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# PARTIAL COVERING OF A SPHERE WITH RANDOM NUMBER OF SPHERICAL CAPS

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ABSTRACT. In this paper we study sphere coverage issue. A sphere of radius one in a 3-dimensional Euclidean space is given. We consider random location of N spherical caps on a sphere, assuming that N is a discrete stochastic variable with a Poisson distribution. Using suitable difference equation, the expected area of the covered region is investigated.

### 1. Introduction

Since the 1930s there appear in the literature works on stochastic models of geometric problems, such as random arrangement of elements (points, lines, segments, arcs, circles, etc.) on the sections of straight, curves (e.g., districts), flat shapes, second degree surfaces (e.g., sphere in  $\mathbb{R}^3$ ) and the associated random distribution of the respective areas. Most of them has discrete character. More details can be found in the papers by Holst [10], Koschitzki [14], Maćkowiak-Lybacka [15], Mannion [19], Moran and Fazekas [20], and monographs of Kendall and Moran [13], Madria [18], and Santalo [21]. The topic of randomly covered surface area was considered by many authors, for example, by Dvoretzki [2], Flatto [5] and [6], Gilbert [7], Stevens [22], and Yadin and Zaks [24]. Proper mathematical models of such processes could be difference equations.

The background for difference equations can be found in the well-known monograph [1] by Agarwal, as well as in Elaydi [4], and Kelley and Peterson [12].

Recently, in [3], D u m e r studied covering of a given sphere of any radius in an n-dimensional Euclidean space with solid spheres of radius one. The author's

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objective was to design a covering of the lowest covering density, which defines the average number of solid spheres covering a point in a bigger sphere.

In [16] and [17], M a e h a r a considered N spherical caps, each of area  $4\pi p(N)$ , randomly distributed on the surface of a unit sphere  $S^n$  in the (n+1)-dimensional Euclidean space. The author investigated the intersection graphs of these random caps. For a simple graph  $G_E$  with edge set E, he considered independent random points on  $S^n$ , and studied the minimum value of the spherical distance between them.

The sphere coverage problem was also studied by S u g i m o t o and T a n e - m u r a, see [23]. The authors considered the sequential covering of identical spherical caps, so that none of them contains the center of another one.

The aim of this work is to obtain the probability characteristics of the random covering sphere areas by spherical caps. Sphere covering is considered according to the number of spherical caps. The expected area  $E_A$  of union of n spherical caps randomly distributed on the sphere is studied. This process is here described by the following difference equation

$$\Delta E_A(N=n) = 4\pi \sin^2 \frac{\alpha}{2} \cos^{2n} \frac{\alpha}{2},$$

where  $\Delta$  denotes the forward difference operator defined in the usually way, i.e.,  $\Delta x_n = x_{n+1} - x_n$  for  $x \colon \mathbb{N} \to \mathbb{R}$ .

There are numerous uses that constitute motivation for this research. For instance, the results are applicable in biology. Influenza virus can be considered as a sphere of radius  $50\mu\mu$ . This virus is attacked by antibodies, which are understood as the cylinders of radii  $37\mu\mu$  and heights that can be skipped. The antibodies attach themselves perpendicularly to the virus surface. Suppose that n such antibodies are attached to the surface of the virus, and the points of contact are randomly and independently placed on the sphere. The question is to find probability of sphere coverage by spherical caps, which allows the evolution of the likelihood of the virus destruction by the antibodies. Another example of application can be found in astronomy. Let  $X_i$  be the position of observatories on the surface of the Earth and  $\alpha$  be the angle of observations of these observatories. The sphere coverage area corresponds to the covering of the observation area on the sphere (in the sky).

Let  $S^2$  be a sphere of radius one in  $\mathbb{R}^3$  and points  $X_i, i \in \{1, 2, ..., n\}$  are randomly distributed over  $S^2$ . A spherical cap is a portion of a sphere cut off by a plane. By  $D(X_i, \alpha)$  we mean a spherical cap of the angular radius  $\alpha \in (0, \pi)$ with center at point  $X_i$  with no boundary (see Fig. 2 in [23]).

The probability space  $(M(S^2), B(M), P)$  is an appropriate model of distribution of random number N of spherical caps on sphere. Here  $M(S^2)$  is a space of Lebesgue measurable subsets on  $S^2 \subset \mathbb{R}^3$  with finite Lebesgue measure  $\mu$  of its subsets. An outcome is the result of a single execution of the model.

