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ON A LATTICE OF TETRACHORDS

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ABSTRACT. The importance of considering of tetrachords lies on the fact that they are the simplest structured tone systems in the music theory. We show that the set of all tetrachords is naturally structured as a lattice. There are 156 superparticular tetrachords, where nine of them can be nontrivially expressed in the form of the geometrical generalized sequence. We show how these 9 tetrachords construct tone systems of 12 or 5 notes (i.e., qualitative musical degrees).

1. Introduction

The importance of considering of *tetrachords* lies on the fact that they *are the simplest structured tone systems in the music theory.* For instance, the theory of pentatonics is much more complicated, cf. [1]. By a *tetrachord* we mean a quadruple of real numbers $T = (\tau_0, \tau_1, \tau_2, \tau_3)$, such that $1 = \tau_0 < \tau_1 < \tau_2 < \tau_3 = 4/3$. (Without loss of generality, from the viewpoint of tuning of music instruments, we identify tone pitches with their relative frequencies.)

Every discrete tone system can be expressed as a geometrical generalized sequence (abbreviation: GGS). This idea, to describe tone systems via GGS, we applied to the following tone systems used in the western music: Pythagorean Tuning, [6]; Ptolemaic Tuning, [5]; Praetorius (1/4-meantone) Tuning, [4]; Diatonic scales, [3]; (and trivially to Equal Temperament).

In this paper we show that the set of all tetrachords is naturally structured as a lattice. There are 156 superparticular tetrachords, where nine of them can be expressed nontrivially in the form of the GGS. *We show how these 9 tetrachords construct tone systems of 12 or 5 notes (i.e., qualitative musical degrees).*

Denote by $\mathbb{N} = \{0, 1, 2, \dots,$ $\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$. If we denote by $\mathcal{L} = \big((0,\infty), \cdot, 1, \leq\big)$ the usual multiplicative group on reals with the natural order,

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then the $\mathcal{L}$-length b/a of the interval (a,b), $0 < a \le b < \infty$, is called *the musical interval.* For $n \in \mathbb{N}$, we say that a sequence $\langle \tau_i \rangle$ is an *n-geometrical generalized sequence* if there exist

$$\alpha \in \mathcal{L}\,, \quad X = (X_1, X_2, \cdots, X_n) \in \mathcal{L}^n\,, \qquad \nu_{i,k} \in \mathbb{Z}\,,$$

such that

$$\tau_i = \alpha \cdot X_1^{\nu_{i,1}} X_2^{\nu_{i,2}} \dots X_n^{\nu_{i,n}}\,, \quad \nu_{i,k} \le \nu_{i+1,k}\,,$$
$$k = 1, 2, \dots, n\,, \quad \nu_{i,1} + \nu_{i,2} + \cdots + \nu_{i,n} = i\,, \quad i \in \mathbb{Z}\,.$$

2. Examples of tetrachords

There are examples of tetrachords of the various type and construction:

(1) *Lydian Pythagorean*: $(1, 9/8, 81/64, 4/3)$,

(2) *Frygian Pythagorean*: $(1, 9/8, 32/27, 4/3)$,

(3) *Dorian Pythagorean*: $(1, 256/243, 32/27, 4/3)$,

(4) *Dorian Aristoxenos*: $(1, 16/15, 32/27, 4/3)$,

(5) *Chromatic Aristoxenos*: $(1, 16/15, 10/9, 4/3)$,

(6) *Enharmonic Aristoxenos*: $(1, 32/31, 16/15, 4/3)$,

(7) *Archytas' Chromatic*: $(1, 28/27, 9/8, 4/3)$,

(8) *Lyra Tuning*: $(1, 28/27, 32/27, 4/3)$,

(9) *Mohajira-type*: $(1, 59/54, 11/9, 4/3)$,

(10) *Al-Farabi Diatonic*: $(1, 8/7, 64/49, 4/3)$.

