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ASYMPTOTIC PROPERTIES OF THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

Blanka Baculíková — Jozef Džurina

ABSTRACT. We present new criteria guaranteeing that all nonoscillatory solutions of the third-order functional differential equation

$$\Big[r(t)\big[x'(t)\big]^{\gamma}\Big]'' + p(t)x^{\beta}\big(\tau(t)\big) = 0$$

tend to zero. Our results are based on the suitable comparison theorems. We consider both delay and advanced case of studied equation. The results obtained essentially improve and complement earlier ones.

1. Introduction

We deal with the oscillatory and asymptotic behavior of all solutions of the third-order functional differential equations

$$\Big[r(t)\big[x''(t)\big]^{\gamma}\Big]'' + p(t)x^{\beta}\big(\tau(t)\big) = 0. \tag{E}$$

In the paper, we will assume $r, p \in C\big([t_0,\infty)\big)$, $\tau \in C^1\big([t_0,\infty)\big)$ and

(H_1) γ, β are the ratios of two positive odd integers,

(H_2) $r(t) > 0$, $p(t) > 0$, $\tau'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

Moreover, it is assumed that the equation (E) is in a canonical form, i.e.,

$$R(t) = \int_{t_0}^{t} r^{-1/\gamma}(s)\,\mathrm{d}s \to \infty \qquad \text{as} \quad t \to \infty.$$

By a solution of the equation (E) we mean a function $x(t) \in C^1\big([T_x,\infty)\big)$, $T_x \ge t_0$, which has the property $r(t)\big(x'(t)\big)^{\gamma} \in C^2\big([T_x,\infty)\big)$ and satisfies the equation (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of the equation (E)

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which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise it is called to be nonoscillatory.

In this paper we offer new comparison principles, in which we deduce properties of the third order differential equation from that of the second order differential inequality and this reduction essentially simplifies the investigation of the properties of third order differential equations.

Our results complement and extend earlier ones presented in [1]–[19].

Remark 1. All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Main results

We start with the classification of the possible nonoscillatory solutions of (E).

Lemma 1. *Let $x(t)$ be a nonoscillatory solution of* (E)*. Then $x(t)$ satisfies, eventually, one of the following conditions:*

$$(\mathrm{C}_1) \qquad x(t)x'(t) < 0, \quad x(t)\Big[r(t)\big[x'(t)\big]^\gamma\Big]' > 0, \quad x(t)\Big[r(t)\big[x'(t)\big]^\gamma\Big]'' < 0;$$

$$(\mathrm{C}_2) \qquad x(t)x'(t) > 0, \quad x(t)\Big[r(t)\big[x'(t)\big]^\gamma\Big]' > 0, \quad x(t)\Big[r(t)\big[x'(t)\big]^\gamma\Big]'' < 0.$$

P r o o f. The proof follows immediately from the canonical form of (E). □

We recall the following definition:

Definition 1. We say that (E) enjoys property (A) if every its nonoscillatory solution satisfies (C_1).

We offer new technique for investigation of property (A) of (E) based on the comparison theorems that reduce property (A) of (E) to the absence of certain positive solution of the suitable second order differential inequality. We deal with both delay and advanced case of (E). At first, we establish criteria for property (A) of advanced differential equation. We start with the following auxiliary result.

Lemma 2. *Let $\tau(t) \geq t$. Assume that $x(t)$ satisfies* (C_2)*. Then for any $k \in (0,1)$,*

$$\big|x\big(\tau(t)\big)\big| \geq k\frac{R(\tau(t))}{R(t)}|x(t)|. \tag{2.1}$$

P r o o f. Assume that $x(t) > 0$. The monotonicity of $w(t) = r(t)\big[x'(t)\big]^{\gamma}$ implies that

$$x\big(\tau(t)\big) - x(t) = \int_t^{\tau(t)} x'(s)\,\mathrm{d}s = \int_t^{\tau(t)} w^{1/\gamma}(s) r^{-1/\gamma}(s)\,\mathrm{d}s$$

$$\geq w^{1/\gamma}(t) \int_t^{\tau(t)} r^{-1/\gamma}(s)\,\mathrm{d}s = w^{1/\gamma}(t)\Big[R\big(\tau(t)\big) - R(t)\Big].$$

