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REMARKS ON TWO TYPES OF EXTENDED AGGREGATION FUNCTIONS

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ABSTRACT. In this paper, the reverse functions of an Extended Quasi-linear Weighted Mean, EQLWM, and of an extended Ordered Weighted Averaging operator, OWA, are considered. Firstly, we characterize when these reverse functions are Extended Aggregation Functions, EAF, by means of a right descending and a right regular triangle of weights respectively. After that, the conditions to obtain and EOWA and an EQLWM as well as their reverse functions which are EAF are given. Finally, in the last section, two inequalities close to the self-identity of each one of these aggregations are studied and these inequalities are analyzed for the two types of Extended Aggregation Functions, i.e., EQLWM and EOWA operators.

1. Introduction

In the last years the problem of information aggregation has been a main topic of study for a lot of researchers. Already, different classes of operators which are very useful in the aggregation field exist, for instance the OWA operators ([16], [17]), the EAF ([12], [4], [5]) and other as [13], etc. Futhermore, EOWA operators that are a generalization of OWA operators and that are very related to EAF [12] occur. The extended operators are multi-dimensional operators, that is, they can be considered for n-dimensional lists for all $n > 1$. Here, the reverse functions of two types of aggregation functions are studied and the conditions needed so that *Extended Quasi-linear Weighted Mean*, *Extended Ordered Weighted Averaging operator* and their reverse functions are extended aggregation functions are obtained. The conditions obtained allow us to introduce the *right* (*doubly*) *regular* and *right* (*doubly*) *descending triangle of weights*. Consequently, an analysis as where to find these types of triangles and some examples are given. Particularly, triangles which are doubly regular and doubly

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descending are mentioned, as for instance the triangles of the Sierpiński family, etc.

In the last section, two inequalities close to the self-identity of an aggregation function are considered, for this reason they are called the *super-self-identity* and *sub-self-identity* properties. Futhermore, two characterizations are given if the generator of the mean is an increasing function, otherwise only necessary conditions are found.

2. Preliminaries

In this section the definitions and basic results that will be used in this work are presented. Most of them are related with the concept of an Extended Aggregation Function introduced in [12].

Let $(L,\le)$ be a lattice with minimum 0 and maximum 1. For each $n \ge 1$, consider the set L^n of n-dimensional ordered lists $(x_1,\dots,x_n)$ with $x_i \in [0,1]$, $i=1,\dots,n$. Finally $E = \bigcup_{n\ge1} L^n$ is the set of all ordered lists that can be constructed from L.

Definition 1. Let $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_m)$ be elements of $\bigcup_{n\ge1}[0,1]^n$, the following orderings on E can be considered

$$x \le_\pi y \text{ if and only if } n=m \text{ and } x_1 \le y_1,\dots,\ x_n \le y_n \ (\textit{product order}),$$

$$x \le_\alpha y \text{ if and only if } \begin{cases} x_1 \le y_1,\dots,x_n \le y_n, & \text{if } n \le m,\\ \sup(x_1,\dots,x_n) \le \inf(y_{n+1},\dots,y_m), & \text{if } n<m, \end{cases}$$

and

$$x \le_\beta y \text{ if and only if } \begin{cases} x_1 \le y_1,\dots,x_m \le y_m, & \text{if } n \ge m,\\ \sup(x_{m+1},\dots,x_n) \le \inf(y_1,\dots,y_m), & \text{if } n>m. \end{cases}$$

Example 1. If $L=[0,10]$ (with the usual order) then we have:

$$(2,1,7,1) <_\beta (2,1,7) <_\alpha (2,2,7,7) <_\beta (8,7) <_\alpha (8,7,8),$$
$$(8,8,5,5) <_\pi (8,8,6,5) <_\alpha (8,8,6,6) <_\beta (8,8,7).$$

Definition 2. Let $A\colon E \longrightarrow L$ be a mapping from E to L. A is *monotonic* if $x \le_i y$ implies $A(x) \le A(y)$, where $i \in \{\alpha,\beta\}$.

Obviously monotonicity can also be considered for only one of the orderings $\le_\alpha$ or $\le_\beta$. In both cases a useful characterization of α and β-monotonicity can be quoted in the following proposition.

Proposition 1. *Given $A\colon E \longrightarrow L$ monotonic with respect to $\le_\pi$, then*

(1) *A is monotonic with respect to $\le_\alpha$ iff $A(x_1,\dots,x_n) \le A(x_1,\dots,x_n,\sup(x_1,\dots,x_n))$.*

(2) *A is monotonic with respect to $\leq_\beta$ iff* $A(x_1,\dots,x_n,\inf(x_1,\dots,x_n)) \leq A(x_1,\dots,x_n)$,

for all lists $(x_1,\dots,x_n) \in E$.

Next the main concept of Extended Aggregation Function as a mapping from the collection, E of ordered lists (of any dimension) of elements in a given lattice L (space in which the values are aggregated) to the same space L is introduced. The definition is as follows.