We say that a tone system is *based on prime numbers* 1, 2, 3, 5, 7, etc. if the acoustical intervals the (unison) octave, fifth, third, seventh, etc. are used in its construction. These numbers are the numbers of harmonics in the Fourier series of the tone.

Tetrachords, the archaic tone systems, have also a philosophical context of their construction. The *Pythagoreans* built tone systems based on numbers 1, 2 and 3, because they asserted that only these numbers are perfect. Tone systems of the *Aristoxenos School* contain also musical intervals based on 5, 7, etc. There is also a large group of tetrachords coming from the Islamic world (*Al-Farabi*, *Mohajira*). For a present music theory survey of tetrachords and tetrachordal scales, see [2].

3. The set of all tetrachords

Consider the subset of all GGSs given by $n = 4$ and the Diophantine equation system:

$$\begin{aligned} \nu_{1,k} \leq \nu_{2,k} \leq \nu_{3,k} \leq \nu_{4,k}\,, \\ \nu_{k,1} + \nu_{k,2} + \nu_{k,3} + \nu_{k,4} = k\,, \\ k = 1,2,3,4\,. \end{aligned} \tag{1}$$

Denote by

$$A = \begin{bmatrix} \nu_{1,1} & \nu_{1,2} & \nu_{1,3} & \nu_{1,4} \\ \nu_{2,1} & \nu_{2,2} & \nu_{2,3} & \nu_{2,4} \\ \nu_{3,1} & \nu_{3,2} & \nu_{3,3} & \nu_{3,4} \\ \nu_{4,1} & \nu_{4,2} & \nu_{4,3} & \nu_{4,4} \end{bmatrix}$$

the matrix of integers satisfying (1). According to the commutativity, the all $4! \cdot 15 = 360$ possibilities of A satisfying (1) are reduced to the following 15 matrixes:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}, \quad A_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}, \quad A_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}, \quad A_{14} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}, \quad A_{15} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

With the matrix A we associate the following equation system:

$$
\begin{aligned}
\tau_1 &= X_1^{\nu_{1,1}} X_2^{\nu_{1,2}} X_3^{\nu_{1,3}} X_4^{\nu_{1,4}} \\
\tau_2 &= X_1^{\nu_{2,1}} X_2^{\nu_{2,2}} X_3^{\nu_{2,3}} X_4^{\nu_{2,4}} \\
\tau_3 &= X_1^{\nu_{3,1}} X_2^{\nu_{3,2}} X_3^{\nu_{3,3}} X_4^{\nu_{3,4}} = 4/3 \\
\tau_4 &= X_1^{\nu_{4,1}} X_2^{\nu_{4,2}} X_3^{\nu_{4,3}} X_4^{\nu_{4,4}} = 3/2 \\
& 1 < \tau_1 < \tau_2 < \tau_3 \,.
\end{aligned}
\tag{2}
$$

The proofs of the following theorems are easy.

Theorem 1. *According to the commutativity, the Diophantine equation system* (1) *has the unique solution* A *such that* $\det A \neq 0$, *namely* A_{15}.

Theorem 2. *Let* $A = A_1, A_2, \ldots, A_{15}$, $1 < \tau_1 < \tau_2 < 4/3$. *Then the equation system* (2) *has the solutions collected in Table 1* (*the intervals for the parameters are in Table 2*).

Table 1. Constructing intervals

	X_1	X_2	X_3	X_4
A_1	–	–	–	–
A_2	$\sqrt[3]{4/3}$	$9/8$	–	–
A_3	$9/8$	$508/243$	–	–
A_4	$\sqrt{32/27}$	$9/8$	–	–
A_9	$9/8$	$\sqrt{32/27}$	–	–
A_{10}	$256/243$	$9/8$	–	–
A_6	$9/8$	$508/243$	–	–
A_7	$\sqrt{32/27}$	$9/8$	–	–
A_5	t	$4/(3t^2)$	$9/8$	–
A_8	t	$4/(3t^2)$	$9/8$	–
A_{11}	t	$\sqrt{4/(3t)}$	$9/8$	–
A_{12}	$9/8$	$8t/9$	$4t/3$	–
A_{13}	t	$9/8$	$32/(27t)$	–
A_{14}	t	$32/(27t)$	$9/8$	–
A_{15}	t	v/t	$4/(3v)$	$9/8$

