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THE PROBLEM OF DETERMINING THE INTERNAL STRUCTURE OF GEOMETRIC DIAGRAMS ON THEMATIC MAPS

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The main objective of this paper is to design a method by means of which it is possible to determine the internal structure of geometric diagrams used on thematic maps. Note that this structure should map the internal structure of human-geographical entities, as for instance of settlements, industrial centres or of whole areas. However, in order to reach this goal, the precondition is to determine first the size of these diagrams. Therefore, the first part of the paper is briefly concerned with this problem. The principle being used in determining the size of geometric diagrams is the principle of direct proportionality, which means that the size of diagrams has to be directly proportional to the size of corresponding human-geographical entities or areas. The principle is mathematically formulated and then a general formula for the size of geometric diagrams is derived. Moreover, besides the general formula, formulae for the size of the most frequently used geometric diagrams, e. g. circle, square and equilateral triangle are also derived. The second part of the paper is explicitly concerned with determining the internal structure of geometric diagrams. The same principle, that is the principle of direct proportionality, is also used in this case. It means that the internal structure of geometric diagrams has to be directly proportional to the internal structure of human-geographical entities or areas. Again, besides the derivation of general formula, the formulae for circle, square and equilateral triangle are derived.

Key words: thematic maps, size and internal structure of geometric diagrams

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INTRODUCTION

The size and internal structure of geometric diagrams, mapping on thematic maps the size and internal structure of human-geographical entities (as for instance settlements, industrial plants or the whole industrial centres), eventually of whole areas, are among the most important attributes of those diagrams. While the problem of determining their size was already discussed in a specific paper (Paulov 1996), the problem of determining their internal structure was not till now. It is the purpose of this paper, which, however, is closely connected with the previous one.

DETERMINING THE SIZE OF GEOMETRIC DIAGRAMS: A SHORT RECAPITULATION

Determining the internal structure of geometric diagrams presupposes determining first their size. It is evident: if one wants to divide geometric diagrams, and their internal division is a measure of their internal structure, one simply needs to know their size; their size must already be given. Moreover, deriving relation, on the basis of which one determines the internal structure of geometric diagrams, presupposes knowledge of the relation on the basis of which their size was determined. Therefore let us turn first our attention, but only to the extent necessary for correct comprehension of the subject being discussed in this paper, to determining the size of geometric diagrams.

The rule we find principal for determining the size of geometric diagrams is the rule of direct proportionality. It says that the size of geometric diagrams has to be directly proportional to the size of corresponding human-geographical entities or areas. If one denotes the size of that entity or area as A, which is common in thematic cartography, and the size of geometric diagram to be determined as x, then the mathematical formulation of this rule is as follows

$$x = kA \tag{1}$$

where A is an independent variable, x dependent variable and k the constant of proportionality. It follows from (1) that

$$k = \frac{x}{A} \tag{2}$$

It is evident that relation (2) has to be, in all cases under consideration, preserved, that is

$$\frac{x_1}{A_1} = \frac{x_2}{A_2} = \dots = \frac{x_n}{A_n} \tag{3}$$

Observe now the consequences following from relation (3). If from the whole series one selects, for instance, $x_1 : A_1 = x_2 : A_2$ then $x_1A_2 = x_2A_1$. If x_2 is the size of the geometric diagram to be determined then

$$x_2 = \frac{x_1A_2}{A_1} \tag{4}$$

In relation (4), A_1 and A_2 are independent, exogenously given variables, but the size of x_1 is not given. It is therefore necessary, in order to determine x_2, to choose x_1 at the beginning. If one does it then the size of the other diagrams will no longer be arbitrary, but determined according to (4), that is

$$x_3 = \frac{x_1 A_3}{A_1},\ x_4 = \frac{x_1 A_4}{A_1},\ \ldots,\ x_n = \frac{x_1 A_n}{A_1} \tag{5}$$