DEFINITION 3. $A\colon E \longrightarrow L$ is an *Extended Aggregation Function* if it verifies the following two conditions:

(i) A is monotonic with respect to $\leq_\alpha$ and $\leq_\beta$.

(ii) A is idempotent $A(\overbrace{x,\dots,x}^{n-\text{times}}) = x$ for all $x \in L$ and any $n \geq 1$.

Remark 1. The condition (ii) of the above definition can be replaced by the following one:

$$A(x) = x, \qquad \text{for all} \quad x \in [0,1].$$

Since, using this condition and the last proposition we can deduce the idempotency of A.

In the previous definitions and results we can obviously consider $L = [0,1]$. Recall now the definition of *Extended Ordered Weighted Averaging* operators introduced by G. Mayor et al. in 1997 [12] as extension of the concept introduced by Yager [16] in 1988. Examples and results that relate these operators with the *Extended Aggregation Functions* are given.

DEFINITION 4. $A\colon \bigcup\limits_{n\geq 1} [0,1]^n \longrightarrow [0,1]$ is an *Extended Ordered Weighted Averaging* operator (EOWA operator) if the restriction of A to each $[0,1]^n$ is a n-dimensional OWA operator. In the other words, if for each n there exists

$$W_n = (w_1^n, w_2^n, \dots, w_n^n) \in [0,1]^n \qquad \text{with} \quad \sum_{i=1}^{n} w_i = 1,$$

such that $A(x_1,\dots,x_n) = W_n \cdot R(x_1,\dots,x_n)$ for all $(x_1,\dots,x_n) \in [0,1]^n$, where $R(x_1,\dots,x_n) = (y_1,\dots,y_n)$ is the list consisting of the arguments of $(x_1,\dots,x_n)$ in descending order, and $W_n \cdot R(x_1,\dots,x_n) = \sum\limits_{i=1}^{n} w_i y_i$.

Observe that an EOWA operator as a function on $\bigcup\limits_{n\geq 1} [0,1]^n$ is idempotent, symmetric and monotonic with respect to $\leq_\pi$.

Example 2.

(1) $A(x_1,\ldots,x_n) = \min(x_1,\ldots,x_n)$ is an EOWA operator with $W_n = (\overbrace{0,0,\ldots,0}^{(n-1)-\text{times}},1)$ as weighting list for each n. Similarly $A(x_1,\ldots,x_n) = \max(x_1,\ldots,x_n)$ is an EOWA operator with $W_n = (1, \overbrace{0,\ldots,0}^{(n-1)-\text{times}})$ for each n. Both operators are also EAF.

(2) $A(x_1,\ldots,x_n) = \frac{x_1+\cdots+x_n}{n}$ is an EOWA operator with $W_n = (\overbrace{\frac{1}{n},\ldots,\frac{1}{n}}^{n-\text{times}})$ for all n. It is an EAF.

(3) Consider $A(x_1,\ldots,x_n) = \text{median}(x_1,\ldots,x_n)$, that is:

$$\text{med}(x_1,\ldots,x_n) = \begin{cases} y_{\frac{(n+1)}{2}}, & n \text{ odd},\\ \frac{y_{n/2}+y_{(n/2)+1}}{2}, & n \text{ even},\end{cases}$$

where $(y_1,\ldots,y_n) = R(x_1,\ldots,x_n)$. A is an EOWA operator with $W_n = (0,0,\ldots,1,\ldots,0,0)$ when n is odd, and $W_n = (0,0,\ldots,1/2,1/2,\ldots,0,0)$ when n is even. It is a bit harder than in previous examples to see that this operator is also an *Extended Aggregation Function.*

Now a global definition that collects the weights of any weighting list $W_n = (w_1^n, w_2^n,\ldots,w_n^n)$, $n \geq 1$ is given.

Definition 5. A *probabilistic triangle* or *triangle of weights* is a set of numbers $w_i^n \in [0,1]$, for $i = 1,\ldots,n$ and $n \geq 1$, such that: $\sum_{i=1}^{n} w_i^n = 1$, for all $n \geq 1$. This can be represented by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & w_1^2 & & w_2^2 & & \\ & w_1^3 & & w_2^3 & & w_3^3 & \\ w_1^4 & & w_2^4 & & w_3^4 & & w_4^4 \\ & & & \cdots & & & \end{array}$$

and it will be denoted by $\triangle w_i^n$.

The next result, presented in [12], gives a characterization of EOWA operators that are EAF.

Proposition 2. *Let A be an* EOWA *operator with weighting list $W_n = (w_1^n, w_2^n,\ldots,w_n^n)$ for each $n \geq 1$. Then A is an Extended Aggregation Function if and only if the following inequalities hold for each n and all $p = 1,\ldots,n$*

$$\sum_{i=1}^{p} w_i^{n+1} \leq \sum_{i=1}^{p} w_i^n \leq \sum_{i=1}^{p+1} w_i^{n+1}. \tag{1}$$

The probabilistic triangle that satisfies the last condition is called *left regular*.