Theorem 3. *For every* T_i, $i \in \{2, 3, \ldots, 15\}$, *there exists* $k \in \{1, 2, 3, 4\}$ *such that* $X_k = 9/8$ (*the Pythagorean whole tone*).

TABLE 2. The set of all tetrachords

	τ_1	τ_2	
T_1	–	–	
T_2	$\sqrt[3]{4/3}$	$\sqrt[3]{16/9}$	
T_3	$9/8$	$81/64$	
T_4	$\sqrt{32/27}$	$32/27$	
T_6	$9/8$	$32/27$	
T_7	$\sqrt{32/27}$	$\sqrt{3/2}$	
T_9	$9/8$	$\sqrt{3/2}$	
T_{10}	$256/243$	$32/27$	
T_5	t	t^2	$1 < t < \sqrt{4/3}$
T_8	t	$4/(3t)$	$1 < t < 4/3$
T_{11}	t	$\sqrt{4t/3}$	$1 < t < 4/3$
T_{12}	$9/8$	t	$9/8 < t < 4/3$
T_{13}	t	$9t/8$	$9/8 < t < 4/3$
T_{14}	t	$32/27$	$1 < t < 32/27$
T_{15}	t	v	$1 < t < v < 4/3$

THEOREM 4. *Let $T_1, T_2, \ldots, T_{15}$ be tetrachords corresponding to $A_1, A_2, \ldots, A_{15}$, respectively.*

Then the expressions of T_1, T_2, $\ldots, T_{15}$ are collected in Table 2.

THEOREM 5. *For arbitrary τ_1, τ_2 $(1 < \tau_1 < \tau_2 < 4/3)$, and A_{15}, the unique solution of the equation system (2) is the following: $(X_1, X_2, X_3, X_4) = \big(\tau_1, \tau_2/\tau_1, 4/(3\tau_2), 9/8\big)$.*

Thus, we obtained the expression (possible, not unique) of each tetrachord by a GGS. Since $A_1, A_2, \ldots, A_{15}$ are (according to the commutativity) the all possibilities of powers for GGS, solving the system (2) for $A_1, A_2, \ldots, A_{15}$, respectively, we obtain *a classification of all tetrachords*, i.e., Table 2.

4. Tetrachord lattice

DEFINITION 1. The vector function $\Theta(\cdot) = \big(1, \tau_1(\cdot), \tau_2(\cdot), 4/3\big)$ is said to be the *F-tetrachord* if the quadruple $\Theta(M) = \big(1, \tau_1(M), \tau_2(M), 4/3\big)$ is a tetrachord for every $M \in D_\Theta$, where D_Θ is the support of Θ ($\emptyset$, 0-, 1-, or 2-dimensional, see Table 2).

Let $\mathcal{T} = \{\Theta_1, \Theta_2, \dots \Theta_{15}\}$, where $\Theta_i = \{T_i(M) \colon M \in D_{\Theta_i}\}$, D_{Θ_i} is the support of Θ_i, θ_i corresponds to T_i in Table 2, $i = 1, 2, \dots, 15$. For $\Phi, \Psi \in \mathcal{T}$, we define $\Phi \leq \Psi$ if and only if $\Phi \subseteq \Psi$. In Table 2, we see that each couple of elements of $\mathcal{T}$ has both the supremum $\wedge$ and the infimum $\vee$.

