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Tatra Mountains
Mathematical Publications

PROBLEMS

1. Density measures

(Proposed by Mark F e y.)

Definitions and motivation. We denote by dA the asymptotic density of a set $A \subseteq \mathbb{N}$. Let μ be a *density measure*: a finitely additive measure on $2^{\mathbb{N}}$ which extends d, i.e., $\mu(A) = d(A)$ if $d(A)$ exists.

Fix $A \subseteq \mathbb{N}$ and denote $s(A) = \sup\{dB\colon B \subseteq A;\, dB \text{ exists}\}$, $t(A) = \inf\{dB\colon B \supseteq A;\, dB \text{ exists}\}$. Clearly $s(A) \leqslant \mu(A) \leqslant t(A)$.

Question: For a fixed A and all possible density measures what are all possible values of $\mu(A)$?

Comment: This can be related to some results of [BUCK] or [VAN DOUWEN].

Answer: (Miloš Z i m a n) By Theorem 3.2.9 in [BHARAO] it is clear that a density measure $\mu(A)$ of a set A can reach any value between $\mu_i(A)$ and $\mu_e(A)$, where

$$\mu_i(A) = \sup \frac{\sum\limits_{i=1}^{n} d(A_i) - \sum\limits_{j=1}^{m} d(B_j)}{k}.$$

The supremum is taken over all finite collections $A_1, A_2, \dots, A_n$, $B_1, B_2, \dots B_m$ of sets having density and positive integers k such that

$$k\chi_A + \sum_{j=1}^{m} \chi_{B_j} \geqslant \sum_{i=1}^{n} \chi_{A_i}.$$

Similarly,

$$\mu_e(A) = \inf \frac{\sum\limits_{i=1}^{n} d(A_i) - \sum\limits_{j=1}^{m} d(B_j)}{k}.$$

The infimum is taken over all finite collections $A_1, A_2, \dots, A_n$, $B_1, B_2, \dots B_m$ of sets having density and positive integers k such that

$$k\chi_A + \sum_{j=1}^{m} \chi_{B_j} \leqslant \sum_{i=1}^{n} \chi_{A_i}.$$

2. Sum of the binary digits

(Proposed by A. O. Gelfond, presented by Michel Mendès–France.)

Definitions and motivation. In his article [GELFOND] A. O. Gelfond asks the following question (among many others). Let $S(n)$ be the sum of the binary digits of the positive integer n. Is it true that the density of those n for which $S(n^2)$ is even, equals $1/2$? (Of course the problem holds for any basis and any modulus $m > 1$, and any polynomial $P(n)$ of degree > 1).

The sequence $S(n^2)$ is obviously quite complex (no mather how one understands "complexity"). In particular, J. P. Allouche proved that it is not an automatic sequence. See his paper [ALLOUCHE]. But that does not solve Gelfond's problem. For a partial result, see [DARTEN].

3. Density set for exponential density

(Proposed by Georges Grekos.)

Definitions and motivation. Properties of the density sets have been studied in [GREVOL]. One can ask similar questions for *exponential density* given by $\overline{\varepsilon}A = \limsup \frac{\log A(n)}{\log n}$ and $\underline{\varepsilon}A = \liminf \frac{\log A(n)}{\log n}$. Let $E(A) = (\overline{\varepsilon}A, \underline{\varepsilon}A)$ be the density point associated with a subset $A \subseteq \mathbb{N}$. For any $A \subseteq \mathbb{N}$ we denote $S(A) = \{E(B) \colon B \subseteq A\}$ and $T(A) = \{E(C) \colon C \supseteq A\}$. It is known (see thesis [GREKOS]) that $T(A)$ is the trapezium given by the points $(\overline{\varepsilon}A, \underline{\varepsilon}A)$, $(\overline{\varepsilon}A, \overline{\varepsilon}A)$, $(1,1)$ and $(1, \underline{\varepsilon}A)$, and that the set $S(A)$ is closed and the upper boundary of this set is a continuous function $[0, \overline{\varepsilon}A] \to [0, \underline{\varepsilon}A]$.

Questions:

1. Is the set $S(A)$ convex?
2. Find a characterization of $S(A)$.

4. Strong limit points and shift invariant ideals

(Proposed by Martin Mačaj.)

Definitions and motivation. If $\mathbb{Q} = \{r_1, \dots, r_n, \dots\}$ is the set of all rational numbers and $P = \{p_1 < p_2 < \dots < p_n < \dots\}$ is the set of all primes (in the natural order) then the sequence (x_n) given by

$$x_n = \begin{cases} r_k & \text{if } n = p_k\,; \\ 0 & \text{otherwise} \end{cases}$$

has the set of all limit points equal to $\mathbb{R}$. But one can feel that 0 is a "more important limit point" of this sequence than other rationals, in fact it is the "most important" one. The following characterization of limit points is known: L is a limit point of (x_n) if and only if $\mathcal{I}\text{-}\lim x_n = L$ for some admissible ideal $\mathcal{I}$.