EXAMPLE 3. Apart from the examples of EOWA operators which are given in Example 2, a curious example of an EOWA operator that is an *Extended Aggregation Function* is mentioned here as an application of the above proposition. It is the "normalized" Pascal triangle which already appears in [12]

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1/2 & & 1/2 & & & \\ & & 1/4 & & 2/4 & & 1/4 & & \\ & 1/8 & & 3/8 & & 3/8 & & 1/8 & \\ 1/16 & & 4/16 & & 6/16 & & 4/16 & & 1/16 \\ & & & & \cdots & & & & \\ & & & & \cdots \binom{n-1}{k}/2^{n-1} \cdots, & & & & \end{array}$$

where $k = 0, \ldots, n-1$ and let A be the EOWA operator defined by this triangle, that is with weighting lists $W_n = \frac{1}{2^{n-1}}\left(\binom{n-1}{0}, \binom{n-1}{1}, \ldots, \binom{n-1}{n-1}\right)$ for each $n \geq 1$. Using Proposition 2 it can be easily shown that this operator is an *Extended Aggregation Function.*

Now, following the idea of an EOWA, the definition of an *Extended Quasi-Linear Weighted Mean* is quoted.

DEFINITION 6. Let $f\colon [0,1] \to [-\infty, +\infty]$ be a real continuous, strictly monotonic function such that $\{f(0), f(1)\} \neq \{-\infty, +\infty\}$ and let $\triangle w_i^n$, be a triangle of weights with $0 < w_i^n < 1$. An *Extended Quasi-Linear Weighted Mean*, EQLWM, A is a mapping from $\bigcup_{n\geq 1} [0,1]^n$ into $[0,1]$ given by

$$A(x_1, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^{n} w_i^n f(x_i)\right).$$

These means are clearly idempotent and monotonic with respect to the product order, but they are symmetric only if all weights are identical. Symmetric quasi-linear weighted means are usually called quasi-arithmetic means [1] and the most important ones are: arithmetic, geometric, harmonic and quadratic means.

It is interesting to quote that these means can be defined on any real interval $[a,b]$. Consequently, all proofs that are presented here are also true in the case of the real interval.

Remark 2. A special example of an EQLWM is the class of idempotent compensatory operators generated by a continuous Archimedean t-norm T

with an additive generator f and introduced by A. Kolesárová and M. Komorníková [10] as follows:

$$A_T(x_1, x_2, \ldots, x_n) = T\big(f_{n,1}(x_1), f_{n,2}(x_2), \ldots, f_{n,n}(x_n)\big),$$

where $f_{n,i}(x_i) = f^{-1}\big(w_i^n f(x_i)\big)$, for all $n > 1$, $i \in \{1, \ldots, n\}$ and $0 < w_i^n < 1$ such that $\sum_{i=1}^{n} w_i^n = 1$. For instance, the operator A_T with the Łukasiewicz T_L, the product T_P and the Hamacher T_H t-norms and parameters $w_i^n = \frac{n-1}{n}$ with $n = 2$ and $i \in \{1, 2\}$ defines respectively the arithmetic, geometric and harmonic means.

The characterization so that an EQLWM is an EAF appears in the following result.

Theorem 1. *An Extended Quasi-Linear Weighted Mean A is an* EAF *if and only if the triangle of weights verifies*

$$w_i^{n+1} \leq w_i^n, \tag{2}$$

for $i = 1, \ldots, n$ and $n \geq 1$.

The probabilistic triangle that satisfies the last condition, is called here *left descending*, although in [4] it was just called descending triangle.

3. Reverses of an EOWA operators

In this section a study of when the reverse function of an Extended Ordered Weighted Averaging operator is made an EAF is given. Conditions on the weights of the probabilistic triangle associated with this function are found. Consequently, another class of probabilistic triangle is introduced and at the same time several ways to generate it are seen. For that the characterization given by Mayor and Calvo in [12] is considered to see when an EOWA operator and its reverse operator are *Extended Aggregation Functions* at the same time. First, we recall the definition of a reverse EOWA operator.

Given an *Extended Ordered Weighted Averaging* operator, EOWA, A, with a weighting lists $W_n = (w_1^n, \ldots, w_n^n)$ for all $n \geq 1$, the reverse of A, denoted by A^r, is an EOWA operator with weighting lists $U_n = (u_1^n, \ldots, u_n^n)$ where $u_j^n = w_{n-j+1}^n$, for all $j = 1, \ldots, n$ and $n \geq 1$.