THEOREM 6. *The set $\mathcal{T}$ equipped with the operations $\wedge, \vee$ is a non-modular atomic lattice with atoms $\Theta_2, \Theta_3, \Theta_4, \Theta_6, \Theta_7, \Theta_9, \Theta_{10}$.*

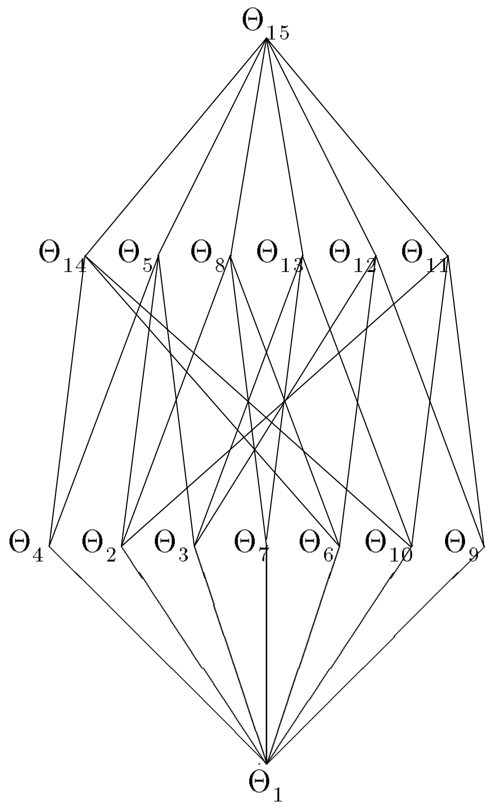

FIGURE 1. Lattice of F-tetrachords

P r o o f. We see that $\mathcal{T}$ is a lattice, see Figure 1. Since it contains a pentagon, it is a non-modular, hence non-distributive lattice. The assertion about atoms is easy. $\square$

The *maximal* F-tetrachord Θ_{15} is given by A_{15}, see Theorem 5. The *minimal* F-tetrachord Θ_1 is given by A_1 and it is the empty set of tetrachords. Indeed, A_1 yields a contradiction: $X_1^3 = 4/3$, $X_1^4 = 3/2$.

The F-tetrachords Θ_1 and Θ_{15} we will call *trivial*.

The support of Θ_1 is $\emptyset$, and $T \in \Theta_1$ have no GGS constructing intervals.

The *nontrivial* 13 F-tetrachords $\Theta_2, \Theta_3, \dots, \Theta_{14}$ are of two types. The first one contains seven F-tetrachords $(\Theta_2, \Theta_3, \Theta_4, \Theta_6, \Theta_7, \Theta_9, \Theta_{10})$ with one individual tetrachord with 2 GGS constructing intervals, i.e., X_1, X_2. Each of these F-tetrachords has a 0-dimensional support.

The second type contains 6 F-tetrachords $(\Theta_5, \Theta_8, \Theta_{11}, \Theta_{12}, \Theta_{13}, \Theta_{14})$ with the 1-dimensional support. Individual tetrachords have 3 GGS constructing intervals, i.e., X_1, X_2, X_3.

The support of Θ_{15} is a 2-dimensional set. Individual tetrachords $T \in T_{15}$ have 4 GGS constructing intervals, i.e., X_1, X_2, X_3, X_4.

5. S-tetrachords

We say that a tetrachord T is *superparticular* if τ_1, τ_2/τ_1, $4/(3\tau_2)$ are of the form $(n+1)/n$ for some $n = 1, 2, \dots$, see [1]. Tetrachords no. 4, 5, 6, 8, and 10 in Section 2 are superparticular.

Clearly Θ_{15} contains all superparticular tetrachords (see Theorem 5), Θ_1 contains none. But what about nontrivial *F-tetrachords* $\Theta_2, \dots, \Theta_{14}$?

Definition 2. A superparticular tetrachord which can be expressed as a 2- or 3-geometrical generalized sequence we will call the *S-tetrachord.* The set of all S-tetrachords will be denoted by W.