That is,

$$\frac{x(\tau(t))}{x(t)} \geq 1 + \frac{w^{1/\gamma}(t)}{x(t)}\Big[R\big(\tau(t)\big) - R(t)\Big]. \tag{2.2}$$

On the other hand, since $x(t) \to \infty$ as $t \to \infty$, then for any $k \in (0, 1)$ there exists a t_1 large enough, such that

$$kx(t) \leq x(t) - x(t_1) = \int_{t_1}^{t} w^{1/\gamma}(s) r^{-1/\gamma}(s)\,\mathrm{d}s$$

$$\leq w^{1/\gamma}(t) \int_{t_1}^{t} r^{-1/\gamma}(s)\,\mathrm{d}s \leq w^{1/\gamma}(t) R(t)$$

or equivalently,

$$\frac{w^{1/\gamma}(t)}{x(t)} \geq \frac{k}{R(t)}. \tag{2.3}$$

Using (2.3) in (2.2), we get

$$\frac{x(\tau(t))}{x(t)} \geq 1 + \frac{k}{R(t)}\Big[R\big(\tau(t)\big) - R(t)\Big] \geq k\frac{R(\tau(t))}{R(t)}.$$

This completes the proof. □

Let us denote

$$p_1(t) = \frac{R^{\beta}(\tau(t))}{R^{\beta}(t)} p(t). \tag{2.4}$$

Theorem 1. *Let $\tau(t) \geq t$. If for some $c \in (0, 1)$ the second-order differential inequality*

$$\left(\frac{1}{p_1^{1/\beta}(t)}\big(z'(t)\big)^{1/\beta}\right)' \operatorname{sgn} z(t) + c\,\frac{t^{1/\gamma}(t)}{r^{1/\gamma}}\big|z^{1/\gamma}(t)\big| \leq 0 \tag{E$_1$}$$

has no solution satisfying

$$z(t) > 0, \quad z'(t) < 0, \quad \left(\frac{1}{p_1^{1/\beta}(t)}\big(z'(t)\big)^{1/\beta}\right)' < 0, \tag{P$_1$}$$

then (E) *has property (A).*

P r o o f. Assume the contrary, let $x(t)$ be a nonoscillatory solution of equation (E), satisfying (C_2). We may assume that $x(t) > 0$, for $t \ge t_0$. Setting (2.1) into (E), we obtain

$$\Big[r(t)\big[x'(t)\big]^{\gamma}\Big]'' + k^{\beta} p(t) \frac{R^{\beta}(\tau(t))}{R^{\beta}(t)} x^{\beta}(t) \le 0. \tag{2.5}$$

On the other hand, it follows from the monotonicity of $y(t) = \big[r(t)\,[x'(t)]^{\gamma}\big]'$, that

$$r(t)\big[x'(t)\big]^{\gamma} \ge \int_{t_1}^{t} y(s)\,\mathrm{d}s \ge y(t)(t-t_1) \ge c_1^{\gamma/\beta} t\, y(t), \tag{2.6}$$

eventually, where $c_1 \in (0,1)$ is an arbitrary chosen constant. Evaluating $x'(t)$ and then integrating from $t_1 \ge t_0$ to t, we are led to

$$x(t) \ge c_1^{1/\beta} \int_{t_1}^{t} \frac{s^{1/\gamma}}{r^{1/\gamma}(s)} y^{1/\gamma}(s)\,\mathrm{d}s. \tag{2.7}$$

Setting to (2.5), we have

$$y'(t) + c_1\, k^{\beta} p_1(t) \left[\int_{t_1}^{t} \frac{s^{1/\gamma}}{r^{1/\gamma}(s)} \left(\Big[r(s)\big[x'(s)\big]^{\gamma}\Big]' \right)^{1/\gamma} \mathrm{d}s \right]^{\beta} \le 0.$$

Integrating from t to ∞, one gets

$$y(t) \ge c \int_{t}^{\infty} p_1(s) \left[\int_{t_1}^{s} \frac{u^{1/\gamma}}{r^{1/\gamma}(u)}\, y^{1/\gamma}(u)\,\mathrm{d}u \right]^{\beta} \mathrm{d}s, \tag{2.8}$$

where $c = c_1\, k^{\beta}$. Let us denote the right hand side of (2.8) by $z(t)$. Then $y(t) \ge z(t) > 0$ and $z(t)$ satisfies (P_1) and moreover,

$$\left(\frac{1}{p^{1/\beta}(t)} \big(z'(t)\big)^{1/\beta} \right)' + c\, \frac{t^{1/\gamma}(t)}{r^{1/\gamma}(t)} y^{1/\gamma}(t) = 0.$$