Corollary 1. *The reverse of an* EOWA *operator A^r is an* EAF *if and only if the probabilistic triangle of A verifies the following inequalities for each $n \geq 1$ and $q = n+1, \ldots, 2$*

$$\sum_{j=q}^{n+1} w_j^{n+1} \leq \sum_{j=q}^{n+1} w_{j-1}^n \leq \sum_{j=q-1}^{n+1} w_j^{n+1}. \tag{3}$$

In this case, the probabilistic triangle $\triangle w_i^n$ is called *right regular.*

Proposition 3. *If a probabilistic triangle $\triangle w_i^n$ of an EOWA operator A is left regular then the triangle of A^r is right regular and vice versa.*

Consequently, we have

Theorem 2. *Let A be an EOWA operator. Then A and A^r are EAF if and only if the probabilistic triangle of A, $\triangle w_i^n$, fulfills:*

$$\sum_{i=1}^{p} w_i^{n+1} \leq \sum_{i=1}^{p} w_i^n \leq \sum_{i=1}^{p+1} w_i^{n+1}, \tag{4}$$

for all $p = 1, \ldots, n$ and

$$\sum_{j=q}^{n+1} w_j^{n+1} \leq \sum_{j=q}^{n+1} w_{j-1}^n \leq \sum_{j=q-1}^{n+1} w_j^{n+1}, \tag{5}$$

for all $q = n+1, \ldots, 2$.

Proof. For the first inequality see [12]. In the second one, the reverse of A, A^r is assumed to be an EAF then

$$\sum_{i=1}^{p} u_i^{n+1} \leq \sum_{i=1}^{p} u_i^n \leq \sum_{i=1}^{p+1} u_i^{n+1}$$

which is equivalent to

$$\sum_{j=q}^{n+1} w_j^{n+1} \leq \sum_{j=q}^{n+1} w_{j-1}^n \leq \sum_{j=q-1}^{n+1} w_j^{n+1},$$

for all $q = n+1, \ldots, 2$.

Note that we take the values of q from the biggest $n+1$ to the smallest 2, since this order is associated with the geometrical idea of the triangle and from which we choose the name of this type of triangle, i.e., *right regular triangle.* □

A triangle that satisfies the two conditions it will be called *doubly regular*.

Example 4. The following triangles are *doubly regular*:

(1) The triangle associated with the Median

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1/2 & & 1/2 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 1/2 & & 1/2 & & 0 & \\ 0 & & 0 & & 1 & & 0 & & 0 \\ & & & & \cdots & & & & \end{array}$$

(2) The triangle generated by the quantifier $Q(x) = x^2$ [17] or by the normalized sequence of odd numbers.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1/4 & & 3/4 & & & \\
 & & 1/9 & & 3/9 & & 5/9 & & \\
 & 1/16 & & 3/16 & & 5/16 & & 7/16 & \\
1/25 & & 3/25 & & 5/25 & & 7/25 & & 9/25 \\
 & & & & \dots & & & &
\end{array}$$

(3) The "normalized" Pascal triangle that appears in Example 3 can be mentioned.

(4) The triangles of the Sierpiński family that appear in the below examples.

Remark 3. A quantifier $Q\colon [0,1] \to [0,1]$ is an increasing function, such that $Q(0) = 0$, $Q(1) = 1$. Moreover, according to Yager [17] a probabilistic triangle can be defined by means of a quantifier as follows

$$w_j^n = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right) \qquad \text{for all} \quad n \geq 1 \quad \text{and} \quad j = 1, \dots, n\,.$$

Proposition 4. *A quantifier always defines a right regular triangle.*

Proof. It is sufficient to see that the weights of the triangle generated by a quantifier Q verify the relations that appear in Theorem 4. First we calculate

$$\sum_{j=q}^{n+1} w_j^{n+1} = 1 - Q\left(\frac{q-1}{n+1}\right),$$

for all $q = n+1, \dots, 2,\ n \geq 1$.

Then the proof of the conditions of Theorem 4 is equivalent to showing the following ones

$$1 - Q\left(\frac{q-1}{n+1}\right) \leq 1 - Q\left(\frac{q-2}{n}\right) \leq 1 - Q\left(\frac{q-2}{n+1}\right),$$

that can be written as follows

$$Q\left(\frac{q-2}{n+1}\right) \leq Q\left(\frac{q-2}{n}\right) \leq Q\left(\frac{q-1}{n+1}\right).$$

These inequalities are true, the first one due to increasingness of Q and the following relation

$$\frac{q-2}{n+1} \leq \frac{q-2}{n}$$

and the second one because of

$$\frac{q-2}{n} \le \frac{q-1}{n+1},$$

is true for $q \le n+2$ and also for the increasingness of Q. □

Consequently, any quantifier determines a *doubly regular probabilistic triangle.*

PROPOSITION 5. *A sequence of positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ defines a right regular triangle if and only if the sequence $\{\frac{S_{n+1}}{S_n}\}$ is decreasing, with $S_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.*

Proof. Suppose that the triangle is right regular. Then

$$\sum_{j=q}^{n+1} w_j^{n+1} \le \sum_{j=q}^{n} w_{j-1}^{n} \le \sum_{j=q-1}^{n+1} w_j^{n+1},$$

for all $q = n+1, \ldots, 2$.

According to the latter condition, we have

$$w_j^n = \frac{\lambda_j}{S_n},$$

which is

$$\sum_{j=q}^{n+1} \frac{\lambda_j}{S_{n+1}} \le \sum_{j=q}^{n} \frac{\lambda_{j-1}}{S_n} \le \sum_{j=q-1}^{n+1} \frac{\lambda_j}{S_{n+1}},$$

for all $q = n+1, \ldots, 2$.

