Tatra Mt. Math. Publ. **16** (1999), 117–125

Tatra Mountains
Mathematical Publications

A BOOLEAN GENERALIZATION OF THE DEMPSTER-SHAFER CONSTRUCTION OF BELIEF AND PLAUSIBILITY FUNCTIONS

George Markakis

ABSTRACT. The present paper focuses on the relation of boolean fuzzy sets with the Dempster-Shafer theory of evidence. It is shown that a B-fuzzy set is actually a boolean basic probability assignment in X. This defines B-possibility and B-necessity measures which play the role of B-upper and B-lower probabilities. If $(\mathbf{B}, p)$ is a probability algebra then we have a way to quantify these functions and these are shown to be the general plausibility and belief measures used as the basis of the theory of evidence. In the particular case where $\mathbf{B}$ is a power set algebra, we also show that the above construction coincides with the one presented in Dempster's original work.

1. Boolean powers and B-fuzzy sets

In this section we summarize the basic results from [Dr] and [DM1], i.e., the theory of Boolean Powers and its relation to B-fuzzy sets. For more on the above theory the reader can find useful references also in [DM2], [M1], [M2] and [Man].

1.1 The duality of the function concept.

In [DM1] we gave an interpretation of the dual aspects of a function, as follows: To any function $f_1 : T \to X$ there corresponds a set valued function defined by

$$f_2(a) := \{t \in T \colon f_1(t) = a\}.$$

It is clear that f_2 partitions T, for the following relations are easily checked:

(1) $a \neq b \Rightarrow f_2(a) \cap f_2(b) = \emptyset$, and

(2) $\bigcup_{a \in X} f_2(a) = T$.

2000 Mathematics Subject Classification: Primary 04A72, 28E10 Secondary: 03E40, 03B50.

Keywords: possibility theory, belief and plausibility functions, Boolean powers, Boolean fuzzy sets.

Considering $\mathcal{P}(T)$ as a boolean algebra, we can say that this function partitions unity in this algebra. The converse is also true, since to any set valued function f_2 which partitions unity, there corresponds a unique point function f_1 defined by

$$f_1(t) := a \qquad \text{iff} \qquad t \in f_2(a)\,.$$

Recall that in [DM1] we interpret any point function f_1 as a T-variable element of the set X and the corresponding f_2 as a $\mathcal{P}(T)$-fuzzy element of X. The set of all $\mathcal{P}(T)$-valued functions partitioning unity is the Boolean Power of X with respect to the boolean algebra $\mathcal{P}(T)$. If we assume that $(\mathbf{B}, \vee, \wedge, \neg, 1_B, 0_B)$ is generally a complete Boolean algebra, and X any set we define:

1.2 Definition.

The set of all functions $f\colon X \to \mathbf{B}$ satisfying

(1) $x \neq y \Rightarrow f(x) \wedge f(y) = 0_B$, and

(2) $\bigvee_{x \in X} f(x) = 1_B$

is the *Boolean Power* of X (see [Man], [VH]). We denote this set by $X[\mathbf{B}]$ or $X^{\#}$. All elements of X can be embedded in $X^{\#}$ by means of constant functions, i.e.,

$$\hat{a}(x) = \begin{cases} 1_B & \text{iff } x = a\,, \\ 0_B & \text{iff } x \neq a\,. \end{cases}$$

These elements are the standard or crisp elements of $X^{\#}$.

If in addition R is a (binary) relation defined on X, then its extension to $X^{\#}$ is given by

$$R^{\#}\colon X^{\#} \times X^{\#} \longrightarrow \mathbf{B}\,,$$

where

$$R^{\#}(f,g) := \bigvee_{\substack{x,y \in X;\\ R(x,y)}} f(x) \wedge g(y)\,.$$

This is a Boolean-valued model, in the following sense: A truth function $\|\phi\|$ is defined for all formulas in the appropriate language describing the new model, taking values in $\mathbf{B}$, i.e., for atomic formulas

$$\big\| R(f,g) \big\| := R^{\#}(f,g)\,,$$

and for any formula by induction as follows

$$\begin{aligned} \| \phi \wedge \psi \| &= \| \phi \| \wedge \| \psi \|\,, \\ \| \neg \phi \| &= \neg \| \phi \|\,, \quad \text{and} \\ \big\| \exists x, \phi(x) \big\| &= \bigvee_{f \in X^{\#}} \big\| \phi(f) \big\|\,. \end{aligned}$$