This motivates the following two definitions. We say that $\mathcal{I}$ is a *shift invariant ideal* if it is admissible and $\mathcal{I}\text{-}\lim x_n = \mathcal{I}\text{-}\lim x_{n+1}$ whenever at least one of the $\mathcal{I}$-limits exists. Equivalently we can define a shift invariant ideal as an admissible ideal which fulfils the condition $A \in \mathcal{I} \iff A+1 \in \mathcal{I}$.

We say that L is a strong limit point of (x_n) if $\mathcal{I}\text{-}\lim x_n = L$ for some shift invariant ideal $\mathcal{I}$.

Questions:

1. Is it possible to have 2 or more strong limit points?

2. Find a property $\mathcal{P}$ such that the following proposition holds: l is a strong limit point of a sequence $(x_n)_{n=1}^{\infty}$ if and only if $\forall \varepsilon > 0$ $\mathcal{P}(A_\varepsilon)$. (We denote $A_\varepsilon = \{x \in \mathbb{R} \colon \|x - l\| < \varepsilon\}$).

Partial answer: (Martin Sleziak). The answer to the first question is affirmative. Let us put $B = \bigcup_{k \in \mathbb{N}_0} \{2^{2k}, \dots, 2^{2k+1}-1\}$, i.e., B is the set of all positive integers such that their binary representation has odd number of digits. We denote by $\mathcal{I}_1$ the smallest shift invariant ideal containing B and by $\mathcal{I}_2$ the smallest shift invariant ideal containing $\mathbb{N} \setminus B$. These ideals are proper and if we take a sequence $x_n = \chi_B(n)$ then we get $\mathcal{I}_1\text{-}\lim x_n = 1$ and $\mathcal{I}_2\text{-}\lim x_n = 0$. So the $\mathcal{I}$-limit for shift invariant ideals is not necessarily unique, and this is an example of a sequence which has at least 2 strong limit points.

So it is natural to ask if we could improve the original definition of strong limit point in a "reasonable" way. We can also investigate the set of strong limit points of a sequence of real numbers.

5. Uniformly distributed subsequences

(Proposed by Ladislav Mišík.)

Definitions and motivation. Let (x_n) be a given sequence and let us define $\mathcal{U} = \{A \subseteq \mathbb{N} \colon (x_n)_{n \in A}$ is uniformly distributed mod $1\}$.

Question: What can be said about the set $\mathcal{U}$? (Hausdorff dimension, Baire class, porosity of the image of this set under the mapping $A \mapsto \sum_{a \in A} 2^{-a}$ into the interval $[0, 1)$ and other similar questions.)

Comments: This question could be related to some characterizations of deterministic numbers obtained in [RAUZY]. Probably the answer could be that A is a characteristic set of a deterministic number.

Comment by Michel Mendès–France: Y. Dupain and J. Lesca have shown that the set of densities of those A in $\mathcal{U}$ is a closed interval $[0, d]$, where d is at most 1. They compute d in terms of the distribution function of (x_n). See [DUPLES]. Y. Dupain and A. Thomas have also shown that if $d > 0$, then for all strictly increasing $f\colon \mathbb{N} \to \mathbb{N}$, there is a uniformly distributed sequence $(x_{g(n)})$ such that $g(n) \sim f(n)$. See [DUPTHOM].

6. Density sets

(Proposed by the late Professor Stylianus C. Pichorides, presented by Georges Grekos.)

Definitions and motivation. Suppose that we are given two regions $S_1, S_2 \subseteq T$, where T is the triangle $(0,0)$, $(1,0)$, $(1,1)$. (By a region we always mean a closed set of points, a subset of T lying between an initial segment on the x-axis and some increasing continuous function.) It is known that if S_1, S_2 such that $S_1 \subseteq S_2$ are both convex regions, then there exist $A_1 \subseteq A_2 \subseteq \mathbb{N}$ such that $\mathcal{S}(A_i) = S_i$ for $i = 1, 2$, [GREVOL] (the symbol $\mathcal{S}(A)$ denotes the density set $\mathcal{S}(A) = \{(\underline{d}B, \overline{d}B)\colon B \subseteq A\}$).

Question: Let S be a convex region such that $S \subseteq T$. Does there exist a set $A \subseteq \mathbb{N}$ with $S(A) = S$ and such that for each convex region $R \subseteq S$ one can find $B = B(R) \subseteq A$ with $\mathcal{S}(B) = R$?

N. B. — This problem is now solved by François Hennecart, *On the regularity of density sets*, this issue.

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