Computing the above sums, it follows that

$$\frac{S_{n+1} - S_{q-1}}{S_{n+1}} \le \frac{S_n - S_{q-2}}{S_n} \le \frac{S_{n+1} - S_{q-2}}{S_{n+1}},$$

then

$$1 - \frac{S_{q-1}}{S_{n+1}} \le 1 - \frac{S_{q-2}}{S_n} \le 1 - \frac{S_{q-2}}{S_{n+1}},$$

which is equivalent to

$$\frac{S_{q-2}}{S_{n+1}} \le \frac{S_{q-2}}{S_n} \le \frac{S_{q-1}}{S_{n+1}}.$$

Taking $q = n+1$ in the second of the above inequalities, we have

$$\frac{S_{n+1}}{S_n} \le \frac{S_n}{S_{n-1}}.$$

Then the sequence $\{\frac{S_{n+1}}{S_n}\}$ is decreasing. The converse can be shown by simple calculations. □

This condition is the same for a *left regular probabilistic triangle* and hence it can be concluded that if the sequence of positive real numbers fulfills the above condition then it determines a *doubly regular triangle.*

4. Reverses of an EQLWM operators

We study when the reverse function of an Extended Quasi-Linear Weighted Mean is made an EAF is given. Similarly to the case of an EOWA operator, we find conditions on the weights of the probabilistic triangle associated with these functions. Consequently, another class of probabilistic triangle is introduced and we present different ways to generate the above class of triangle.

Definition 7. Given an *Extended Quasi-Linear Weighted Mean*, EQLWM, A with a weighting lists $W_n = (w_1^n, \dots, w_n^n)$ for all $n \geq 1$ and a generator f, the *reverse* of A, denoted by A^r, is defined as the EQLW mean with weighting lists $U_n = (u_1^n, \dots, u_n^n)$ where $u_j^n = w_{n-j+1}^n$, for all $j = 1, \dots, n$, $n \geq 1$ and with the same generator f of A.

Easily, it can be seen that

Proposition 6. *If an* EQLWM A *is an* EAF *then the probabilistic triangle associated with the reverse of* A *fulfills:*

$$u_{j+1}^{n+1} \leq u_j^n ,$$

for $j = 1, \dots, n$.

Taking into account Theorem 1, we obtain easily the condition of when the reverse of A, A^r is an EAF.

Corollary 2. *The reverse of an* EQLWM A^r *is an* EAF *if and only if the probabilistic triangle of* A *satisfies*

$$w_{j+1}^{n+1} \leq w_j^n , \tag{6}$$

for $j = 1, \dots, n$ *and* $n \geq 1$.

The triangle that fulfills the above condition is called *right descending*.

According to this result and the definition of a right descending triangle some examples can be mentioned.

Example 5. The following triangles are *right descending*:

(1) The triangle associated with the operators: min, max and Arithmetic Mean which are respectively as follows

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 1 & & \\ & 0 & & 0 & & 1 & \\ 0 & & 0 & & 0 & & 1 \\ & & & \dots & & & \end{array}$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 0 & & \\
 & 1 & & 0 & & 0 & \\
1 & & 0 & & 0 & & 0 \\
 & & & \cdots & & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \frac{1}{2} & & \frac{1}{2} & & \\
 & \frac{1}{3} & & \frac{1}{3} & & \frac{1}{3} & \\
\frac{1}{4} & & \frac{1}{4} & & \frac{1}{4} & & \frac{1}{4} \\
 & & & \cdots & & &
\end{array}
$$

(2) The triangle generated by the Sugeno negation, with $\lambda = 1$ (see Remark 4).

(3) The geometric progressions.

(4) The triangle of the Sierpiński family (see Remark 4).

Remark 4.

(i) A probabilistic triangle generated by means of a negation, N (see [5]), is defined as follows

$$w_j^n = N\left(\frac{j-1}{n}\right) - N\left(\frac{j}{n}\right),$$

for all $j = 1, \dots, n$ and $n \geq 1$. Considering the Sugeno negation, i. e.,

$$N_\lambda(x) = \frac{1-x}{1+\lambda x}, \qquad \lambda > -1.$$

for $\lambda = 1$, the following weights are obtained

$$w_j^n = N_1\left(\frac{j-1}{n}\right) - N_1\left(\frac{j}{n}\right) = \frac{2n}{(n+j)(n+j-1)} = \frac{n}{\binom{n+j}{2}}.$$

Consequently, we have the triangle

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 4/2\cdot 3 & & 4/3\cdot 4 & & \\
 & 6/3\cdot 4 & & 6/4\cdot 5 & & 6/5\cdot 6 & \\
8/4\cdot 5 & & 8/5\cdot 6 & & 8/6\cdot 7 & & 8/7\cdot 8 \\
 & & & \cdots & & &
\end{array}
$$