Consequently, $z(t)$ is a solution of the differential inequality (E_1), which contradicts our assumption. □

Now we turn our attention to delay differential equations. Let us denote

$$p_2(t) = \frac{p(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}. \tag{2.9}$$

Theorem 2. *Let $\tau(t) \le t$. If for some $c \in (0,1)$ the second order differential inequality*

$$\left(\frac{1}{p_2^{1/\beta}(t)}\big(z'(t)\big)^{1/\beta}\right)'\operatorname{sgn} z(t) + c\,\frac{t^{1/\gamma}}{r^{1/\gamma}(t)}\big|z^{1/\gamma}(t)\big| \le 0 \tag{E$_2$}$$

has no solution satisfying

$$z(t) > 0, \quad z'(t) < 0, \quad \left(\frac{1}{p_2^{1/\beta}(t)}\big(z'(t)\big)^{1/\beta}\right)' < 0, \tag{P$_2$}$$

then (E) *has property* (A).

Proof. Assume the contrary, let $x(t)$ be a positive solution of the equation (E), satisfying (C_2). An integration of (E) from t to ∞, yields

$$\begin{aligned}\Big[r(t)\big[x'(t)\big]^{\gamma}\Big]' &\ge \int_t^{\infty} p(s)x^{\beta}\big(\tau(s)\big)\,\mathrm{d}s\\ &= \int_{\tau(t)}^{\infty} \frac{p(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}x^{\beta}(s)\,\mathrm{d}s \ge \int_t^{\infty} \frac{p(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}x^{\beta}(s)\,\mathrm{d}s.\end{aligned}$$

Using (2.7), one can see that $y(t) = \big[r(t)\,[x'(t)]^{\gamma}\big]'$ satisfies

$$y(t) \ge c\int_t^{\infty} p_2(s)\left[\int_{t_1}^{s} \frac{u^{1/\gamma}}{r^{1/\gamma}(u)}\,y^{1/\gamma}(u)\,\mathrm{d}u\right]^{\beta}\mathrm{d}s. \tag{2.10}$$

Let us denote the right hand side of (2.10) by $z(t)$. Then similarly as in the proof of Theorem 1, we can verify that $z(t)$ is a positive solution of (E_2) and moreover, it satisfies (P_2), which contradicts our assumption. □

Theorems 1 and 2 reduce property (A) of the equation (E) into certain asymptotic behavior of the differential inequalities (E_i). Now we are prepared to eliminate solutions of (E_i) satisfying (P_i), $i = 1, 2$, to obtain sufficient conditions for property (A) of the equation (E). Since (E_1) and (E_2) have the same form, we present just one general criterion and then, we adapt them for both (E_i). We consider the noncanonical differential inequality

$$\Big(a(t)\big(z'(t)\big)^{\alpha}\Big)'\operatorname{sgn} z(t) + b(t)|z^{\delta}(t)| \le 0, \tag{E*}$$

where

(H_3) α, δ are the ratios of two positive odd integers,

(H_4) $a(t) > 0$, $b(t) > 0$.

Let us denote

$$\varrho(t) = \int_t^\infty a^{-1/\alpha}(s)\,\mathrm{d}s.$$

Theorem 3. *Assume that $\alpha \ge \delta$. If*

$$\int_{t_0}^\infty \frac{1}{a^{1/\alpha}(u)} \left(\int_{t_0}^u b(s)\,\mathrm{d}s \right)^{1/\alpha} \mathrm{d}u = \infty \tag{2.11}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\varrho^\alpha(s) b(s) - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(\frac{\alpha}{\delta} \right)^\alpha \frac{1}{\varrho(s) a^{1/\alpha}(s)} \right] \mathrm{d}s > 0, \tag{2.12}$$

then (E*) *has no solution satisfying*

$$z(t) > 0, \quad z'(t) < 0, \quad \Big(a(t)\big(z'(t)\big)^\alpha\Big)' < 0, \tag{P*}$$

