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CONDITIONS FOR FACTORIZATION OF LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

Klara R. Janglajew — Kim G. Valeev

ABSTRACT. The paper deals with a linear system of differential equations of the form

$$\frac{dX(t)}{dt} = AX(t) + \mu \sum_{k=1}^{n} A_k X(t+\tau_k)$$

with constant coefficients, a small parameter and complex deviating argument. Sufficient conditions for factorizing of this system are presented. These conditions are obtained by construction of an integral manifold of solutions to the considered system.

1. Introduction and preliminaries

Factorization of linear differential and difference operators is an effective method for analyzing linear ordinary differential and difference equations, see for instance [8], [10], [14].

Methods of factorization are dedicated to analytical and algebraic approaches to the problem of the integration of ordinary differential equations (see [2] and the references given there).

The contents of this paper is connected to the study of asymptotic properties of delay differential equations. There is a number of important papers on the subject, and we only mention a few: [3], [4], [11], [12].

The theory of linear differential equations with deviating argument is well elaborated. The background for linear functional-differential equations can be found in the well known monographs: [1], [9].

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Consider the linear system of equations with a complex deviating

$$\frac{dX(t)}{dt} = AX(t) + \mu \sum_{k=1}^{n} A_k X(t + \tau_k), \tag{1}$$

where μ is a small parameter, τ_k, $(k = 1, \dots n)$ are complex constants, and A, A_k, are nonsingular constant matrices, $\dim X(t) = m$. $A, A_k, X(t)$ are supposed to be complex.

We seek a solution of system (1) in the form

$$X(t) = e^{pt} K, \qquad K = const.$$

For the complex number p we obtain the equation

$$\det \left(pI - A - \mu \sum_{k=1}^{n} A_k e^{p\tau_k} \right) = 0.$$

In what follows we search for the factorization

$$pI - A - \mu \sum_{k=1}^{n} A_k e^{p\tau_k} = \big(pI - C(\mu)\big)\big(I + \mu G(p, \mu)\big), \tag{2}$$

where $G(p, \mu)$ is a matrix with integer elements. Equation (2) could be written in the form

$$C(\mu) - A - \mu \sum_{k=1}^{n} A_k e^{p\tau_k} = p\mu G(p, \mu) - \mu C(\mu) G(p, \mu).$$

By setting $\mu B(\mu) = C(\mu) - A$ we get the matrix equation

$$B(\mu) - pG(p, \mu) = F(p, \mu), \tag{3}$$

where

$$F(p, \mu) \equiv \sum_{k=1}^{n} A_k e^{p\tau_k} - \big(A + \mu B(\mu)\big) G(p, \mu).$$

The matrix equation (3) defines the matrices $G(p, \mu)$, $B(\mu)$ and it may be solved by the method of successive approximations. A direct application of the contraction mapping theorem to the equations (3) seems not possible.

It is shown that for a sufficient small values of $|\mu| > 0$ there is a system without deviating

$$\frac{dX(t)}{dt} = \big(A + \mu B(\mu)\big) X(t), \tag{4}$$

whose solutions are solutions to system (1).

2. Successive approximations

The matrix equation (3) can be solved by the method of successive approximations.

We start this process by setting $G_0(p,\mu) = 0$, $B_0(\mu) = 0$ and

$$\begin{gathered} B_{j+1}(\mu) - pG_{j+1}(p,\mu) = F_j(p,\mu), \\ F_j(p,\mu) \equiv \sum_{k=1}^{n} A_k e^{p\tau_k} - \big(A + \mu B_j(\mu)\big)G_j(p,\mu), \quad \text{for } j = 0,1,2,\ldots \end{gathered} \tag{5}$$

From this we obtain the matrices

$$\begin{gathered} B_{j+1}(\mu) = F_j(o,\mu) = \frac{1}{2\pi}\int_0^{2\pi} F_j(re^{i\varphi},\mu)d\varphi, \\ G_{j+1}(p,\mu) = \frac{1}{p}\big(B_{j+1}(\mu) - F_j(p,\mu)\big). \end{gathered} \tag{6}$$

