \rightleftharpoons \boldsymbol{\delta}_i$, where $i \in \{0, 1\}$, the equation $T_{q^*}(\lambda) y = 0$ is equivalent to the problem

$$-y'' = \lambda y,$$

$$y'(0) - \big[k_0^2 + (1 - i)\big]\, y(0) = y'(1) + \big[k_1^2 + i\big]\, y(1) = 0.$$

Therefore, we have

$$\frac{\lambda_1(\boldsymbol{\delta}_i) - k_0^2 k_1^2 - k_{1-i}^2}{k_0^2 + k_1^2 + 1} = \sqrt{\lambda_1(\boldsymbol{\delta}_i)} \cot \sqrt{\lambda_1(\boldsymbol{\delta}_i)},$$

so $m_1^+ = \lambda_1(\boldsymbol{\delta}_1)$.

**3.5. Proof of Theorem 1.3.4.** Consider some potential $q \in -\Gamma_1$, and some positive eigenfunction $y \in \ker T_q\big(\lambda_1(q)\big)$. Then for any $\lambda > \lambda_1(q)$, according to 2.3.3, we have

$$\begin{aligned} 0 &> \int_0^1 \big[(y')^2 + (q-\lambda)\, y^2\big]\, dx + k_0^2\, y^2(0) + k_1^2\, y^2(1) \\ &\geq \int_0^1 \big[(y')^2 - \lambda y^2\,\big]\, dx \, - \sup_{x\in[0,1]} y^2(x) + k_0^2\, y^2(0) + k_1^2\, y^2(1). \end{aligned}$$

It follows that there exists $\zeta \in [0,1]$ such that

$$\int_0^1 \big[(y')^2 + (-\boldsymbol{\delta}_\zeta - \lambda)\, y^2\big]\, dx + k_0^2\, y^2(0) + k_1^2\, y^2(1) < 0.$$

So for any $\lambda > m_1^-$ there exists $\zeta \in [0,1]$ such that $\lambda_1(-\boldsymbol{\delta}_\zeta) < \lambda$. Hence, using 2.3.3, we get $m_1^- = \inf_{x\in[0,1]} \lambda_1(-\boldsymbol{\delta}_x)$. This equality is equivalent, according to 2.4.1, to the following fact: $F(m_1^-, x)$ is defined for all $x \in [0,1]$ and satisfies $\sup_{x\in[0,1]} F(m_1^-, x) = -1$.

For any fixed value $\mu \in \mathbb{R}$ we consider the conditions

$$F(\mu,\zeta) \;<\; 0, \tag{39}$$

$$\partial F(\mu,\zeta)/\partial\zeta \;=\; 0. \tag{40}$$

It is clear that some point $\zeta \in (0,1)$ can satisfy the equalities $F(\mu,\zeta) = \sup_{x\in[0,1]} F(\mu,x) = -1$ only if (39) and (40) hold.

Suppose $\mu > 0$. Then, according to (38), (26), (32) and (30), for any point $\zeta \in (0,1)$ satisfying (39) the condition (40) holds if and only if the problem

$$-y'' = \mu y \qquad \text{at } \ (0,\zeta) \cup (\zeta,1), \tag{41}$$

$$\begin{aligned} y'(0) - k_0^2 y(0) &= 2y'(\zeta - 0) + F(\mu,\zeta) y(\zeta) \\ &= 2y'(\zeta+0) - F(\mu,\zeta) y(\zeta) = y'(1) + k_1^2 y(1) = 0 \end{aligned} \tag{42}$$

has a continuous positive solution. Besides, for any point $\zeta \in (0,1)$ satisfying (39) and (40) we have

$$\alpha_\mu > \zeta > 1 - \beta_\mu. \tag{43}$$

Therefore, according to (38) and (25), this stationary point $\zeta$ is a strict maximum of $F(\mu,\cdot)$. Since for any $x \in [0,1]$, using (43), we get

$$-\pi/2 < -\sqrt{\mu}\alpha_\mu \leq \sqrt{\mu}\cdot(x-\alpha_\mu) < \sqrt{\mu}\cdot(x - 1 + \beta_\mu) \leq \sqrt{\mu}\beta_\mu < \pi/2,$$

it follows from the proposition 2.4.3 that the function $F(\mu,\cdot)$ is defined everywhere on $[0,1]$.

Suppose $\mu = 0$. Let us use the same method as in the previous case, changing (26) to (27), and (32) to (33). Then we get that for any point $\zeta \in (0,1)$ satisfying (39) the condition (40) holds if and only if the problem (41), (42) has a continuous positive solution. Using (27) we also get that the second derivative of $F(0,\cdot)$ is negative. Hence any stationary point $\zeta \in (0,1)$ is a strict maximum of $F(0,\cdot)$.