It can be seen that x_1 and A_1 repeat themselves, that is they become constants. Note that it is also appropriate to set $x_1 = 1$ and to give to the corresponding A_1 specific notation, say $A_1 = M$. After doing that relation (5) can be rewritten as follows

$$x_3 = \frac{A_3}{M},\ x_4 = \frac{A_4}{M},\ \ldots,\ x_n = \frac{A_n}{M} \tag{6}$$

or, generally, as

$$x_i = \frac{A_i}{M} \tag{7}$$

or, after the mutual correspondence between A_i and x_i is clear, without subscript i, that is

$$x = \frac{A}{M} \tag{8}$$

Relation (7) or (8) represents a basic formula for determining the size of geometric diagrams. The rule of direct proportionality is really contained in them since they can also be rewritten as $x_i = (1/M)A_i$ or $x = (1/M)A$, where $1/M$ is constant, k. Therefore one can write

$$x_i = kA_i,\ \text{or}\ \ x = kA \tag{9}$$

Relation (9), as can be seen, is identical to relation (1) representing mathematical formulation of the rule.

Repeat, for convenience, that $1{:}M$ is a general expression for map scale, where the first member, 1, denotes the unit length on the map and the second one, M, the corresponding length on the earth's surface. It is similar in our case: the first member, 1, denotes the unit size of a geometric diagram and the second one, M, the corresponding size of the human-geographical entity or area. Therefore, in fact $1{:}M$ represents the scale of the geometric diagrams also here.

Follow now how relation (8) will manifest itself in determining the size of the three, most frequently used geometric diagrams: the circle, square and equilateral triangle, if the size of them will be measured by their area. However, the goal is to compute a variable which makes it possible to construct the corresponding figure with a given area.

The area of a circle, as is well known, is πr^2. This must equal the size of geometric diagrams, x, which, however, equals A/M.

So one can write

$$x = \pi r^2 = \frac{A}{M} \Rightarrow r = \sqrt{\frac{A}{\pi M}} = \sqrt{\frac{1}{\pi}}\sqrt{\frac{A}{M}} \tag{10}$$

where r denotes the radius of a circle.

The corresponding relation for a square is

$$x = a^2 = \frac{A}{M} \Rightarrow a = \sqrt{\frac{A}{M}} \tag{11}$$

where a denotes the side of a square.

The corresponding relation for an equilateral triangle is

$$x = \frac{\sqrt{3}}{4} s^2 = \frac{A}{M} \Rightarrow s = \sqrt[4]{\frac{16}{3}}\sqrt{\frac{A}{M}} \tag{12}$$

Relations (10), (11) and (12) make it possible to determine also A, which, however, is known for the designer of a thematic map, but not known for the reader of this map. This one can measure r, a, or s on the map, but A must be computed by him. It follows from relation (10) that $A = M\pi r^2$, from relation (11) $A = Ma^2$ and from relation (12)

$$A = M\frac{\sqrt{3}}{4} s^2$$

The situation when the size of geometric diagrams on thematic maps is determined otherwise, not on the principle of direct proportionality, is specifically discussed elsewhere (Paulov 1996).

DETERMING THE INTERNAL STRUCTURE OF GEOMETRIC DIAGRAMS

Internal structure of geometric diagrams maps, as already said, the internal structure of human-geographical entities and areas. Analogically, as in case of determining the size of these diagrams, we find it reasonable to use the rule of direct proportionality also in this case. It means that the internal structure of geometric diagrams has to be directly proportional to the internal structure of corresponding human-geographical entities or areas. In other words: the same share taken by the structural component in the human-geographical entity or area has this component taken in the geometric diagrams. Or still otherwise: in the same quotient in which are the size of human-geographical entity or area and the size of geometric diagrams must be the size of structural component in human-geographical entity or area and the size of structural component in geometric diagram.

The mathematical formulation of this rule is as follows

$$z_i : x = B_i : A \tag{13}$$

where z_i is the size of the structural component in the geometric diagram, x the size of the geometric diagram, B_i the size of the structural component in the human-geographical entity or area and A the size of the human-geographical entity or area. It must, of course, hold that

$$\sum_{i=1}^{n} z_i = x \tag{14}$$

$$\sum_{i=1}^{n} B_i = A \tag{15}$$

It follows from relation (13) that

$$z_i A = B_i x \tag{16}$$

and from relation (16) that

$$z_i = \frac{B_i x}{A} = \frac{x}{A} B_i \tag{17}$$

Since x and A are fixed their quotient is also fixed; so we can write

$$z_i = kB_i \tag{18}$$

It is clear from relation (18) that z_i is directly proportional to B_i as required at the beginning.