(ii) Triangles of the Sierpiński family are determined following a process similar to the one in the generation of two well known fractals, as the

Sierpiński triangle and Sierpiński carpet [5]. That is, the following triangle belongs to the Sierpiński family [5], where $a < b$ and $\frac{a}{b} \neq \frac{b-a}{b}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & \frac{a}{b} & & \frac{b-a}{b} & & \\ & \frac{a}{b} & & (\frac{b-a}{b})(\frac{a}{b}) & & (\frac{b-a}{b})^2 & \\ \frac{a}{b} & & (\frac{b-a}{b})(\frac{a}{b}) & & (\frac{b-a}{b})^2(\frac{a}{b}) & & (\frac{b-a}{b})^3 \\ & & & \cdots & & & \end{array}$$

The weights of this triangle are

$$w_i^n = \frac{a}{b} \cdot \Big(\frac{b-a}{b}\Big)^{i-1}, \qquad i = 1, \dots, n-1 \quad \text{and } w_n^n = \Big(\frac{b-a}{b}\Big)^{n-1}.$$

Note, that the weights of this triangle determine a geometric progression for each $n > 1$ and for this reason the aggregation operators associated with this type of triangles can be called "geometric operators". If $\frac{a}{b} = \frac{1}{4}$ or $\frac{a}{b} = \frac{1}{9}$ the triangles of Example 6 are obtained, also known as the Sierpiński triangle and the Sierpiński carpet.

Easily it follows that if a triangle $\triangle w_i^n$ is *right descending* then the triangle $\triangle u_i^n$, where $u_i^n = w_{n-i+1}^n$ for all $i = 1, \dots, n$ is a *left descending* triangle and vice-versa. Consequently

THEOREM 3. *Let A be an Extended Quasi-Linear Weighted Mean,* EQLWM, *then A and A^r are* EAF *if and only if the probabilistic triangle of A fulfills:*

$$w_i^{n+1} \leq w_i^n, \qquad i = 1, \dots, n, \tag{7}$$

and

$$w_{i+1}^{n+1} \leq w_i^n, \qquad i = 1, \dots, n. \tag{8}$$

That is, the triangle $\triangle w_i^n$ of A is right descending and left descending.

DEFINITION 8. A probabilistic triangle that satisfies the conditions (7) and (8) will be called *doubly descending*. That is, the triangle is a *right* and *left descending* triangle.

Remark 5. Given a sequence of positive real numbers $\lambda_1, \lambda_2, \dots \geq 0$, such that $\lambda_1 > 0$, a probabilistic triangle can be defined in the following way:

$$w_i^n = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n},$$

for all $n \geq 1$. Observe that $\sum\limits_{i=1}^{n} w_i^n = \frac{\sum_{i=1}^{n} \lambda_i}{\lambda_1 + \cdots + \lambda_n} = 1$, therefore we obtain a probabilistic triangle.

Proposition 7. *Any sequence of positive real numbers $\lambda_1, \lambda_2, \dots \geq 0$ generates a right descending triangle in the following way:*

$$w_i^n = \frac{\lambda_{n-i+1}}{\lambda_1 + \cdots + \lambda_n},$$

for all $i = 1, \dots, n$.

We can define also a right descending triangle as the reverse of a left descending triangle. In this case, the weights must be defined by means of a sequence of positive real numbers as follows

$$w_i^n = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n},$$

for all $i = 1, \dots, n$.

Example 6. Any probabilistic triangle of the Sierpiński family [5] is *doubly descending*, for instance

$$\begin{array}{c} 1 \\ \\ 1.\frac{1}{4} \quad 3.\frac{1}{4} \\ \\ 1.\frac{1}{4} \quad 3.\frac{1}{4^2} \quad 3^2.\frac{1}{4^2} \\ \\ 1.\frac{1}{4} \quad 3.\frac{1}{4^2} \quad 3^2.\frac{1}{4^3} \quad 3^3.\frac{1}{4^3} \\ \\ \dots \end{array}$$

and

$$\begin{array}{c} 1 \\ \\ 1 \cdot \frac{1}{9} \quad 8 \cdot \frac{1}{9} \\ \\ 1 \cdot \frac{1}{9} \quad 8 \cdot \frac{1}{9^2} \quad 8^2 \cdot \frac{1}{9^2} \\ \\ 1 \cdot \frac{1}{9} \quad 8 \cdot \frac{1}{9^2} \quad 8^2 \cdot \frac{1}{9^3} \quad 8^3 \cdot \frac{1}{9^3} \\ \\ \dots \end{array}$$

Moreover, all the triangles mentioned in the case of *right descending* triangles are also *left descending*, therefore they are *doubly descending*.