Suppose $\mu \in \bigl(-k_0^4, 0\bigr)$. Then, using (28), we get

$$\frac{\partial F(\mu,\zeta)}{\partial \zeta} \equiv -\mu \Bigl\{ \sinh^{-2}\Bigl(\sqrt{|\mu|}\zeta + \alpha_\mu\Bigr) - \sinh^{-2}\Bigl(\sqrt{|\mu|}(1-\zeta) + \beta_\mu\Bigr)\Bigr\},$$

where

$$\alpha_\mu \rightleftharpoons \frac{1}{2}\ln\frac{k_0^2 + \sqrt{|\mu|}}{k_0^2 - \sqrt{|\mu|}}, \quad \beta_\mu \rightleftharpoons \frac{1}{2}\ln\frac{k_1^2 + \sqrt{|\mu|}}{k_1^2 - \sqrt{|\mu|}}.$$

Therefore, according to (34), for any point $\zeta \in (0,1)$ satisfying (39) the condition (40) holds if and only if the problem (41), (42) has a continuous positive solution. Since $\partial^2 F(\mu,\zeta)/\partial\zeta^2 < 0$, it follows that any stationary point $\zeta \in (0,1)$ is a strict maximum of $F(\mu,\cdot)$.

Suppose $0 > \mu = -k_0^4 = -k_1^4$. Then the function $F(\mu,\cdot)$ is a negative constant, and for any point $\zeta \in (0,1)$ problem (41), (42) has a continuous positive solution.

Suppose $\mu \in \bigl[-k_1^4, -k_0^4\bigr]$, also $\mu < 0$ and $k_1 > k_0$. Then from (28) and (34) it follows that $\partial F(\mu,\zeta)/\partial\zeta < 0$, and the problem (41), (42) has no positive solutions for any $\zeta \in (0,1)$.

Suppose $\mu < -k_1^4$. Then, using (28), we get

$$\frac{\partial F(\mu,\zeta)}{\partial \zeta} \equiv \mu \Bigl\{ \cosh^{-2}\Bigl(\sqrt{|\mu|}\zeta + \alpha_\mu\Bigr) - \cosh^{-2}\Bigl(\sqrt{|\mu|}(1-\zeta) + \beta_\mu\Bigr)\Bigr\},$$

where

$$\alpha_\mu \rightleftharpoons \frac{1}{2}\ln\frac{\sqrt{|\mu|} + k_0^2}{\sqrt{|\mu|} - k_0^2}, \quad \beta_\mu \rightleftharpoons \frac{1}{2}\ln\frac{\sqrt{|\mu|} + k_1^2}{\sqrt{|\mu|} - k_1^2}.$$

Therefore, according to (34), for any point $\zeta \in (0,1)$ satisfying (39) the condition (40) holds if and only if problem (41), (42) has a continuous positive solution. Since $\partial^2 F(\mu,\zeta)/\partial\zeta^2 > 0$, it follows that any stationary point $\zeta \in (0,1)$ is a strict minimum of $F(\mu,\cdot)$.

From the proposition 2.4.1 we also get that for any $\mu \le 0$ the function $F(\mu,\cdot)$ is defined everywhere on $[0,1]$.

Combining all this, we obtain the following: the existence of a continuous positive solution to the problem (12), (13) for some $\mu \ge -k_0^4$ and $\zeta \in (0,1)$ implies that $F(\mu,\zeta) = -1$, the function $F(\mu,\cdot)$ is defined everywhere on $[0,1]$, and $\sup_{x\in[0,1]} F(\mu,x) \le -1$. Therefore, $m_1^- = \lambda_1(-\boldsymbol{\delta}_\zeta)$. In converse, if for any $\mu \ge -k_0^4$ and $\zeta \in (0,1)$ the positive solution of (12), (13) does not exist, we get $m_1^- = \inf\bigl\{\lambda_1(-\boldsymbol{\delta}_0), \lambda_1(-\boldsymbol{\delta}_1)\bigr\}$. From the equation

$$\lambda_1(-\boldsymbol{\delta}_i) - k_0^2 k_1^2 + k_{1-i}^2 = \bigl(k_0^2 + k_1^2 - 1\bigr)\cdot\psi\bigl(\lambda_1(-\boldsymbol{\delta}_i)\bigr),$$

where $i \in \{0,1\}$ and

$$\psi(x) \rightleftharpoons \begin{cases} \sqrt{x}\cot\sqrt{x} & \text{for } x > 0, \\ 1 & \text{for } x = 0, \\ \sqrt{|x|}\coth\sqrt{|x|} & \text{for } x < 0, \end{cases}$$

we obtain that $\inf\big\{\lambda_1(-\boldsymbol{\delta}_0), \lambda_1(-\boldsymbol{\delta}_1)\big\} = \lambda_1(-\boldsymbol{\delta}_0)$.

**3.6.** Now we get some conditions for the existence of a continuous positive solution to the problem (12), (13) considered in Theorem 1.3.4.