The Łos Theorem is a very useful tool for the calculation of the truth value of any formula with n free variables:

$$\|\phi(f_1, f_2, \ldots, f_n)\| = \bigvee_{\substack{a_1, a_2, \ldots, a_n \in X; \\ \phi(a_1, a_2, \ldots, a_n)}} f_1(a_1) \wedge f_2(a_2) \cdots \wedge f_n(a_n).$$

We say that a proposition ϕ holds if $\|\phi\| = 1_B$.

1.3 B-fuzzy sets and membership functions.

From the above definition and the interpretation given by means of the power set algebra $\mathcal{P}(T)$, any element of $\mathcal{P}(X)^{\#}$ is a B-fuzzy subset of X. Elements of $\mathcal{P}(X)^{\#}$ are functions $f\colon \mathcal{P}(X) \to \mathbf{B}$ partitioning unity in $\mathbf{B}$.

A convenient notation for F is the "sum-product" notation (see [Man], [DM2])

$$F = \sum_{i \in I} A_i \cdot t_i, \tag{1}$$

where

$$F(A) = \begin{cases} t_i & \text{iff } A = A_i, \\ 0_B & \text{else}. \end{cases}$$

Note also that if F is a B-fuzzy subset of X, we have

$$F(A) = \|F = \hat{A}\|.$$

In the following we denote by capital letters $F, G, \ldots$ all the B-fuzzy subsets of X. Elements of $X^{\#}$, i.e., functions $f\colon X \longrightarrow \mathbf{B}$ partitioning unity, are B-fuzzy elements of X and are denoted by small letters. For any $F \in \mathcal{P}(X)^{\#}$, we define its membership function by

$$M_F(f) := \|f \in F\| = \bigvee_{\substack{x \in X, A \subseteq X; \\ x \in A}} f(x) \wedge F(A).$$

The element f may be crisp or fuzzy element of X. Moreover, this function may be restricted only to crisp elements of X, by

$$m_F(x) := \|\hat{x} \in F\| = \bigvee_{A \subseteq X; x \in A} F(A) = \bigvee_{i \in I; x \in A_i} t_i.$$

This restriction called reduced membership function determines the membership function (M_F) uniquely as it is proved in [DM1].

In our treatment, B-fuzzy sets were derived from the generalization of both equality and membership relation and they may contain both crisp and fuzzy elements. Recall also from [DM1] that Zadeh's extension principle is simply the truth value of the proposition defining new (B-fuzzy) mathematical objects.

B-possibility and B-necessity measures

In this section, we recall the basic definitions and results from [M2].

2.1 Definition.

Let $\mathcal{C}$ be a collection of subsets of X. If X is finite we usually take $\mathcal{C} = \mathcal{P}(X)$, otherwise it can be a σ-algebra of "events". We recall that in [M2], we define a *B-possibility measure* to be any function $\Pi\colon \mathcal{C} \to \mathbf{B}$, satisfying

(i) $\Pi(X) = 1_B$, $\Pi(\emptyset) = 0_B$,

(ii) for any countable collection $\{A_i\}_{i\in I}$ of sets in $\mathcal{C}$,

$$\Pi\Big(\bigcup_{i\in I} A_i\Big) = \bigvee_{i\in I} \Pi(A_i)\,. \tag{2}$$

To every B-possibility measure there corresponds a B-necessity measure, which is defined by

$$N(A) = 1_B - \Pi\big(X \setminus A\big)\,. \tag{3}$$

2.2 Remarks.

In [M2] it is shown that to any B-fuzzy set which is clearly different from $\emptyset$, (i.e., $\|F \neq \hat{\emptyset}\| = 1_B$ or equivalently $F(\emptyset) = 0_B$) there corresponds a B-possibility measure by

$$\Pi_F(A) = \bigvee_{x\in A} m_F(x) = \bigvee_{x\in A} \|\hat{x} \in F\|\,. \tag{4}$$

The corresponding B-necessity measure N_F is defined by (3).