5. The super-self-identity and the sub-self-identity of an EQLWM and EOWA operator

Two properties of Extended Quasi-Linear Weighted Means and EOWA operators are studied. These properties are close to Proposition 1 since an EQLWM is always between the Minimum and the Maximum then why not study the relations that appear in this proposition replacing the Minimum and the Maximum for an EQLWM or an EOWA operator respectively. The name given to these two properties is due to their relationship with *the self identity property.*

THEOREM 4. *Let A be an Extended Quasi-Linear Weighted Mean,* EQLWM, *with an increasing generator f and a probabilistic triangle $\triangle w_i^n$. Then A fulfills*

$$A\big(x_1,\dots,x_n,A(x_1,\dots,x_n)\big)\le A(x_1,\dots,x_n) \tag{9}$$

for all $(x_1,\dots,x_n)$ if and only if

$$w_i^{n+1}\le w_i^n\big(1-w_{n+1}^{n+1}\big)\,, \tag{10}$$

for $i=1,\dots,n$.

P r o o f. Suppose that A satisfies the above inequality then

$$f^{-1}\left(\sum_{\substack{j=1\\ j\neq i}}^{n} w_j^{n+1}f(x_j)+w_i^{n+1}f(x_i)+w_{n+1}^{n+1}\left(\sum_{\substack{j=1\\ j\neq i}}^{n} w_j^n f(x_j)+w_i^n f(x_i)\right)\right)$$
$$\le f^{-1}\left(\sum_{\substack{j=1\\ j\neq i}}^{n} w_j^n f(x_j)+w_i^n f(x_i)\right).$$

Setting $x_i=1, x_j=0$ for all $j\neq i$ in the last expression by a simple reordering we have

$$\big(w_i^{n+1}+w_{n+1}^{n+1}w_i^n-w_i^n\big)\,f(1)\le\left(-\sum_{\substack{j=1\\ j\neq i}}^{n} w_j^{n+1}-w_{n+1}^{n+1}\sum_{\substack{j=1\\ j\neq i}}^{n} w_j^n+\sum_{\substack{j=1\\ j\neq i}}^{n} w_j^n\right)f(0)\,.$$

But the coefficient of $f(0)$, P, is equal to the coefficient of $f(1)$, Q, since

$$\begin{aligned}P&=-\big(1-w_i^{n+1}-w_{n+1}^{n+1}\big)-w_{n+1}^{n+1}(1-w_i^n)+(1-w_i^n)\\&=(1-w_i^n)(1-w_{n+1}^{n+1})-(1-w_{n+1}^{n+1})+w_i^{n+1}\\&=-\big(1-w_{n+1}^{n+1}\big)w_i^n+w_i^{n+1}=-w_i^n+w_i^n w_{n+1}^{n+1}+w_i^{n+1}=Q\,.\end{aligned}$$

So that, $P(f(1)-f(0)) \leq 0$ and as $f(1)-f(0) \geq 0$, then $P \leq 0$.

Therefore, $w_i^{n+1} \leq w_i^n(1-w_{n+1}^{n+1}), i=1,\ldots,n$.

Conversely, it is enough to see that

$$\left(\sum_{j=1}^{n} w_j^{n+1} f(x_j) + w_{n+1}^{n+1} \sum_{j=1}^{n} w_j^n f(x_j)\right) \leq \left(\sum_{j=1}^{n} w_j^n f(x_j)\right) .$$

But, taken into account condition (10), the first term of this inequality fulfills

$$\begin{aligned} &\left(\sum_{j=1}^{n} w_j^{n+1} f(x_j) + w_{n+1}^{n+1} \sum_{j=1}^{n} w_j^n f(x_j)\right) \\ \leq &\left(\sum_{j=1}^{n} w_j^n (1-w_{n+1}^{n+1}) f(x_j) + w_{n+1}^{n+1} \sum_{j=1}^{n} w_j^n f(x_j)\right) \\ = &\left(\sum_{j=1}^{n} w_j^n f(x_j)\right) . \end{aligned}$$

This concludes the proof. □

Similarly

THEOREM 5. *Let A be an Extended Quasi-Linear Weighted Mean,* EQLWM, *with an increasing generator f and a probabilistic triangle $\triangle w_i^n$. Then A fulfills*

$$A(x_1,\ldots,x_n,A(x_1,\ldots,x_n)) \geq A(x_1,\ldots,x_n) \tag{11}$$

for all $(x_1,\ldots,x_n)$ if and only if

$$w_i^{n+1} \geq w_i^n(1-w_{n+1}^{n+1}), \tag{12}$$

for $i=1,\ldots,n$.