Suppose $\mu_0(\zeta)$, where $\zeta \in (0,1]$, is the minimal eigenvalue of the problem

$$-y'' = \lambda y, \tag{44}$$

$$y'(0) - k_0^2 y(0) = 2y'(\zeta) - y(\zeta) = 0, \tag{45}$$

and suppose $\mu_1(\zeta)$, where $\zeta \in [0,1)$, is the minimal eigenvalue of the problem (44) and

$$2y'(\zeta) + y(\zeta) = y'(1) + k_1^2 y(1) = 0.$$

It is clear that for some $\mu \in \mathbb{R}$ and $\zeta \in (0,1)$ a continuous positive solution to (12), (13) exists if and only if the equalities $\mu_0(\zeta) = \mu_1(\zeta) = \mu$ hold.

**3.6.1.** *If $k_0^2 = 1/2$, then $\mu_0(\zeta) \equiv -1/4$.*

*If $k_0^2 > 1/2$, then the function $\mu_0$ strictly decreases and satisfies*

$$\lim_{\zeta\to 0} \mu_0(\zeta) = +\infty \quad \textit{and} \quad \mu_0(1) > -1/4.$$

*If $k_0^2 < 1/2$, then for any $\zeta \in (0,1]$ the inequality $\mu_0(\zeta) < -1/4$ holds.*

P r o o f. Suppose $k_0^2 = 1/2$. Then for any $\zeta \in (0,1]$ the problem (44), (45) has the positive eigenfunction $y(x) \equiv e^{x/2}$ corresponding to the eigenvalue $-1/4$.

Suppose $k_0^2 > 1/2$. Since the eigenvalues of the problem (44), (45) increase by $k_0^2$, it follows that $\mu_0(\zeta) > -1/4$. Then let $y_0 \in W_2^1[0,\zeta]$ be an eigenfunction of the problem (44), (45) corresponding to the eigenvalue $\mu_0(\zeta)$. Continuing the function $y_0$ for any $\theta \in (\zeta, 1]$ to the interval $(\zeta,\theta]$ in the form $y(x) \rightleftharpoons y_0(\zeta)e^{(x-\zeta)/2}$, for the obtained function $y \in W_2^1[0,\theta]$ we get

$$\int_0^\theta \big[(y')^2 - \mu_0(\zeta)\, y^2\big]\, dx + k_0^2\, y^2(0) - \frac{y^2(\theta)}{2} = \big[-1/4 - \mu_0(\zeta)\big] \cdot \big[e^{\theta-\zeta} - 1\big] \cdot y^2(\zeta) < 0,$$

hence $\mu_0(\theta) < \mu_0(\zeta)$.

Finally, for $\zeta \to 0$ we have uniform by $y \in W_2^1[0,\zeta]$ asymptotic estimate

$$\begin{aligned}&\int_0^\zeta (y')^2\,dx + k_0^2\,y^2(0) - \frac{y^2(\zeta)}{2}\\ &= \left[\int_0^\zeta \frac{(y')^2}{2}\,dx + (k_0^2 - 1/2)\,y^2(0)\right] + \left[\int_0^\zeta \frac{(y')^2}{2}\,dx + \frac{y^2(0) - y^2(\zeta)}{2}\right]\\ &\geq \frac{k_0^2 - 1/2 + o(1)}{\zeta}\int_0^\zeta y^2\,dx - \frac{1}{2}\int_0^\zeta y^2\,dx,\end{aligned}$$

therefore, $\mu_0(\zeta) \geq [k_0^2 - 1/2 + o(1)] \cdot \zeta^{-1}$.

The inequality $\mu_0(\zeta) < -1/4$ for the case $k_0^2 < 1/2$ is proved likewise the inequality $\mu_0(\zeta) > -1/4$ for the case $k_0^2 > 1/2$. □

**3.6.2.** *If $k_1^2 = 1/2$, then $\mu_1(\zeta) \equiv -1/4$.*

*If $k_1^2 > 1/2$, then the function $\mu_1$ strictly increases and satisfies*

$$\lim_{\zeta\to 1} \mu_1(\zeta) = +\infty \quad \textit{and} \quad \mu_1(0) > -1/4.$$

*If $k_1^2 < 1/2$, then for any $\zeta \in [0,1)$ the inequality $\mu_1(\zeta) < -1/4$ holds.*

The proposition 3.6.2 is proved likewise 3.6.1.

Combining 3.6.1 and 3.6.2, we get the last proposition:

**3.6.3.** *The problem (12), (13) has a continuous positive solution for some $\mu \geq -k_0^4$ and $\zeta \in (0,1)$ if and only if this condition holds:*

$$k_0^2 > 1/2 \quad \textit{or} \quad k_0^2 = k_1^2 = 1/2.$$

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*Elena S. Karulina*
*Moscow State University of Economics*
*Statistics and Informatics*
*Nezhinskaja 7*
*Moscow 119501*
*RUSSIA*
*E-mail*: karulinaes@yandex.ru

*Anton A. Vladimirov*
*Dorodnitsyn Computing Center of RAS*
*Vavilova 40*
*Moscow 119991*
*RUSSIA*
*E-mail*: vladimi@mech.math.msu.su