Any of the B-measures defined in 2.1 can be easily extended to B-fuzzy subsets of X, by

$$\Pi(G) = \bigvee_{A\subseteq X} \Pi(A) \wedge \|G = \hat{A}\|$$

(where G is a B-fuzzy subset of X).

Note that in (4), the restricted membership function m_F plays the role of a B-possibility distribution defined on X.

The following proposition gives us a way to interpret the B-possibility and the B-necessity of any subset of X.

2.3 Proposition.

If F is a B-fuzzy subset of X and A any subset of X (crisp or B-fuzzy), then

$$\Pi_F(A) = \|F \cap A \neq \hat{\emptyset}\|\,, \quad \textit{and} \tag{5}$$

$$N_F(A) = \|F \subseteq A\|\,. \tag{6}$$

For proof, see [M2].

2.4 Remarks.

From Proposition 2.3 and (4) one can see that if A is a crisp subset of X, the following equation holds

$$\Pi_F(A) = \| F \cap \hat{A} \neq \hat{\emptyset} \| = \bigvee_{\substack{B \subseteq X;\\ B \cap A \neq \emptyset}} F(B) = \bigvee_{\substack{i \in I;\\ A \cap A_i \neq \emptyset}} t_i\,. \tag{7}$$

This equation gives us an interpretation of Π_F as well as a way to calculate it for any crisp subset of X.

3. Evidence theory and B-evidence theory

In this section as well as in the following ones (4 and 5) we may assume for simplicity that X is a finite set. Similar results can be obtained if X is countable, however some modifications are needed if X is not countable. We leave the possibility of this generalization to another work.

3.1 Basic probability assignments.

Evidence theory, otherwise called the Dempster-Shafer Theory [De], [S], is based on a basic probability assignment (b.p.a.) on X, i.e., a function $d\colon \mathcal{P}(X) \to [0,1]$, such that

$$\sum_{A \subseteq X} d(A) = 1\,.$$

This function takes strictly positive values in a collection $\{A_i\}_{i \in I}$ of subsets of X and the elements of this collection are called focal elements. Writing d_i instead of $d(A_i)$, upper and lower probabilities are defined for all elements of $\mathcal{P}(X)$ by

$$P^*(C) = \sum_{A_i \cap C \neq \emptyset} d_i\,, \qquad \text{and} \qquad P_*(C) = \sum_{A_i \subseteq C} d_i\,, \tag{8}$$

respectively. One can prove that these measures are Plausibility and Belief measures (see for example [DP] or [KW]).

3.2 Basic boolean probability assignments.

We can then interpret any B-fuzzy subset of X, clearly different from $\emptyset$, as a basic B-probability assignment (b.B.p.a.), i.e., a function $F\colon \mathcal{P}(X) \to \mathbf{B}$, such that

$$\bigvee_{A \subseteq X} F(A) = 1_B\,.$$

This function is an assignment of "qualitative" probabilities to all subsets of X. It is clear from (1) that this assignment is defined on a class $\{A_i\}_{i\in I}$ of B-focal elements, such that the B-probability of each A_i is t_i. Then we can define the upper and lower B-probabilities of any subset of X, by

$$BP^*(C) = \bigvee_{A_i\cap C\neq\emptyset} t_i\,, \qquad \text{and} \qquad BP_*(C) = \bigvee_{A_i\subseteq C} t_i\,, \tag{9}$$

which are B-valued functions defined on $\mathcal{P}(X)$.

3.3 Proposition.

The functions defined by (9), are the B-possibility and the B-necessity measures defined by the fuzzy set F, or

$$BP^*(C) = \Pi_F(C)\,, \qquad \text{and} \qquad BP_*(C) = N_F(C)\,. \tag{10}$$

P r o o f . Easily follows from (5), (6) and (7). □

4. Quantitative treatment

If $(\mathbf{B}, p)$ is a probability algebra (as in [DM2], [M1]) we have a way to quantify the elements of the boolean algebra by means of the probability p. All "truth statements" can take a value in the $[0,1]$ interval.

4.1 Definition.

For any subset of X, we define ϕ_1 and ϕ_2 by

$$\phi_1(A) = p\big(\Pi_F(A)\big)\,, \qquad \text{and} \qquad \phi_2(A) = p\big(N_F(A)\big)\,.$$

These are functions defined in $P(X)$ taking values in the interval $[0,1]$.