Proof. First note that (2.12) implies that there exists $\varepsilon > 0$, such that

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\varrho^\alpha(s) b(s) - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(\frac{\alpha}{\delta} \right)^\alpha \frac{1}{\varrho(s) a^{1/\alpha}(s)} \right] \mathrm{d}s > \varepsilon. \tag{2.13}$$

Let $z(t)$ be a positive solution of (E*) satisfying (P*). Then there exists $\lim_{t\to\infty} z(t) = \ell$. We claim that $\ell = 0$. If not, then $z(t) \ge \ell > 0$, eventually. An integration of (E*) from t_1 to t, yields

$$-a(t)\big(z'(t)\big)^\alpha \ge \int_{t_1}^t b(s) z^\delta(s)\,\mathrm{d}s \ge \ell^\delta \int_{t_1}^t b(s)\,\mathrm{d}s.$$

Evaluating $z'(t)$ and integrating once more from t_1 to t, we get

$$z(t_1) \ge \int_{t_1}^t \frac{1}{a^{1/\alpha}(u)} \left(\int_{t_1}^u b(s)\,\mathrm{d}s \right)^{1/\alpha} \mathrm{d}u$$

which letting $t \to \infty$ contradicts (2.11) and we conclude that $\lim_{t\to\infty} z(t) = 0$.

We define

$$w(t) = \frac{a(t)(z'(t))^\alpha}{z^\delta(t)}.$$

Then $w(t) < 0$ and, moreover,

$$\begin{aligned} w'(t) &= \frac{(a(t)(z'(t))^\alpha)'}{z^\delta(t)} - \delta w(t) \frac{z'(t)}{z(t)} \\ &\le -b(t) - \delta w(t) \frac{a^{1/\alpha}(t) z'(t)}{z(t)} \frac{1}{a^{1/\alpha}(t)}. \end{aligned} \tag{2.14}$$

Since $\lim_{t\to\infty} z(t) = 0$ and $\alpha \ge \delta$, we derive $z(t) \le z^{\delta/\alpha}(t)$. Setting it to (2.14), one gets

$$w'(t) \le -b(t) - \delta w^{1+1/\alpha}(t)a^{-1/\alpha}(t). \tag{2.15}$$

On the other hand, noting like that $-\big(a(t)(z'(t))^{\alpha}\big)^{1/\alpha}$ is positive and increasing, we see that

$$z(t) \ge \int_t^{\infty} -z'(s)\,\mathrm{d}s = \int_t^{\infty} -\Big(a(s)\big(z'(s)\big)^{\alpha}\Big)^{1/\alpha} a^{-1/\alpha}(s)\,\mathrm{d}s \ge -\Big(a(t)\big(z'(t)\big)^{\alpha}\Big)^{1/\alpha}\varrho(t),$$

or equivalently

$$z^{\alpha}(t) \ge -a(t)\big(z'(t)\big)^{\alpha}\varrho^{\alpha}(t),$$

which in view of $\lim_{t\to\infty} z(t) = 0$ and $\alpha \ge \delta$ implies

$$\varepsilon > z^{\alpha-\beta}(t) \ge -w(t)\varrho^{\alpha}(t), \tag{2.16}$$

Multiplying (2.15) by $\varrho^{\alpha}(t)$ and then integrating from t_1 to t, we are led to

$$w(t)\varrho^{\alpha}(t) - w(t_1)\varrho^{\alpha}(t_1) + \alpha\int_{t_1}^{t} \frac{\varrho^{\alpha-1}(s)}{a^{1/\alpha}(s)}\,w(s)\,\mathrm{d}s$$
$$\le -\int_{t_1}^{t} b(s)\varrho^{\alpha}(s)\,\mathrm{d}s - \delta\int_{t_1}^{t}\frac{\varrho^{\alpha}(s)}{a^{1/\alpha}(s)}\,w^{1+1/\alpha}(s)\,\mathrm{d}s,$$

which in view of (2.16) yields

$$\int_{t_1}^{t} b(s)\varrho^{\alpha}(s) + \left[\alpha\,\frac{\varrho^{\alpha-1}(s)}{a^{1/\alpha}(s)}\left(w(s) + \frac{\delta}{\alpha}\,\varrho(s)w^{1+1/\alpha}(s)\right)\right]\mathrm{d}s \le \varepsilon.$$