Observe now how relation (17) will manifest itself in three most frequently used geometric diagrams, viz. circle, square and equilateral triangle.

If it generally holds that

$$z_i = \frac{x}{A} B_i$$

then in case of a circle it is

$$z_i = \frac{\pi r^2}{M \pi r^2} B_i = \frac{B_i}{M} = \frac{1}{M} B_i \tag{19}$$

In case of a square

$$z_i = \frac{a^2}{Ma^2} B_i = \frac{B_i}{M} = \frac{1}{M} B_i \tag{20}$$

In case of an equilateral triangle

$$z_i = \frac{\frac{\sqrt{3}}{4} s^2}{M \frac{\sqrt{3}}{4} s^2} B_i = \frac{B_i}{M} = \frac{1}{M} B_i \tag{21}$$

As can be seen, expression 1:M appears here in all three cases which again can be interpreted as scale. And again, as

$$\frac{1}{M} = const$$

the rule of direct proportionality is also present here.

Let us now deepen the resolution level and turn specific attention to the internal division of geometric diagrams since by their internal division one expresses precisely their internal structure.

The most frequent ways of internal division of a circle are division in sectors and circular rings. However, the question arises which way, from the point of view of thematic cartography, is more suitable: Evidently the one which makes it possible to estimate internal structure more precisely. In this respect division into sectors is to be preferred since the area of a sector depends only on the central angle φ_i, and whether one φ_i is larger or smaller than another one can be relatively well estimated. Thus, if one prefers the division of a circle in to sectors, the goal is to determine the size of central angle φ_i. It is evident that the area of a sector with $\varphi_i = 1°$ will be $\pi r^2/360$. It means that the area of a sector will generally be $(\pi r^2/360)\varphi_i$, where φ_i must be given in angle degrees.

If $\varphi_i = 360$ then the area of a sector will, of course, be identical to the area of a whole circle, that is πr^2. One knows, however, from relation (19) that the size of structural component z_i is B_i/M. As z_i is now represented by an area of a sector one can write

$$z_i = \frac{\pi r^2}{360}\varphi_i = \frac{B_i}{M} \Rightarrow \varphi_i = \frac{360 B_i}{M\pi r^2} \qquad (22)$$

One also knows that $M\pi r^2 = A$, so relation (22) can be rewritten as

$$\varphi_i = \frac{360}{A} B_i \qquad (23)$$

Since

$$\frac{360}{A} = const = k$$

one can also write

$$\varphi_i = kB_i \qquad (24)$$

It follows from relation (23) that in case that one works with only one structural component covering the whole area of a circle, then $B_i = A$, from where it follows that $\varphi_i = 360$. This shows that the determination of φ_i is correct.

The square can be divided in different ways. Analogically as in case of a circle two ways are found to be most frequent: division in rectangles and division in forms that structurally resemble circular rings, the so called “square rings“. Their mutual comparison, however, supports preferring the first division, into rectangles. Then

$$z_i = ab = \frac{B_i}{M} \Rightarrow b = \frac{1}{a}\frac{B_i}{M} = \frac{1}{a}\frac{1}{M}B_i \tag{25}$$

where a and b are sides of a rectangle, a being identical to the side of a square. Since

$$\frac{1}{a} \text{ and } \frac{1}{M}$$

are constants their product is also constant, k, thus

$$b = kB_i \tag{26}$$

If only one structural component is considered covering the whole area of a square then $B_i = A$. Since in the case of a square $A = Ma^2$ it follows from (25) that $b = a$. This is an evidence that the size of b has been determined correctly.