For $|p| = r$ it follows that

$$\begin{gathered} \left\|B_{j+1}(\mu)\right| \le \max_{0\le\varphi\le 2\pi} \left\|F_j(re^{i\varphi},\mu)\right\|, \\ \max_{|p|=r} \|G_{j+1}(p,\mu)\| \le \frac{2}{r}\left\|F_j(re^{i\varphi},\mu)\right\|. \end{gathered} \tag{7}$$

Now we have to establish the boundedness for the sequences

$$\big\{\|B_j(\mu)\|\big\}, \qquad \left\{\max_{|p|=r}\|G_j(p,\mu)\|\right\}.$$

Let us denote

$$\tau = \max_{|p|=r}\sum_{k=1}^{n}\|A_k\|\, e^{p\tau_k} = \sum_{k=1}^{n}\|A_k\|\, e^{r|\tau_k|}, \qquad \tau = \tau(r). \tag{8}$$

Then from the estimate (7) we get

$$\|B_{j+1}(\mu)\| \le \tau + \big(\|A\| + |\mu|\,\|B_j(\mu)\|\big)\frac{2}{r}\,\|B_j(\mu)\|.$$

Theorem 1. *If the inequality*

$$\frac{2}{r} - \sqrt{2r\,|\mu|\,\tau} \ge \|A\| \tag{9}$$

is satisfied, then the sequences

$$\big\{\|B_j(\mu)\|\big\}, \qquad \left\{\max_{|p|=r}\|G_j(p,\mu)\|\right\}$$

are bounded.

P r o o f. The sequence of matrices $B_j(\mu)$ (j=0,1,2,...) is bounded in norm, if the equation in z

$$z = \tau + \|A\| \frac{2}{r} z + \frac{2}{r} |\mu| z^2 \tag{10}$$

has two positive roots. To this end it suffices to satisfy the inequality

$$1 - \frac{2}{r} \|A\| > 2\sqrt{\frac{r}{2} |\mu| \tau},$$

which coincides with the condition (9).

For any given matrix A we can find $r > 2\|A\|$ and a value μ_0 such that for $|\mu| \le \mu_0$ the inequality (9) is fulfilled.

For $z = \max_j \|B_j(\mu)\|$, from (10) we obtain

$$z = \sqrt{\frac{r\tau}{2|\mu|}}, \qquad \|B_j(\mu)\| \le \sqrt{\frac{r\tau}{2|\mu|}}. \tag{11}$$

Under the condition (9), from the inequality (7), it follows that the sequence $\max_{|r|=r} \|G_{j+1}(p,\mu)\|$, (n=0,1,2,...) is bounded

$$\max_{|r|=r} \|G_{j+1}(p,\mu)\| \le \frac{2}{r}\sqrt{\frac{r\tau}{2|\mu|}} = \sqrt{\frac{2\tau}{r|\mu|}}, \tag{12}$$

where τ is given by (8). □

3. Convergence of successive approximations

We prove of convergence of the matrix sequences $\{B_j(\mu)\}$ and $\{G_j(p,\mu)\}$.

Theorem 2. *If the inequality*

$$\frac{2}{r} - \sqrt{2r|\mu|\tau} > \|A\| \tag{13}$$

is satisfied, then the sequences $\{B_j(\mu)\}$ and $\{G_j(p,\mu)\}$ ($j = 0, 1, 2, \dots$) converge when $|p| = r$.