Since individual outcomes might be of little practical use, more complex events are used to characterize groups of outcomes. The collection of all such events is a  $\sigma$ -algebra. We consider Borel algebra B(M) of the class of all Lebesgue measurable sets on  $M(S^2)$ . There is a need to specify likelihood of occurrence of each event. This can be done using the probability measure function P. Here P is the assignment of geometrical probabilities to the events A, that is, a function from events to probability levels defined as follows

$$P(A) = \frac{\mu(A)}{\mu(S^2)}.$$
(1)

Note that stochastic geometry emphasizes the random geometrical objects themselves. For instance, there are different models for random lines, for random tessellations of the plane, and for random sets of points of a spatial Poisson process that can be centers of discs. So, the Poisson distribution can be applied to systems with a large number of possible events, each of which is rare. Such distribution is used in this paper. The Poisson distribution arises as the distribution of counts of occurrences of events in multidimensional intervals.

In Poisson point processes the probability that random value  $\xi \colon M(S^2) \to \mathbb{R}$ equals *i* is given by the following formula

$$P(\xi(A) = i) = \frac{(\lambda \mu(A))^i e^{-\lambda \mu(A)}}{i!},$$
(2)

where  $\mu(A)$  is the area of the region A. The positive real number  $\lambda \mu(A)$  is equal to the expected value of  $E(\xi(A))$ . So, we have

$$E(\xi(A)) = \lambda \mu(A). \tag{3}$$

The following law of total probability is explored in our investigations.

**LEMMA 1.** The event A in the probability space is given. Hence

$$P(A) = \sum_{i=0}^{\infty} P(A/\xi = i) P(\xi = i),$$
(4)

where  $P(A|\xi = i)$  is conditional probability and  $P(\xi = i)$  is probability of event  $\xi = i$ .

For discrete random variable  $\xi$  the expected value  $E(\xi(A))$  can be found using the following formula

$$E(\xi(A)) = \sum_{i=1}^{\infty} E(\xi(A)/\xi = i)P(\xi = i).$$
 (5)

Let  $X_i \neq X_j$  where  $i, j \in \{1, 2, ..., n\}$  and  $D(X_i, \alpha) \cap D(X_j, \alpha) \neq \emptyset$ . We call the points which belong to the boundary of both spherical caps the boundary intersection points. Any two spherical caps have either two boundary intersection

points or none. Boundary intersection point is not covered by any of these caps. Such point could be covered only by another spherical cap. Sphere  $S^2$  is covered by spherical caps if each boundary intersection point is covered by a spherical cap.

By a random variable  $\xi \colon M(S^2) \to \mathbb{R}$  of event A we mean a number of boundary intersection points in a fixed region  $A \subset S^2$  ( $\xi(A) < \infty$ ).

# 2. Covering a sphere with caps

We will investigate the expected value of the covered area by spherical caps randomly and independently distributed on the sphere. We assume that the Lebesgue measure of the cap is its area. Curved surface area of each spherical cap denoted by  $\mu(D(X_i, \alpha))$  equals

$$\mu(D(X_i,\alpha)) = 2\pi(1-\cos\alpha) = 4\pi\sin^2\frac{\alpha}{2}.$$
(6)

**LEMMA 2.** Let Y be any point on  $S^2$ . Probability p of the event A that  $Y \in D(X_i, \alpha)$  for exactly one i where  $i \in \{1, 2, 3, ..., N\}$  is

$$p = P(A) = \sin^2 \frac{\alpha}{2}.$$
(7)

Proof. Utilizing (6) and  $\mu(S^2) = 4\pi$  in (1), we get the thesis.

**LEMMA 3.** Probability P(C) of event C that point Y is covered by at least one of n spherical caps is

$$P(C) = 1 - (1 - p)^n \quad and \quad P(C^c) = (1 - p)^n.$$
(8)

Proof. Let  $A^c$  be complement event of A. Since P(A) = p is probability that point Y is covered by one cap then  $P(A^c) = 1 - p$  is probability that Y is not covered by a cap. Thus  $P(C^c) = (1 - p)^n$  is probability that Y is not covered by any cap. This implies that  $P(C) = 1 - (1 - p)^n$  is probability of the event that Y is covered by at least one of n caps.

Let us assume that N spherical caps are on  $S^2$ . The Poisson distribution of number of spherical caps will be used in the following lemma.