Using the estimate

$$w + Aw^{1+1/\alpha} \ge -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{1}{A^{\alpha}}, \qquad w < 0,$$

we have

$$\int_{t_1}^{t}\left[\varrho^{\alpha}(s)b(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{\delta}\right)^{\alpha}\frac{1}{\varrho(s)a^{1/\alpha}(s)}\right]\mathrm{d}s \le \varepsilon,$$

which contradicts (2.13). This finishes our proof. □

Now, we transform condition (2.12) to the simpler form.

Corollary 1. *Let* (2.11) *hold. Assume that* $\alpha \ge \delta$. *If*

$$\liminf_{t\to\infty} \varrho^{\alpha+1}(t) b(t) a^{1/\alpha}(t) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{\delta}\right)^{\alpha}, \tag{2.17}$$

then (E*) *has no solution satisfying* (P*).

P r o o f. It follows from (2.17) that there exists some $k > 0$ such that

$$\varrho^{\alpha+1}(t) b(t) a^{1/\alpha}(t) \ \ge \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{\delta}\right)^{\alpha} + \ k,$$

eventually. That is

$$\varrho^{\alpha}(t) b(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{\delta}\right)^{\alpha} \frac{1}{\varrho(t) a^{1/\alpha}(t)} \ge \frac{k}{\varrho(t) a^{1/\alpha}(t)}.$$

Integrating the above inequality from t_1 to t, one gets

$$\int_{t_1}^{t} \left[\varrho^{\alpha}(s) b(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{\delta}\right)^{\alpha} \frac{1}{\varrho(s) a^{1/\alpha}(s)}\right] \mathrm{d}s = k \ln \frac{\varrho(t_1)}{\varrho(t)}.$$

Taking lim sup on the both sides, we see that (2.12) holds true and the assertion now follows from Theorem 3. □

We combine Theorems 1 and 2 together with Corollary 1 to obtain easily verifiable criteria for property (A) of (E).

Theorem 4. *Let* $\gamma \ge \beta$, $\tau(t) \ge t$. *If*

$$\int_{t_0}^{\infty} \frac{R^{\beta}(\tau(u))}{R^{\beta}(u)} p(u) \left(\int_{t_0}^{u} \frac{s^{1/\gamma}}{r^{1/\gamma}(s)} \mathrm{d}s\right)^{\beta} \mathrm{d}u = \infty, \tag{2.18}$$

and

$$\liminf_{t\to\infty} \left(\int_{t}^{\infty} \frac{R^{\beta}(\tau(s))}{R^{\beta}(s)} p(s)\, \mathrm{d}s\right)^{1+1/\beta} \frac{t^{1/\gamma}}{r^{1/\gamma}(t)} \frac{R^{\beta}(t)}{p(t) R^{\beta}(\tau(t))} > \left(\frac{1}{\beta+1}\right)^{1+1/\beta}\left(\frac{\gamma}{\beta}\right)^{1/\beta}, \tag{2.19}$$

then (E) *has property* (A).

P r o o f. Note that (2.19) implies that some $c \in (0,1)$ exists such that

$$\liminf_{t\to\infty} \left(\int_{t}^{\infty} \frac{R^{\beta}(\tau(s))}{R^{\beta}(s)} p(s)\, \mathrm{d}s\right)^{1+1/\beta} c \frac{t^{1/\gamma}}{r^{1/\gamma}(t)} \frac{R^{\beta}(t)}{p(t) R^{\beta}(\tau(t))} > \left(\frac{1}{\beta+1}\right)^{1+1/\beta}\left(\frac{\gamma}{\beta}\right)^{1/\beta}. \tag{2.20}$$

We set $\alpha = 1/\beta$, $\delta = 1/\gamma$, $\sigma(t) = \tau(t)$, $a(t) = p_1^{-1/\beta}(t)$, and $b(t) = c\,\frac{t^{1/\gamma}}{r^{1/\gamma}(t)}$. Then $\varrho(t) = \int_t^\infty p_1(s)\,\mathrm{d}s$. Since (2.11) and (2.17) reduce to (2.18) and (2.20), respectively, Corollary 1 ensures that (E_1) has no solution satisfying (P_1). The assertion now follows from Theorem 1. □