4.2 Proposition.

Let $(\mathbf{B}, p)$ be a probability algebra and X a finite set. Then to any b.B.p.a. on X, there corresponds a unique b.p.a. on X. The functions ϕ_1, ϕ_2 defined in 4.1 are the plausibility and belief measures respectively, that correspond to this b.p.a.

P r o o f . Suppose that $\{t_i\}_{i\in I}$ is a b.B.p.a. on the family $\{A_i\}_{i\in I}$ of subsets of X (or equivalently is a B-fuzzy subset of X). Then taking the probabilities of the events t_i, we have that $\sum_{i\in I} p(t_i) = 1$, since t_i's constitute a partition of unity of $\mathbf{B}$. So the family $\{p(t_i)\}_{i\in I}$ is a b.p.a. on X focused on the sets $\{A_i\}_{i\in I}$.

In addition, using (9) and (10) we can see that for any $C \subseteq X$,

$$\phi_1(C) = p(\Pi_F(C)) = p\Big(\bigvee_{A_i\cap C\neq\emptyset} t_i\Big) = \sum_{A_i\cap C\neq\emptyset} p(t_i)$$

since t_i's are disjoint elements of $\mathbf{B}$. So, ϕ_1 is a function defined as Dempster's upper probability [De] by equation (8) so it is a Plausibility measure. By similar arguments one can see that

$$\phi_2(C) = p(N_F(C)) = p\Big(\bigvee_{A_i\subseteq C} t_i\Big) = \sum_{A_i\subseteq C} p(t_i)$$

and ϕ_2 is defined as Dempster's lower probability and it is a belief measure. □

4.3 Remarks — special cases.

It is clear that there is a distinction between B-measures and their corresponding $[0,1]$-measures. The measures defined by b.B.p.a. are B-possibilities and B-necessities, while their "quantitative" analogues are generally plausibility and belief measures. Then it is easy to see that if the original information contained in the B-fuzzy set F defined by (1), is nested (i.e., $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \ldots$ or $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \ldots$) then the $[0,1]$ measures are also possibility and necessity. If this information is focused on singletons (i.e., $A_i = \{x_i\}$) these measures lead to the same result which is a probability measure on X. In this particular case the B-possibility measure defined by this B-fuzzy set behaves like a B-probability measure on X, in the following sense: Whenever the collection in (2) is pairwise disjoint, the collection $\{\Pi(A_i)\}_{i\in I}$ is also pairwise disjoint collection of elements in $\mathbf{B}$, and the supremum in (2) is actually a Boolean sum (a disjoint union).

The following diagram summarizes our procedure and the quantification of the B-valued functions we propose

$$\begin{tikzcd}
(\mathcal{P}(X)\setminus\{\emptyset\})^{\#} \arrow[r, "(\text{Eq. } 9)"] \arrow[d, "p"'] \arrow[dr, "\phi_1"] & \mathrm{BPOS}(X) \arrow[d, "p"] \\
\mathrm{BPA}(X) \arrow[r, "(\text{Eq. } 8)"'] & \mathrm{PL}(X)
\end{tikzcd}$$

where $\mathrm{BPA}(X)$ is the set of all b.p.a.'s defined on X, $\mathrm{BPOS}(X)$ the set of all B-possibility measures on X, and $\mathrm{PL}(X)$ the set of all plausibility measures of X.

5. Concluding Remarks ($\mathcal{P}(T)$-evidence theory)

Until now we have developed a basis for a B-evidence theory in terms of a general complete boolean algebra $\mathbf{B}$. In this section we restrict ourselves to the algebra of sets $\mathcal{P}(T)$. Although T may be any set, we may assume that it is countable and there is a probability measure μ defined on $\mathcal{P}(T)$. In this way $(\mathcal{P}(T), \mu)$ becomes a probability algebra $(\mathbf{B}, p)$ as in §4, and we have the possibility of the $[0,1]$-interval quantification. In terms of the duality of the function concept defined in 1.1, we can see that to any function $\Gamma\colon T \longrightarrow \mathcal{P}(X)$ there corresponds a unique function $F\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(T)$ that partitions T and conversely. Clearly, if Γ is a many valued function from T to $\mathcal{P}(X)$, then we may define F by