**LEMMA 4.** Probability of the event that given point is not covered by any of N spherical caps is

$$P(C^c) = e^{-4\lambda\pi p}.$$

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Proof. From (4), we obtain

$$P(C^{c}) = \sum_{n=0}^{\infty} P(C^{c}/N = n)P(N = n).$$

By (8), (2) and Maclaurin expansion of  $e^x$ , using  $\mu(S^2) = 4\pi$ , we get

$$=\sum_{n=0}^{\infty} (1-p)^n \frac{e^{-4\lambda\pi} (4\lambda\pi)^n}{n!} = e^{-4\lambda\pi} e^{4\lambda\pi - 4\lambda\pi p} = e^{-4\lambda\pi p}.$$

### Remark 1.

$$P(C) = 1 - e^{-4\lambda\pi p}.$$

Let  $Y_1, Y_2, \ldots, Y_k$  where  $k \in \{1, 2, 3, \ldots, N\}$  be given k points on  $S^2$ . Let B be an event that such k points are not covered by any spherical cap  $D(X_i, \alpha)$ ,  $i \in \{1, 2, 3, \ldots, N\}$  where distance  $Y_i Y_j$ ,  $i \neq j$ , is greater than or equal to  $2\alpha$ . We will find conditional probability P(B/N = n) of the event B that  $Y_k \notin D(X_i, \alpha)$  for  $i \in \{1, 2, 3, \ldots, N\}$  when the number of caps N equals n. By (4) and (2), we have

$$P(B) = \sum_{n=0}^{\infty} P(B/N = n) \frac{(4\lambda\pi)^n e^{-4\lambda\pi}}{n!}.$$
 (9)

On the other hand, using Remark 1 and independence of events, we get

$$P(B) = \left(1 - e^{-4\lambda\pi p}\right)^n.$$

Hence, by Binomial Formula, we obtain

$$P(B) = \sum_{r=0}^{k} \binom{k}{r} (-1)^{r} e^{-4r\lambda\pi p} = e^{-4\lambda\pi} \sum_{r=0}^{k} \binom{k}{r} (-1)^{r} e^{4\lambda\pi(1-rp)},$$

and Maclaurin expansion of  $e^x$  implies that

$$P(B) = e^{-4\lambda\pi} \sum_{r=0}^{k} {\binom{k}{r}} (-1)^{r} \sum_{n=0}^{\infty} \frac{(4\lambda\pi(1-rp))^{n}}{n!}$$
$$= e^{-4\lambda\pi} \sum_{n=0}^{\infty} \frac{(4\lambda\pi)^{n}}{n!} \sum_{r=0}^{k} {\binom{k}{r}} (-1)^{r} (1-rp)^{n}.$$

Comparing the above formula with (9), it yields

$$P(B/N = n) = \sum_{r=0}^{k} \binom{k}{r} (-1)^{r} (1 - rp)^{n}.$$

Note that S t e v e n s, in [22], proved the above formula in a different way.

Let us assume that exactly N spherical caps are distributed on the unit sphere. If the maximal area of N caps equal to  $4N\pi \sin^2 \frac{\alpha}{2}$  is smaller than the area of the sphere  $S^2$  then probability of the event that sphere  $S^2$  is covered by N spherical caps  $D(X_i, \alpha)$  where i = 1, 2, ... N equals zero. It means that  $4N\pi \sin^2 \frac{\alpha}{2}$  should be greater than or equal to the sphere area for possible covering.

Let us consider one spherical cap  $D(X_i, \alpha)$ . If  $X_j \in D(X_i, 2\alpha)$ , where  $i \neq j$ , then cap  $D(X_j, \alpha)$  has boundary intersection points with cap  $D(X_i, \alpha)$ . By  $N_{D(X_i,2\alpha)}$  we denote the number of spherical caps which have boundary intersection points with spherical cap  $D(X_i, \alpha)$ . By k we denote the number of boundary intersection points of cap  $D(X_i, \alpha)$  and other caps. We consider the event  $A_i$  that i different points  $X_j, j \in \{1, 2, 3, \ldots, N\}$ , are distributed on spherical cap  $D(X_i, 2\alpha)$ . By a random variable  $\xi \colon M(S^2) \to \mathbb{R}$  of this event, we mean the number of boundary intersection points in spherical cap  $D(X_i, 2\alpha)$ . So,  $\xi(A_i) = i$  and  $i = 2N_{D(X_i, 2\alpha)}$ . Hence, the expected value of the number of boundary intersection points of one spherical cap is  $E(\xi(A_i) = i)$ . We have

$$E(\xi(A_i) = i) = 2E(N_{D(X_i, 2\alpha)}).$$

From above and (3), we get

$$E(\xi(A_i) = i) = 8\lambda\pi\sin^2\alpha$$

independently of i.