If we define the *tetrachord sequence* $\langle \tau_i \rangle$ recursively as follows:

$$(\tau_{4k}, \tau_{4k+1}, \tau_{4k+2}, \tau_{4k+3}) = (3/2)^k T,$$

(where T is a tetrachord and k an integer), we obtain a tone system with the period 3/2 (i.e., the perfect fifth).

Definition 3. We say that two tetrachords $T = (\tau_0, \tau_1, \tau_2, \tau_3)$ and $L = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ are *equivalent*, we write $L \approx T$, if for the corresponding tetrachord sequences $\langle \tau_i \rangle$ and $\langle \lambda_i \rangle$, there exist integers p, q and $k \in (0, \infty)$ such that

$$(\tau_p, \tau_{p+1}, \tau_{p+2}, \tau_{p+3}) = k \cdot (\lambda_q, \lambda_{q+1}, \lambda_{q+2}, \lambda_{q+3}).$$

It is easy to see that this definition introduces a relation of *equivalency* on the set of all tetrachords (tetrachord sequences).

Theorem 7. *According to the equivalency of tetrachords, all S-tetrachords are the following nine:*

$$\begin{aligned}
&(1, 9/8, 5/4, 4/3), \quad (1, 9/8, 6/5, 4/3), \quad (1, 16/15, 6/5, 4/3),\\
&(1, 9/8, 7/6, 4/3), \quad (1, 9/8, 21/16, 4/3), \quad (1, 9/8, 8/7, 4/3),\\
&(1, 7/6, 21/16, 4/3), \quad (1, 8/7, 9/7, 4/3), \quad (1, 9/8, 9/7, 4/3).
\end{aligned}$$

TABLE 3. Musical intervals $[\tau_1\,,\ \tau_2/\tau_1\,,\ 4/(3\tau_2)]$ constructing superparticular tetrachords

5/4	17/16	256/255	6/5	15/14	28/27
5/4	18/17	136/135	6/5	16/15	25/24
5/4	19/18	96/95	6/5	19/18	20/19
5/4	20/19	76/75	7/6	9/8	64/63
5/4	21/20	64/63	7/6	10/9	36/35
5/4	22/21	56/55	7/6	12/11	22/21
5/4	24/23	46/45	7/6	15/14	16/15
5/4	26/25	40/39	8/7	8/7	49/48
5/4	28/27	36/35	8/7	9/8	28/27
5/4	31/30	32/31	8/7	10/9	21/20
6/5	11/10	100/99	8/7	13/12	14/13
6/5	12/11	55/54	9/8	10/9	16/15
6/5	13/12	40/39	10/9	11/10	12/11

TABLE 4. S-tetrachords

$T \in T_{12}$	$T \in T_{14}$	$T \in T_{13}$	based on
$(1, 9/8, 21/16, 4/3)$	$(1, 7/6, 32/27, 4/3)$	$(1, 7/6, 21/16, 4/3)$	$2, 3, 7$
$(1, 9/8, 8/7, 4/3)$	$(1, 64/63, 32/27, 4/3)$	$(1, 64/63, 8/7, 4/3)$	$2, 3, 7$
$(1, 9/8, 9/7, 4/3)$	$(1, 8/7, 32/27, 4/3)$	$(1, 8/7, 9/7, 4/3)$	$2, 3, 7$
$(1, 9/8, 7/6, 4/3)$	$(1, 28/27, 32/27, 4/3)$	$(1, 28/27, 7/6, 4/3)$	$2, 3, 7$
$(1, 9/8, 5/4, 4/3)$	$(1, 10/9, 32/27, 4/3)$	$(1, 10/9, 5/4, 4/3)$	$2, 3, 5$
$(1, 9/8, 6/5, 4/3)$	$(1, 16/15, 32/27, 4/3)$	$(1, 16/15, 6/5, 4/3)$	$2, 3, 5$