$$F(A) = \{t \in T\colon \Gamma(t) = A\}$$

and conversely if F partitions T, then $F(A) \cap F(B) = \emptyset$, for all $A, B \subseteq X$ and $\bigcup_{A \subseteq X} F(A) = T$. Then the function Γ is defined by

$$\Gamma(t) = A \qquad \text{iff} \qquad t \in F(A).$$

From our interpretation it follows that F is an element of the boolean power of $\mathcal{P}(X)$ with respect to $\mathcal{P}(T)$, so it is a $\mathcal{P}(T)$-fuzzy subset of X, while Γ is a T-variable subset of X. Recall that Dempster [De] defined upper and lower probabilities for every $C \subseteq X$, by

$$P^*(C) = \mu(C^*) \qquad \text{and} \qquad P_*(C) = \mu(C_*), \tag{11}$$

where the sets C^* and C_* were defined by

$$C^* = \{t \in T\colon \Gamma(t) \cap C \neq \emptyset\}, \tag{12}$$

$$C_* = \{t \in T\colon \Gamma(t) \subseteq C\}. \tag{13}$$

We can see that Dempster's procedure is just a special case of the construction outlined in §3 and §4, by observing that the sets defined by (12) and (13) are just the B-possibility and B-necessity of C with respect to the boolean algebra $\mathcal{P}(T)$, or

$$C^* = \Pi_F(C) \qquad \text{and} \qquad C_* = N_F(C).$$

It is also clear that the upper and lower probabilities defined by (11) are just the compositions ϕ_1 and ϕ_2 defined in 4.1.

Note that in (11) Dempster's correction by $\mu(X^*)$ is replaced by the assumption that $F(\emptyset) = 0_B$ (the fuzzy set F is clearly different from the empty set). This is equivalently expressed as the requirement that Γ can take values on $\mathcal{P}(X) \setminus \{\emptyset\}$ only.

From the above discussion it follows that $[0,1]$-evidence theory could be a special case of a more general one (B-evidence theory), and can be seen as a two step procedure:

(1) The definition of $\mathcal{P}(T)$-possibilities and $\mathcal{P}(T)$-necessities of subsets of X

(2) The quantification of the above measures in the $[0,1]$ interval by the measure μ.

The task of formulating a general B-evidence theory following the steps of Dempster and Shafer, is left for a future work.

Acknowledgments. I thank the anonymous referee for his valuable comments which helped me to improve the original manuscript.

REFERENCES

[De] DEMPSTER, A. P.: *Upper and lower probabilities induced by a multivalued mapping*, Ann. Inst. Statist. Math. **38** (1967), 235–339.

[Dr] DROSSOS, C. A.: *Foundations of fuzzy sets: a nonstandard approach*, Fuzzy Sets and Systems **37** (1990), 287–307.

[DM1] DROSSOS, C. A.—MARKAKIS, G.: *Boolean fuzzy sets*, Fuzzy Sets and Systems **46** (1992), 81–95.

[DM2] DROSSOS, C. A.—MARKAKIS, G.: *Boolean powers and stochastic spaces*, Math. Slovaca **44** (1994), 1–19.

[DP] DUBOIS, D.—PRADE, H.: *Possibility Theory*, Plenum Press, New York, 1988.

[KW] KLIRG.—WANG, B.: *Fuzzy Measure Theory*, Plenum Press, New York, 1992.

[Man] MANSFIELD, R.: *The theory of Boolean ultrapowers*, Ann. Math. Logic **2** (1971), 297–323.

[M1] MARKAKIS, G.: *The Boolean generalization of infinitesimal analysis with applications to fuzzy sets*, PhD dissertation (in Greek), University of Patras, Greece, 1990.

[M2] MARKAKIS, G.: *Boolean fuzzy sets and possibility theory*, Fuzzy Sets and Systems (1998) (to appear).

[S] SHAFER, G.: *A Mathematical Theory of Evidence*, Princeton Univ. Press, Priceton, New York, 1976.

[VH] VOPĚNKA, P.—HÁJEK, P.: *The Theory of Semisets*, North Holland, Amsterdam, 1972.

Received April 3, 1998

Department of Greenhouse
Coprs and Floriculture
Technological Educational Institute of Iraklion
Stavromenos, Heraklion GR–71500
GREECE

E-mail: gmark@lyttos.admin.teiher.gr