An equilateral triangle can as well be divided differently, but also in this case two ways are found most frequent: division into isosceles trapezoids and, again, division into so called “triangle rings“. Mutual comparison again supports preferring the first division, though the advantages are not as evident as in case of the first division of a circle or a square. Therefore, an equilateral triangle is generally not as suitable as a circle or a square for mapping the internal structure of human-geographical entities or areas. If nevertheless it is used for those purposes then it is to be realized that the first isosceles trapezoid, located at the vertex of an equilateral triangle, will transform into an equilateral triangle. Thus

$$z_1 = \frac{\sqrt{3}}{4}s_1^2 = \frac{B_1}{M} \Rightarrow s_1 = \sqrt[4]{\frac{16}{3}}\sqrt{\frac{B_1}{M}} \tag{27}$$

where s_1 is the side of the first equilateral triangle. Relation (27) can be rewritten as

$$s_1 = \sqrt[4]{\frac{16}{3}}\sqrt{\frac{1}{M}}\sqrt{B_1} \tag{28}$$

Since two first members on the right-hand side of (28) are constants their product is also constant, k; so one can write

$$s_1 = k\sqrt{B_1} \tag{29}$$

The second figure, located under the first equilateral triangle, will already be an isosceles trapezoid, but if one joins it to the first figure one gets again an equilateral triangle with the side s_2. Then one can write

$$z_1 + z_2 = \frac{\sqrt{3}}{4}s_2^2 = \frac{B_1}{M} + \frac{B_2}{M} \Rightarrow s_2 = \sqrt[4]{\frac{16}{3}}\sqrt{\frac{B_1 + B_2}{M}} \tag{30}$$

This procedure can be continued until one comes to the original equilateral triangle, now with the side s_n; so one can write

$$z_1 + z_2 + \ldots + z_n = \frac{\sqrt{3}}{4}s_n^2 = \frac{B_1}{M} + \frac{B_2}{M} + \ldots + \frac{B_n}{M} \Rightarrow$$
$$\Rightarrow s_n = \sqrt[4]{\frac{16}{3}}\sqrt{\frac{B_1 + B_2 + \ldots + B_n}{M}} \tag{31}$$

Since, however,

$$\sum_{i=1}^{n} B_i = A$$

relation (31) can be rewritten as

$$s_n = \sqrt[4]{\frac{16}{3}}\sqrt{\frac{A}{M}} \tag{32}$$

Relation (32) would hold if one considers only one structural component covering the whole area of an equilateral triangle, because $s_n = s$, where s is the side of an original equilateral triangle.

If one wants to compute the area of the second structural component, z_2, that is now mapped in the form of an isosceles trapezoid, it is necessary to compute the area difference between an equilateral triangle with the side s_2 and an equilateral triangle with the side s_1. Thus

$$\left(z_1 + z_2\right) - z_1 = z_2 = \frac{\sqrt{3}}{4}s_2^2 - \frac{\sqrt{3}}{4}s_1^2 = \frac{\sqrt{3}}{4}\left(s_2^2 - s_1^2\right) \tag{33}$$

CONCLUSION

The main intention of this paper has been to design a method for determining the internal structure of geometric diagrams used on thematic maps for mapping the internal structure of human-geographical entities or areas. The precondition for reaching this goal has been to determine first the size of these diagrams. The rule for determining the size of geometric diagrams has been the rule of direct proportionality. It says that the size of geometric diagrams has to be directly proportional to the size of the corresponding human-geographical entities or areas. It has been found that it is reasonable to use the same rule in determining the internal structure of geometric diagrams. It means that the internal structure of geometric diagrams has to be directly proportional to the internal structure of the corresponding human-geographical entities or areas. The rule of direct proportionality has also been formulated mathematically, both for determining the size and for determining the internal structure of geometric diagrams. This formulation has been given not only in a general form, but also for the most frequently used geometric diagrams: the circle, square and equilateral triangle. In such a way a basis has been created for determining the structure of geometric diagrams used on thematic maps on a correct exact basis.

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K PROBLÉMU STANOVENIA VNÚTORNEJ ŠTRUKTÚRY GEOMETRICKÝCH DIAGRAMOV NA TEMATICKÝCH MAPÁCH

Veľkosť a vnútorná štruktúra sú najdôležitejšie atribúty geometrických diagramov používaných na tematických mapách. Tieto atribúty slúžia na zobrazenie veľkosti a vnútornej štruktúry korešpondujúcich humánnogeografických objektov (napr. sídel, priemyselných závodov či celých priemyselných centier), resp. oblastí. Kým stanoveniu veľkosti geometrických diagramov bol už venovaný osobitný článok (Paulov 1996), stanoveniu ich vnútornej štruktúry ešte nie, čo je práve úlohou predloženého príspevku.