Proof. In Table 3, there are collected all 26 superparticular ratios, the combinations of the numbers $[\tau_1,\ \tau_2/\tau_1,\ 4/(3\tau_2)]$. To obtain this table, we used the general algorithm for dividing a given interval into a certain number of superparticular steps, [1]. Table 3 is attributed to I. E. Hofmann, [2]. One triple $[\tau_1,\ \tau_2/\tau_1,\ 4/(3\tau_2)]$ yields $3 \times 2 \times 1 = 6$ permutations. So, the set S consists of $6 \times 26 = 156$ superparticular tetrachords. We have to find the intersection

$$W = S \cap \bigcup_{i=2,\dots,14} \ \bigcup_{M \in D_{\Theta_i}} T_i(M)\,,$$

where $D_{\Theta_\cdot}$ is the support of $\Theta_\cdot \in \mathcal{T}$.

(a) *Superparticular tetrachords.*

The interval 9/8 appears in the F-tetrachord T_{12} and also in the combinations [7/6, 9/8, 64/63], [8/7, 9/8, 28/27], [9/8, 10/9, 16/15], see Table 3. So we obtained the first column of Table 4.

Analogously, for T_{13}, T_{14}, we obtain the second and third column of S-tetrachords in Table 4.

It can be verified that the following pairs of tetrachords are equivalent:

$$\begin{aligned}
(1,9/8,7/6,4/3) &\approx (1,28/27,32/27,4/3)\,,\\
(1,9/8,21/16,4/3) &\approx (1,7/6,32/27,4/3)\,,\\
(1,9/8,8/7,4/3) &\approx (1,64/63,32/27,4/3)\,,\\
(1,9/8,9/7,4/3) &\approx (1,8/7,32/27,4/3)\,,\\
(1,9/8,5/4,4/3) &\approx (1,10/9,32/27,4/3)\,,\\
(1,9/8,6/5,4/3) &\approx (1,16/15,32/27,4/3)\,,\\
(1,7/6,21/16,4/3) &\approx (1,64/63,8/7,4/3)\,,\\
(1,8/7,9/7,4/3) &\approx (1,28/27,7/6,4/3)\,,\\
(1,16/15,6/5,4/3) &\approx (1,10/9,5/4,4/3)\,.
\end{aligned}$$

(b) *Non superparticular tetrachords.*

$4/(3\tau_2)$ is not superparticular for T_3.

τ_1 is not superparticular for T_2, T_4, T_7, T_{10}.

τ_2/τ_1 is not superparticular for T_6, T_9.

If $t=(n+1)/n$, then $t^2=(n^2+2n+1)/n^2$ is not superparticular, so T_5 is not superparticular.

If $t=(n+1)/n$, then $\tau_2/\tau_1=\sqrt{4n/(3n+3)}$. Suppose $n=k^2$, then $2k=1+\sqrt{3k^2+3}$ which has no integer solution. Hence T_{11} is not superparticular.

τ_2/τ_1 is not superparticular for T_8. Indeed, $\tau_2/\tau_1=4/(3t^2)$. Hence $4-3t^2=1$ implies $t=-1$ or $+1$, a contradiction. $\square$

We proved also the following statements:

Theorem 8. *No S-tetrachord is Pythagorean (based only on primes 2 and 3).*

Theorem 9. *S-tetrachords are based either on the triplet $[2,3,5]$ or on $[2,3,7]$.*

Theorem 10. *Other primes or triplets of primes different of $[2,3,5]$ and $[2,3,7]$ or n-tuples of primes cannot construct S-tetrachords.*

6. Symmetry: pentatonics or 12-degree systems

Definition 4. We say that a tone system

$$D=\{d_1,d_2,\dots,d_n\colon\, d_1=1<d_2<\dots<d_{n-1}<d_n=2\}$$

has the center of symmetry $\sqrt{2}$ if for every $x \in D$ there exists $y \in D$ such that $xy = 2$.