Theorem 5. *Let $\gamma \geq \beta$, $\tau(t) \leq t$. Assume that* (2.18) *holds. If*

$$\liminf_{t\to\infty} \left(\int_t^\infty p(s)\,\mathrm{d}s\right)^{\frac{\beta+1}{\beta}} \frac{\tau^{1/\gamma}(t)}{r^{1/\gamma}(\tau(t))}\,\frac{\tau'(t)}{p(t)} > \left(\frac{1}{\beta+1}\right)^{\frac{\beta+1}{\beta}} \left(\frac{\gamma}{\beta}\right)^{\frac{1}{\beta}}, \tag{2.21}$$

then (E) *has property* (A).

P r o o f. We set

$$\alpha = 1/\beta,\ \delta = 1/\gamma,\ \sigma(t) = \tau(t),\ a(t) = p_2^{-1/\beta}(t), \quad \text{and} \quad b(t) = c\,\frac{t^{1/\gamma}}{r^{1/\gamma}(t)}.$$

Then

$$\varrho(t) = \int_{\tau^{-1}(t)}^\infty p(s)\,\mathrm{d}s.$$

As (2.17) takes the form (2.21), Corollary 1 guarantees that (E_2) has no solution satisfying (P_2). By Theorem 2, (E) has property (A). □

Remark 2. For $\tau(t) \equiv t$ both conditions (2.19) and (2.21) simplifies to the same condition

$$\liminf_{t\to\infty} \left(\int_t^\infty p(s)\,\mathrm{d}s\right)^{1+1/\beta} \frac{t^{1/\gamma}}{r^{1/\gamma}(t)}\,\frac{1}{p(t)} > \left(\frac{1}{\beta+1}\right)^{1+1/\beta} \left(\frac{\gamma}{\beta}\right)^{1/\beta},$$

for property (A) of (E).

Corollary 2. *Assume that* (E) *enjoys property* (A). If moreover,

$$\int_{t_0}^\infty \frac{1}{r^{1/\gamma}(v)} \left(\int_v^\infty\int_u^\infty p(s)\,\mathrm{d}s\mathrm{d}u\right)^{1/\gamma} \mathrm{d}v = \infty, \tag{2.22}$$

then every nonoscillatory solution of (E) tends to zero as $t \to \infty$.

P r o o f. Since (E) has property (A), every its nonoscillatory solution satisfies (C_1), and what is more, (2.22) ensures that such solution tends to zero as $t \to \infty$. □

Example 1. Consider the third order nonlinear advanced differential equation

$$\Big(t\big(x'(t)\big)^5\Big)'' + \frac{a}{t^4}\, x^3(\lambda t) = 0, \qquad t \ge 1, \tag{E_{x1}}$$

where $a > 0$ and $\lambda > 0$. It is easy verify that (2.18) and (2.22) hold. Moreover, (2.19) for $\lambda \ge 1$ and (2.21) for $\lambda \le 1$ reduce to

$$\lambda^{4/5} a^{1/3} > \frac{3}{4}\left(\frac{5}{4}\right)^{1/3} \qquad \text{and} \qquad \lambda a^{1/3} > \frac{3}{4}\left(\frac{5}{4}\right)^{1/3},$$

respectively, which, according to Theorems 4 and 5, implies that (E_{x1}) enjoys property (A) and, moreover, Corollary 2 guarantees that every nonoscillatory solution of (E_{x1}) tends to zero as $t \to \infty$.

3. Summary

In this paper, we present new comparison principles for deducing property (A) of the third order differential equation from the properties the suitable second order differential inequality. Our results can be applied to both delay and advanced third order differential equations. The criteria obtained are easy verifiable and have been precedented by suitable joint illustrative example.

Our method essentially simplifies the examination of the third order equations and what is more, it supports backward the research on the second order delay/advanced differential equations and inequalities.

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Department of Mathematics
Faculty of Electrical Engineering
and Informatics
Technical University of Košice
Letná 9
SK–042-00 Košice
SLOVAKIA

E-mail: blanka.baculikova@tuke.sk
jozef.dzurina@tuke.sk