Stanovenie vnútornej štruktúry geometrických diagramov predpokladá stanoviť najskôr ich veľkosť; aj príslušné matematické vzťahy, prostredníctvom ktorých sa stanovuje ich vnútorná štruktúra, nadväzujú na vzťahy, prostredníctvom ktorých sa stanovuje ich veľkosť. Preto sa v prvej časti venujeme stručnej rekapitulácii stanovenia veľkosti geometrických diagramov. Pravidlo, na základe ktorého je stanovená ich veľkosť, je pravidlom priamej úmernosti. Toto pravidlo hovorí, že veľkosť geometrických diagramov musí byť priamo úmerná veľkosti humánnogeografických objektov, resp. oblastí. Matematicky je pravidlo vyjadrené vzťahom (1), v ktorom A, ako nezávisle premenná, je veľkosť humánnogeografického objektu, resp. oblasti, x, ako závisle premenná, veľkosť odpovedajúceho geometrického diagramu a k konštanta úmernosti. Z pravidla napokon vyplýva vzťah (8), ktorým je daná veľkosť geometrického diagramu, kde M je veľkosť humánnogeografického objektu, resp. oblasti, ktorý(-á) je zobrazený(-á) geometrickým diagramom o veľkosti 1. Výraz 1:M možno interpretovať ako mierku geometrických diagramov. Zo vzťahu (8) možno tiež odvodiť, aké veľké budú najčastejšie používané geometrické diagramy, ak ich veľkosť vyjadrujeme ich plošným obsahom. Vzťahy (10), (11) a (12) udávajú veľkosti premenných, prostredníctvom ktorých možno skonštruovať uvedené diagramy, t. j. polomer kruhu r, stranu štvorca a a stranu rovnostranného trojuholníka s.

Rovnaké pravidlo ako pri stanovení veľkosti geometrických diagramov, t. j. pravidlo priamej úmernosti, je použité aj pri stanovení ich vnútornej štruktúry. Z uplatnenia tohto pravidla plynie, že veľkosť štruktúrnej zložky v geometrickom diagrame z_i je priamo úmerná veľkosti štruktúrnej zložky v humánnogeografickom objekte, resp. oblasti B_i. Táto skutočnosť je zachytená vzťahom (17), resp. (18). Okrem tohto všeobecného vzťahu sú odvodené príslušné vzťahy aj pre najčastejšie používané geometrické diagramy, a to kruh, štvorec a rovnostranný trojuholník, v ktorých opäť vystupuje výraz pre mierku, t. j. 1:M [vzťahy (19), (20) a (21)].

Keďže sa vnútorná štruktúra geometrických diagramov vyjadruje prostredníctvom ich vnútorného členenia, odvodené sú tiež príslušné vzťahy pre vnútorné členenie kruhu, štvorca a rovnostranného trojuholníka. Pri kruhu je uprednostnené členenie na kruhové výseky, pri štvorci na obdĺžniky a pri rovnostrannom trojuholníku na rovnoramenné lichobežníky. Poznamenajme ale, že rovnoramenný lichobežník sa pri vrchole rovnostranného trojuholníka mení tiež na rovnostranný trojuholník. Ak k nemu pripojíme teraz rovnoramenný lichobežník, dostávame opäť rovnostranný trojuholník. Takýto postup možno uplatniť, až sa pokryje celý pôvodný rovnostranný trojuholník. Zodpovedajúce premenné, prostredníctvom ktorých sa stanovuje plocha štruktúrnej zložky, sú dané vzťahmi (23), (25) a (27), pričom vzťah (23) udáva veľkosť stredového uhla φ_i pri členení kruhu na kruhové výseky, vzťah (25) veľkosť strany b príslušného obdĺžnika pri členení štvorca na obdĺžniky a vzťah (27) veľkosť strany prvého rovnostranného trojuholníka s_1, umiestneného pri vrchole pôvodného rovnostranného trojuholníka.