We join all S-tetrachord sequences based on the number triple [2, 3, 5] into a new tone system as follows:

1. take two tetrachord periods ($k = 0, 1$);

2. group the neighbouring values with the ratio less or equal than 81/80 (Comma of Dydimus) into clusters (qualitative musical degrees);

3. the $\mathcal{L}$-length between the tag points of two different clusters is equal or greater than the diatonic semitone (16/15);

4. consider the octave equivalence for the new tone system.

In this way, we obtain the following 11 degrees (clusters):

I: 16/15 (diatonic semitone);
II: 10/9, 9/8 (the major and Pythagorean tones, respectively);
III: 32/27, 6/5 (the Pythagorean and pure minor third, respectively);
IV: 5/4 (the major third);
V: 4/3 (the perfect fourth);
VI: –
VII: 3/2 (the perfect fifth);
VIII: 8/5 (the minor sixth)
IX: 5/3, 27/16 (the pure and Pythagorean minor sevenths, respectively);
X: 16/9, 9/5 (the Pythagorean and minor sevenths, respectively);
XI: 15/8 (the major seventh);
XII: $2 \approx 1$.

We denote the resulting tone system (the many valued 12-degree system) as follows:

$$D_{2,3,5} = \{1\,, 16/15\,, (10/9, 9/8), (32/27, 6/5), 5/4, 4/3\,,\\ 3/2\,, 8/5\,, (5/3, 27/16)\,, (16/9, 9/5), 2\}\,,$$

where $(\dots)$ denotes the clustering.

Consider also the 12-degree many valued tone system:

$$D^{+}_{2,3,5} = \{1,\ 16/15\,, (10/9, 9/8), (32/27, 6/5), 5/4, 4/3\,,\\ (45/32, 64/45), 3/2, 8/5\,, (5/3, 27/16), (16/9,\ 9/5), 15/8\,, 2\}\,.$$

It is easy to see that the tone system $D_{2,3,5}$ has not the center of symmetry $\sqrt{2}$.

We say that the system D^{+} is the *minimal extension of the system* D *with respect to a property* $(*)$ if (1) $D \subseteq D^{+}$; (2) D^{+} fulfills the property $(*)$; (3) every proper subset D', $D \subseteq D' \subset D^{+}$ does not fulfill the property $(*)$.

THEOREM 11. *The tone system $D^{+}_{2,3,5}$ is the minimal extension of $D_{2,3,5}$ such that it has the center of symmetry $\sqrt{2}$.*

Analogously, we join all S-tetrachord sequences based on the number triple $[2,3,7]$ into a new tone system as follows:

1. take two tetrachord periods ($k = 0,1$);

2. group values having their ratio less or equal than 49/48 into clusters (qualitative musical degrees);

3. the $\mathcal{L}$-length between the tag points of two different clusters is greater than the chromatic semitone (25/24);

4. consider the octave equivalence for the new tone system.

In this way, we obtain the following 5 degrees (clusters):

I: 9/8, 8/7, 7/6, 32/27;
II: 9/7, 21/16, 4/3;
III: 3/2, 32/21, 14/9;
IV: 27/16, 12/7, 7/4, 16/9;
V: 27/14, 63/32, $2 \approx 1$, 64/63, 28/27.

We denote the resulting tone system (the many valued pentatonic) as follows:

$$\begin{aligned} D_{2,3,7} = \big\{&(1, 64/63, 28/27), (9/8, 8/7, 7/6, 32/27), (9/7, 21/16, 4/3), \\ &(3/2, 32/21, 14/9), (27/16, 12/7, 7/4, 16/9), (27/14, 63/32, 2)\big\}, \end{aligned}$$

where $(\dots)$ denotes the clustering.

THEOREM 12. *The tone system $D_{2,3,7}$ has the center of symmetry $\sqrt{2}$.*

Note that 12-degree tone systems are characteristic for the European cultural zone, while 5-degree tone systems are typical for Asia (excluding the Islamic world and India), Africa, Polynesia, Micronesia, Malaysia, etc.

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