Conversely, the above theorems are not true when the generator f of the EQLWM A is a decreasing function. Although, the following results can be mentioned

PROPOSITION 8. *If A is an Extended Quasi-Linear Weighted Mean,* EQLWM, *with a decreasing generator f and a probabilistic triangle $\triangle w_i^n$, such that A fulfills*

$$A(x_1,\ldots,x_n,A(x_1,\ldots,x_n)) \leq A(x_1,\ldots,x_n) \tag{13}$$

then

$$w_i^{n+1} \leq w_i^n(1-w_{n+1}^{n+1}), \qquad \text{for all} \quad i=1,\ldots,n. \tag{14}$$

Proof. We assume that A holds the above inequality and taking into account that f is a decreasing function it follows that

$$\sum_{\substack{j=1\\ j\neq i}}^{n} w_j^{n+1} f(x_j) + w_i^{n+1} f(x_i) + w_{n+1}^{n+1} \left(\sum_{\substack{j=1\\ j\neq i}}^{n} w_j^n f(x_j) + w_i^n f(x_i) \right)$$
$$\geq \sum_{\substack{j=1\\ j\neq i}}^{n} w_j^n f(x_j) + w_i^n f(x_i).$$

Now, taking $x_i = 1$ and $x_j = 0$, for all $j \neq i$,

$$\left(\sum_{\substack{j=1\\ j\neq i}}^{n} w_j^{n+1} + w_{n+1}^{n+1} \sum_{\substack{j=1\\ j\neq i}}^{n} w_j^n - \sum_{\substack{j=1\\ j\neq i}}^{n} w_j^n \right) f(0) \geq \left(w_i^n - w_i^{n+1} - w_{n+1}^{n+1} w_i^n + \right) f(1).$$

For arguments similar to the above theorem it is easy to see that the coefficient of $f(0)$ is equal to the coefficient of $f(1)$. And since $f(0) - f(1) \geq 0$, then $w_i^{n+1} \leq w_i^n (1 - w_{n+1}^{n+1}), i = 1, \dots, n$. □

Similarly

Proposition 9. *If A is an Extended Quasi-Linear Weighted Mean,* EQLWM, *with a decreasing generator f and a probabilistic triangle $\triangle w_i^n$, such that A fulfills*

$$A(x_1, \dots, x_n, A(x_1, \dots, x_n)) \geq A(x_1, \dots, x_n), \tag{15}$$

for all $(x_1, \dots, x_n)$ then

$$w_i^{n+1} \geq w_i^n (1 - w_{n+1}^{n+1}), \qquad \textit{for} \quad i = 1, \dots, n. \tag{16}$$

The above conditions (14) and (16) are not sufficient because it is possible to find an EQLWM, A, whose triangle holds one of these conditions but A does not verify the corresponding relation (13) or (15). For the second case we present the following counterexample:

Example 7. The following triangle fulfills the condition (16) of Proposition 9. However, the *Extended Quasi-Linear Weighted Mean* built by a probabilistic triangle with condition (16) and a positive decreasing generator does not verify the mentioned inequality (15) for all n.

$$\begin{array}{c} 1 \\ 0.2 \quad 0.8 \\ 0.15 \quad 0.45 \quad 0.5 \\ 0.110 \quad 0.320 \quad 0.40 \quad 0.3 \\ \dots \end{array}$$

Based on these results, a sufficient condition for each one of the last two propositions can be mentioned

Proposition 10. *Let A be an* EQLWM *with:*

(1) *A negative decreasing generator f and a probabilistic triangle with weights:*

$$w_i^{n+1} \leq w_i^n \left(1 - w_{n+1}^{n+1}\right), \qquad i = 1, \dots, n\,.$$

(2) *A negative decreasing generator f and a probabilistic triangle with weights:*

$$w_i^{n+1} \geq w_i^n \left(1 - w_{n+1}^{n+1}\right), \qquad i = 1, \dots, n\,.$$

then A satisfies conditions (13) *and* (15) *respectively.*

Proposition 11. *If A is an Extended* OWA *operator,* EOWA, *with a probabilistic triangle $\triangle w_i^n$, such that, it fulfills*

$$A(x_1, \dots, x_n, A(x_1, \dots, x_n)) \geq A(x_1, \dots, x_n)\,,$$

for all $(x_1, \dots, x_n)$ then $\sum\limits_{i=1}^{p} w_i^{n+1} \geq \sum\limits_{i=1}^{p} w_i^n \left(1 - w_{p+1}^{n+1}\right)$, $p = 1, \dots, n$.

Proof. It is easy to obtain the last condition taking $x_i = 1$ for $i = 1, \dots, p$, $x_j = 0$ for $j = p+1, \dots, n$ and considering the definition of an EOWA operator. □

Example 8. The following triangles verify the above relation

(1) The triangles associated with: Minimum, Maximum, Median and Arithmetic Mean.

(2) The triangles of the Sierpiński family.

Moreover,

PROPOSITION 12. *If A is an Extended* OWA *operator,* EOWA, *with a probabilistic triangle $\triangle w_i^n$ such that, it fulfills*

$$A(x_1,\dots,x_n,A(x_1,\dots,x_n)) \le A(x_1,\dots,x_n)$$

then $\sum_{i=1}^{p} w_i^{n+1} \le \sum_{i=1}^{p} w_i^n\left(1-w_{p+1}^{n+1}\right),\ p=1,\dots,n.$

EXAMPLE 9. The following triangles verify the relation of the above proposition

(1) The triangles associated with: Minimum, Maximum, Median and Arithmetic Mean.

(2) The normalized Pascal triangle.

(3) The triangle defined by the normalized sequence of odd numbers.

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