P r o o f. From (6) we have the equalities

$$B_{j+1}(\mu) - B_j(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \big(F_j(re^{i\varphi},\mu) - F_{j-1}(re^{i\varphi},\mu)\big) d\varphi,$$

$$G_{j+1}(p,\mu) - G_j(p,\mu) = \frac{1}{p}\big(B_{j+1}(\mu) - B_j(\mu) - F_j(p,\mu) + F_{j-1}(p,\mu)\big).$$

Let us estimate the difference

$$\begin{aligned} F_j(p,\mu) - F_{j-1}(p,\mu) = & -A\big(G_j(p,\mu) - G_{j-1}(p,\mu)\big) \\ & -\mu B_j(\mu)\big(G_j(p,\mu) - G_{j-1}(p,\mu)\big) \\ & -\mu\big(B_j(\mu) - B_{j-1}(\mu)\big)G_{j-1}(p,\mu). \end{aligned}$$

For $|p| = r$ we obtain the inequalities

$$\begin{aligned} \|B_{j+1}(\mu) - B_j(\mu)\| \leq & \sqrt{\frac{2\,|\mu|\,\tau}{r}}\,\|B_j(\mu) - B_{j-1}(\mu)\| \\ & + \left(\|A\| + \sqrt{\frac{1}{2}r\,|\mu|\,\tau}\right)\|G_j(p,\mu) - G_{j-1}(p,\mu)\|\,, \\ \|G_{j+1}(p,\mu) - G_j(p,\mu)\| \leq & \frac{1}{r}\left(1 + \sqrt{\frac{2}{r}\,|\mu|\,\tau}\right)\|B_j(\mu) - B_{j-1}(\mu)\| \\ & + \frac{1}{r}\left(\|A\| + \sqrt{\frac{1}{2}r\,|\mu|\,\tau}\right)\|G_j(p,\mu) - G_{j-1}(p,\mu)\|\,. \end{aligned}$$

We introduce the notation

$$\alpha = \sqrt{\frac{2}{r}\,|\mu|\,\tau}, \qquad \beta = \|A\| + \sqrt{\frac{1}{2}r\,|\mu|\,\tau},$$

in order to get simple sufficient conditions for convergence of the successive approximations (6). This requires that the eigenvalues of the matrix

$$R = \begin{pmatrix} \alpha & \beta \\ r^{-1}(1+\alpha) & r^{-1}\beta \end{pmatrix}$$

are less than 1 in modulus. And this leads to the inequality $\alpha + \frac{2\beta}{r} < 1$, which is the same as (13). □

4. The main result

Summarizing, we have

Theorem 3. *If the inequality*

$$\frac{2}{r} - \sqrt{2r\,|\mu|\,\tau} > \|A\|\,, \quad \textit{where} \quad \tau = \sum_{k=1}^{n}\|A_k\|\,e^{r|\tau_k|}, \qquad r = |p|\,,$$

is satisfied, then the system (1)

$$\frac{dX(t)}{dt} = AX(t) + \mu\sum_{k=1}^{n}A_kX(t+\tau_k)$$

with complex deviating argument has an integral manifold of solutions given by the system

$$\frac{dX(t)}{dt} = C(\mu)X(t), \quad \textit{where} \quad C(\mu) = \big(A + \mu B(\mu)\big). \tag{14}$$

Remark 1. Under the assumptions of Theorem 3 and provided that the real numbers $\tau_k < 0$ the stability of solutions of the system (1) is equivalent to the stability of solutions of the system (14) ([13]).

The invariant subspace of the perturbed system was explicitly constructed using successive approximations in [6], [13].

Remark 2. The unstable subspace of linear ordinary differential equations persists under small linear perturbations—even in the presence of delays, this has been proved in [5], [7].

By this way one could investigate asymptotic properties of solutions of differential—delay equations.

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Klara R. Janglajew
Institute of Mathematics
University of Białystok
Akademicka 2
15-267 Białystok
POLAND
E-mail: jang@math.uwb.edu.pl

Kim G. Valeev
Department of Mathematics
Kyiv National Economic University
03049 Kyiv
Kurskaya 3/13
UKRAINE
E-mail: kimvaleev36@mail.ru