Let K be a number of all boundary intersection points on the unit sphere  $S^2$ . Due to double counting of each boundary intersection point

$$K = \frac{1}{2} \sum_{i=0}^{N} k_i.$$

Firstly, we assume that any point of unit sphere  $S^2$  is covered by spherical caps. We will find the expected value  $E_K(\lambda)$  of K. Independence of events and (3) implies that

$$E_K(\lambda) = \frac{1}{2} E(\xi(A_i) = i) E(S^2) = 4\lambda\pi \sin^2 \alpha 4\lambda\pi = 16\pi^2 \lambda^2 \sin^2 \alpha.$$

By (7), we have

$$E_K(\lambda) = 16 \pi^2 \lambda^2 4 \sin^2 \frac{\alpha}{2} \left( 1 - \sin^2 \frac{\alpha}{2} \right) = 4\pi^2 \lambda^2 p (1-p).$$
(10)

Next, we will find the expected value E(K/N = n) of all boundary intersection points under condition that N is fixed. Set N = n. Using (5), we get

$$E_K(\lambda) = \sum_{n=0}^{\infty} E(K/N = n) \frac{(4\lambda\pi)^n e^{-\lambda 4\pi}}{n!}.$$
(11)

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On the other hand, by (10), we have

$$E_K(\lambda) = 4\pi^2 \lambda^2 p(1-p) e^{-4\lambda\pi} e^{4\lambda\pi}.$$

Using Maclaurin expansion of  $e^x$ , we obtain

$$E_K(\lambda) = 4p(1-p)e^{-4\lambda\pi} \sum_{n=0}^{\infty} \frac{\lambda^{n+2}(4\pi)^{n+2}}{n!}.$$

It follows

$$E_K(\lambda) = 4p(1-p)e^{-4\lambda\pi} \sum_{n=2}^{\infty} \frac{\lambda^n (4\pi)^n (n-1)n}{n!}$$

Due to the property of Poisson process, we get

$$E_K(\lambda) = 4p(1-p)e^{-4\lambda\pi} \sum_{n=0}^{\infty} \frac{\lambda^n (4\pi)^n (n-1)n}{n!}.$$
 (12)

Comparing (11) and (12) we have

$$E(K/N = n) = 4p(1 - p)n(n - 1).$$

## 3. Main result

The expected value  $E_A(N = n)$  of area of a part of unit sphere  $S^2$ , which is covered by N = n randomly distributed spherical caps  $D(X_i, \alpha), i \in \{1, \ldots, n\}$ , is studied.

**THEOREM 1.** The expected value  $E_A(N = n)$  is given by difference equation

$$\Delta E_A(N=n) = 4\pi \sin^2 \frac{\alpha}{2} \cos^{2n} \frac{\alpha}{2}.$$
(13)

Proof. Probability of the event  $C^c$  means that a given point on unit sphere is not covered by any spherical cap. From Remark 1 we have  $P(C) = 1 - e^{-4\lambda\pi p}$ . Hence, the expected value of covered area of a part of unit sphere  $S^2$ , denoted by  $E_A(\lambda)$ , is

$$E_A(\lambda) = 4\pi \left(1 - e^{-4\lambda\pi p}\right). \tag{14}$$

On the other hand, by (10), we have

$$E_A(\lambda) = \sum_{n=0}^{\infty} E_A(N=n) \frac{\lambda^n (4\pi)^n}{n!}.$$
(15)

Hence, by (14), we get

$$E_A(\lambda) = e^{-4\lambda\pi} e^{4\lambda\pi} 4\pi (1 - e^{-4\lambda\pi p}) = e^{-4\lambda\pi} 4\pi \Big( e^{4\lambda\pi} - e^{4\lambda\pi(1-p)} \Big).$$

Using Maclaurin expansion of  $e^x$ , we obtain

$$E_A(\lambda) = e^{-4\lambda\pi} 4\pi \left( \sum_{n=0}^{\infty} \frac{\lambda^n (4\pi)^n}{n!} - \frac{\lambda^n (4\pi)^n (1-p)^n}{n} \right)$$
  
=  $e^{-4\lambda\pi} 4\pi \sum_{n=0}^{\infty} \frac{\lambda^n (4\pi)^n}{n!} (1 - (1-p)^n).$ 

Comparing the above with (15), we get the equation

$$E_A(N=n) = 4\pi (1 - (1-p)^n),$$

and, by (7), we have

$$E_A(N=n) = 4\pi \left(1 - \left(1 - \sin^2 \frac{\alpha}{2}\right)^n\right).$$

Hence

$$E_A(N=n) = 4\pi \left(1 - \cos^{2n}\frac{\alpha}{2}\right).$$
 (16)

It is easy to check that sequence (16) fulfills difference equation (13).  $\Box$ 

**Remark 2.** Letting n go to  $\infty$  we get

$$\lim_{n \to \infty} E_A(N=n) = 4\pi.$$
(17)

P r o o f. Equality (17) follows directly from (16).

EXAMPLE 1. The following graph shows changes of covered area of unit sphere depending on number of spherical caps N and  $\alpha$ . For small values  $\alpha$  the covering of the sphere is small, even for large values of N.



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The above results are closely connected to those presented by G r o n e k in [8] and [9]. In [8], the random configuration of three spherical caps of constant radii on the unit sphere is considered. In [9], the author studied the case of random configuration of three spherical caps of different radii.

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